Vectors in the Plane

- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.
- Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane’s speed. When measuring a force, such as the thrust of the plane’s engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as or force, are defined in terms of both size (also called *magnitude*) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as **v**.

Note:
Definition
A vector is a quantity that has both magnitude and direction.

Vector Representation

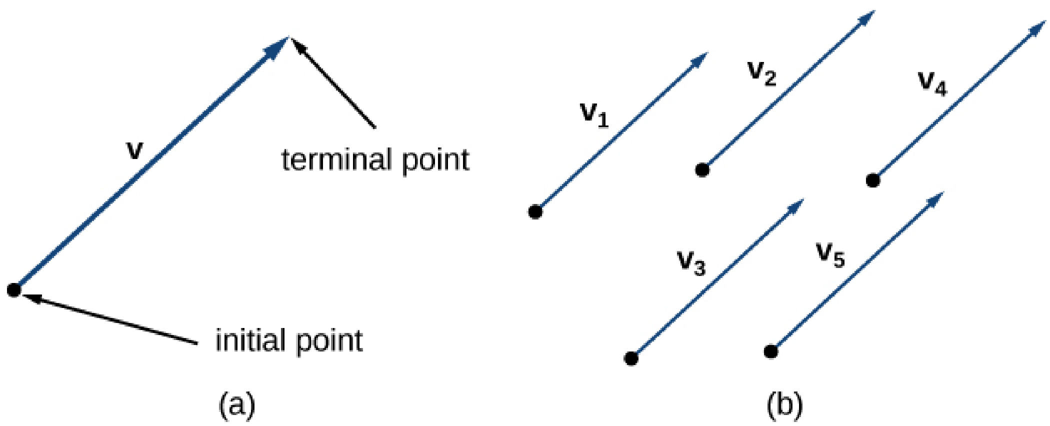
A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\mathbf{v}\|$ to denote the magnitude of the vector $\mathbf{v}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\mathbf{0}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if **v** and **w** are equivalent, we write
Equation:

$$\mathbf{v} = \mathbf{w}.$$

Note:
Definition
Vectors are said to be **equivalent vectors** if they have the same magnitude and direction.

The arrows in [link](b) are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.



(a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\mathbf{v}_1$ through $\mathbf{v}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to simulate boldface type: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\vec{PQ}$ is useful because it indicates the direction and location of the vector.

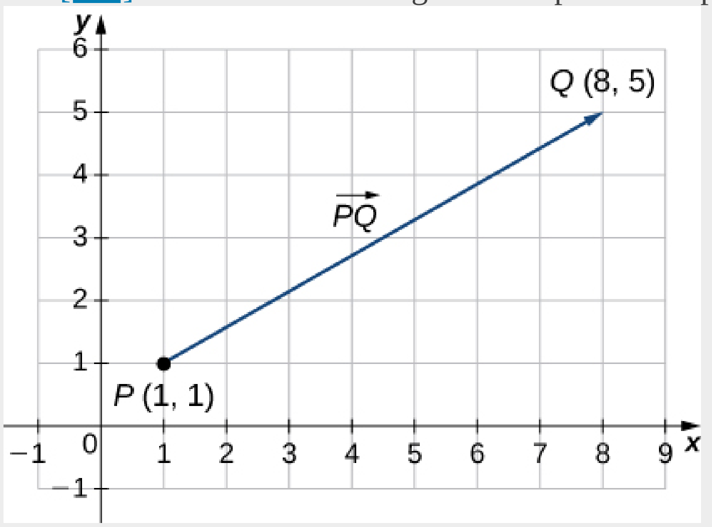
Example:
Exercise:

Problem:
Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution:

See [link]. Because the vector goes from point P to point Q, we name it $\vec{PQ}$.

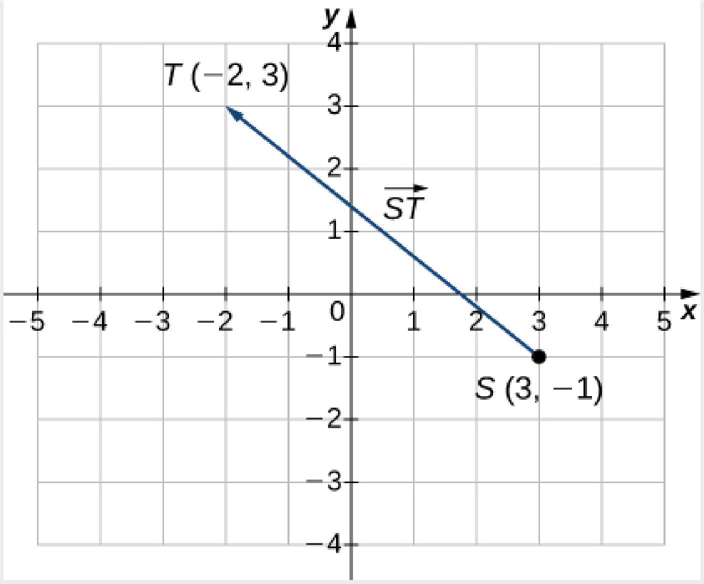


The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\vec{PQ}$.

Note:
Exercise:

Problem: Sketch the vector $\vec{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Solution:



Hint

The first point listed in the name of the vector is the initial point of the vector.

Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

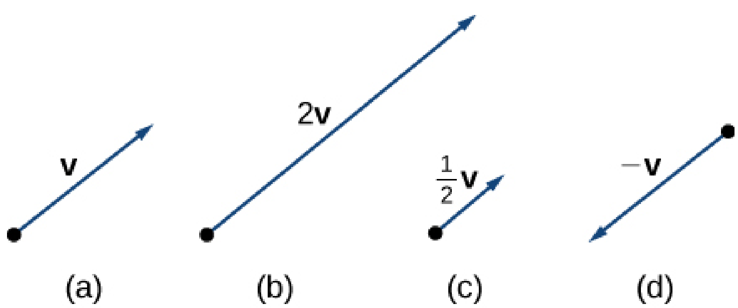
A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Note:
Definition
The product $k\mathbf{v}$ of a vector $\mathbf{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\mathbf{v}$, and with a direction that is the same as the direction of $\mathbf{v}$ if $k > 0$, and opposite the direction of $\mathbf{v}$ if $k < 0$. This is called **scalar multiplication**. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

As you might expect, if $k = -1$, we denote the product $k\mathbf{v}$ as
Equation:

$$k\mathbf{v} = (-1)\mathbf{v} = -\mathbf{v}.$$

Note that $-\mathbf{v}$ has the same magnitude as $\mathbf{v}$, but has the opposite direction ([link]).

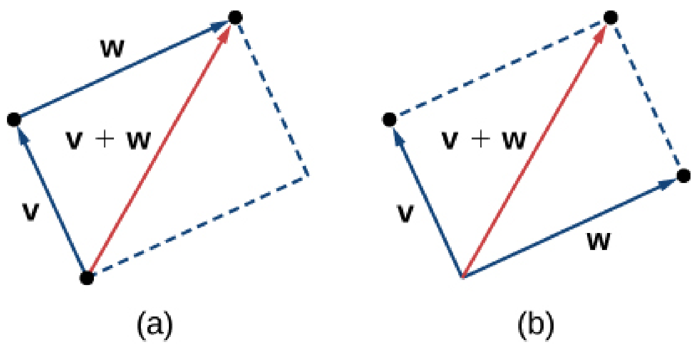


(a) The original vector $\mathbf{v}$ has length n units. (b) The length of $2\mathbf{v}$ equals $2n$ units. (c) The length of $\mathbf{v}/2$ is $n/2$ units. (d) The vectors $\mathbf{v}$ and $-\mathbf{v}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in [link](a). To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\mathbf{v}$, then from the initial point to the terminal point of vector $\mathbf{w}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\mathbf{v} + \mathbf{w}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\mathbf{w}$ was our first vector and $\mathbf{v}$ was our second vector, we would have ended up in the same place. (Again, see [link](a).) Thus, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

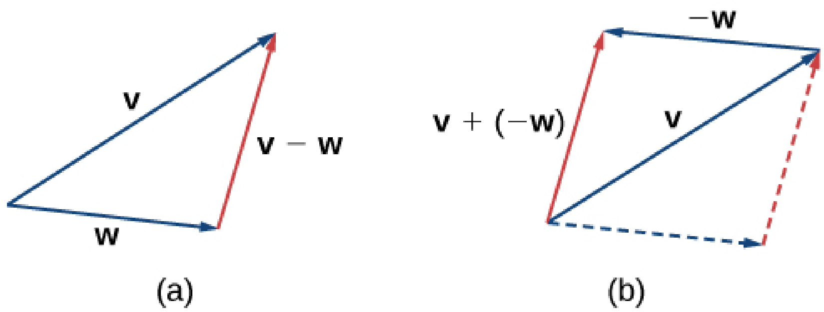
A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in [link](b). The length of the diagonal of the parallelogram is the sum. Comparing [link](b) and [link](a), we can see that we get the same answer using either method. The vector $\mathbf{v}+\mathbf{w}$ is called the **vector sum**.

Note:
Definition
The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$. Then, the vector sum, $\mathbf{v}+\mathbf{w}$, is the vector with an initial point that coincides with the initial point of $\mathbf{v}$ and has a terminal point that coincides with the terminal point of $\mathbf{w}$. This operation is known as **vector addition**.



(a) When adding vectors by the triangle method, the initial point of $\mathbf{w}$ is the terminal point of $\mathbf{v}$. (b) When adding vectors by the parallelogram method, the vectors $\mathbf{v}$ and $\mathbf{w}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\mathbf{v}-\mathbf{w}$ as $\mathbf{v}+(-\mathbf{w})=\mathbf{v}+(-1)\mathbf{w}$. The vector $\mathbf{v}-\mathbf{w}$ is called the **vector difference**. Graphically, the vector $\mathbf{v}-\mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$ ([link]).



(a) The vector difference $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$. (b) The vector $\mathbf{v} - \mathbf{w}$ is equivalent to the vector $\mathbf{v} + (-\mathbf{w})$.

In [link](a), the initial point of $\mathbf{v} + \mathbf{w}$ is the initial point of $\mathbf{v}$. The terminal point of $\mathbf{v} + \mathbf{w}$ is the terminal point of $\mathbf{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

Equation:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

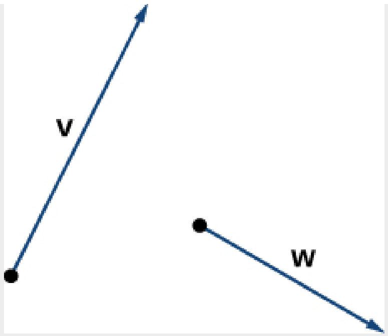
This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\mathbf{u} + \mathbf{v}$ has the same magnitude as the sum of the magnitudes of $\mathbf{u}$ and $\mathbf{v}$. This happens only when $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

Example:
Exercise:

Problem:
Combining Vectors

Given the vectors $\mathbf{v}$ and $\mathbf{w}$ shown in [link], sketch the vectors

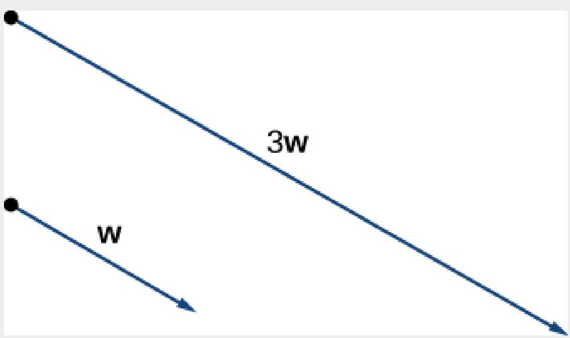
a. $3\mathbf{w}$
b. $\mathbf{v} + \mathbf{w}$
c. $2\mathbf{v} - \mathbf{w}$



Vectors **v** and **w** lie in the same plane.

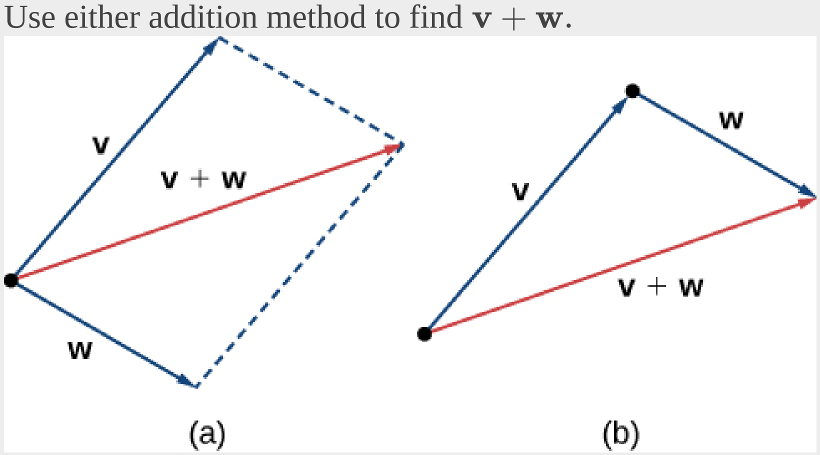
Solution:

a. The vector **3w** has the same direction as **w**; it is three times as long as **w**.



Vector **3w** has the same direction as **w** and is three times as long.

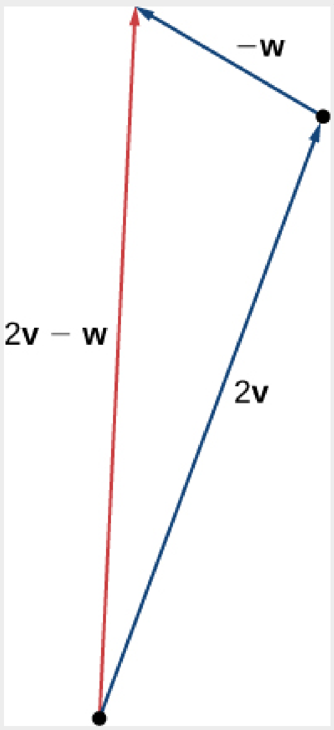
b. Use either addition method to find **v** + **w**.



To find **v** + **w**, align the vectors at their initial points

or place the initial point of one vector at the terminal point of the other. (a) The vector $\mathbf{v}+\mathbf{w}$ is the diagonal of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ (b) The vector $\mathbf{v}+\mathbf{w}$ is the third side of a triangle formed with $\mathbf{w}$ placed at the terminal point of $\mathbf{v}$.

c. To find $2\mathbf{v}-\mathbf{w}$, we can first rewrite the expression as $2\mathbf{v}+(-\mathbf{w})$. Then we can draw the vector $-\mathbf{w}$, then add it to the vector $2\mathbf{v}$.

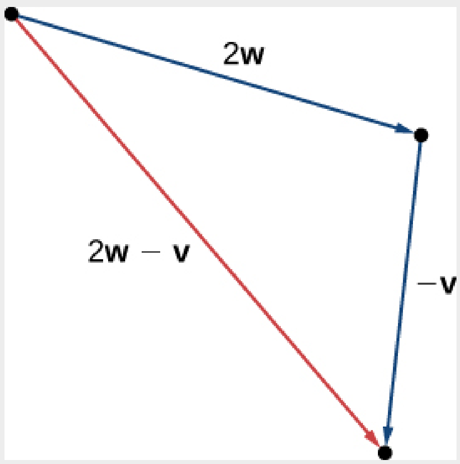


To find $2\mathbf{v}-\mathbf{w}$, simply add $2\mathbf{v}+(-\mathbf{w})$.

Note:
Exercise:

Problem: Using vectors $\mathbf{v}$ and $\mathbf{w}$ from [link], sketch the vector $2\mathbf{w}-\mathbf{v}$.

Solution:



Hint

First sketch vectors $2\mathbf{w}$ and $-\mathbf{v}$.

Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example:
Exercise:

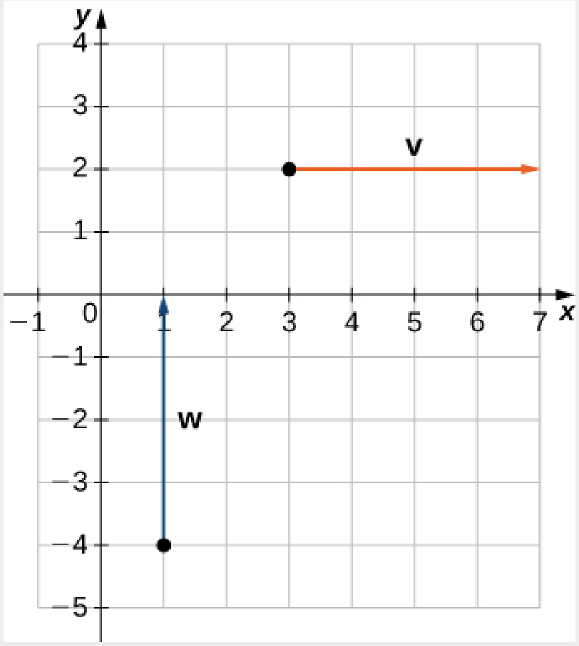
Problem:
Comparing Vectors

Are $\mathbf{v}$ and $\mathbf{w}$ equivalent vectors?

a. $\mathbf{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
$\mathbf{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$
b. $\mathbf{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
$\mathbf{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

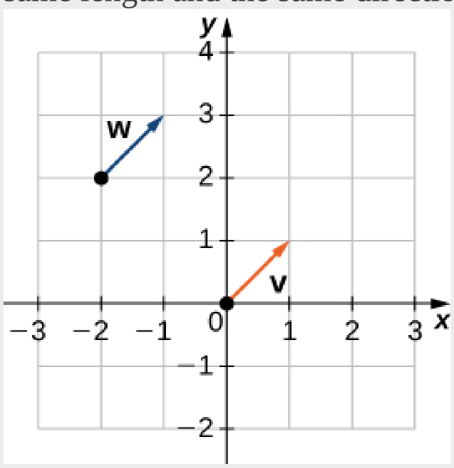
Solution:

a. The vectors are each 4 units long, but they are oriented in different directions. So **v** and **w** are not equivalent ([link]).



These vectors are not equivalent.

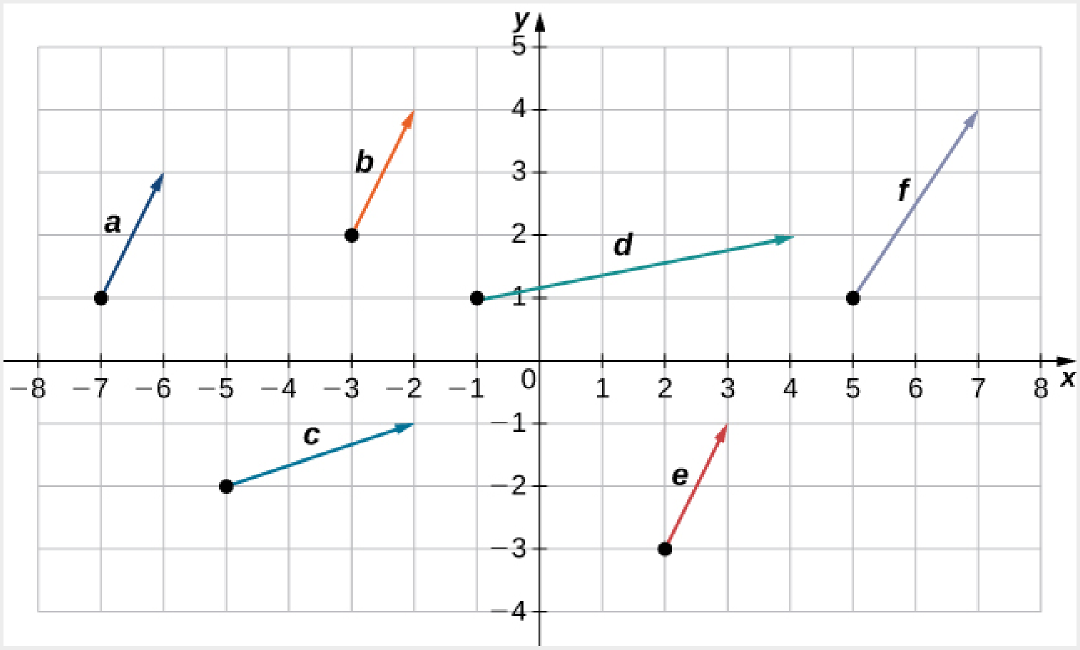
b. Based on [link], and using a bit of geometry, it is clear these vectors have the same length and the same direction, so **v** and **w** are equivalent.



These vectors are equivalent.

Note:
Exercise:

Problem: Which of the following vectors are equivalent?



Solution:

Vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{e}$ are equivalent.

Hint

Equivalent vectors have both the same magnitude and the same direction.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector**. Because the initial point of any vector in standard position is known to be $(0,0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector **v** has its initial point at the origin and its terminal point at (x,y), we write the vector in component form as

Equation:

$$\mathbf{v} = \langle x, y \rangle.$$

When a vector is written in component form like this, the scalars *x* and *y* are called the **components** of $\mathbf{v}$.

Note:
Definition
The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as
Equation:

$$\mathbf{v} = \langle x, y \rangle.$$

The scalars x and y are called the components of $\mathbf{v}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the *x*-coordinate of the initial point from the *x*-coordinate of the terminal point to get the *x* component, and we subtract the *y*-coordinate of the initial point from the *y*-coordinate of the terminal point to get the *y* component.

Note:
Rule: Component Form of a Vector
Let **v** be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express **v** in component form as $\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle$.

Example:
Exercise:

Problem:
Expressing Vectors in Component Form

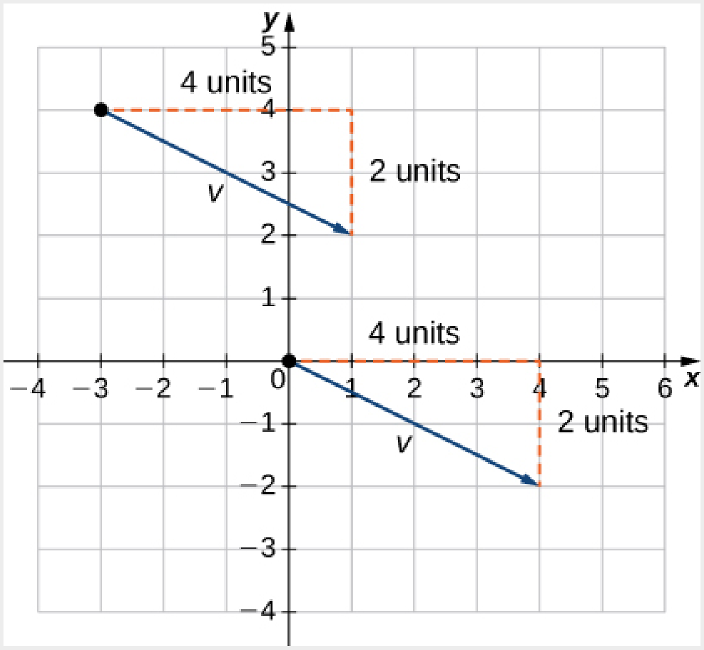
Express vector **v** with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric

1. Sketch the vector in the coordinate plane ([link]).
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:
   Equation:

$$\mathbf{v} = \langle 4, -2 \rangle.$$



These vectors are equivalent.

b. Algebraic
In the first solution, we used a sketch of the vector to see that the terminal point

lies 4 units to the right. We can accomplish this algebraically by finding the difference of the *x*-coordinates:

Equation:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the *y*-coordinates shows the vertical length of the vector.

Equation:

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

Equation:

$$\begin{aligned}\mathbf{v} &= \langle x_t - x_i, y_t - y_i \rangle \\ &= \langle 1 - (-3), 2 - 4 \rangle \\ &= \langle 4, -2 \rangle.\end{aligned}$$

Note:

Exercise:

Problem:

Vector $\mathbf{w}$ has initial point $(-4, -5)$ and terminal point $(-1, 2)$. Express $\mathbf{w}$ in component form.

Solution:

$\langle 3, 7 \rangle$

Hint

You may use either geometric or algebraic method.

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The magnitude of vector $\mathbf{v} = \langle x, y \rangle$ is denoted $\| \mathbf{v} \|$, or $|\mathbf{v}|$, and can be computed using the formula

Equation:

$$\| \mathbf{v} \| = \sqrt{x^2 + y^2}.$$

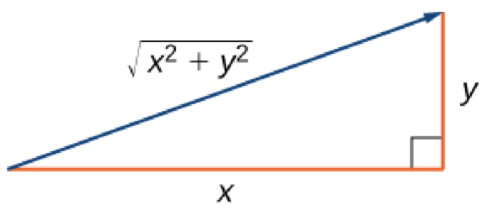
Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

Equation:

$$\begin{aligned} \| \mathbf{v} \| &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Based on this formula, it is clear that for any vector $\mathbf{v}$, $\| \mathbf{v} \| \geq 0$, and $\| \mathbf{v} \| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.



If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Note:

Definition

Let $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication:$k\mathbf{v} = \langle kx_1, ky_1 \rangle$

Vector addition:$\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

Example:

Exercise:

Problem:

Performing Operations in Component Form

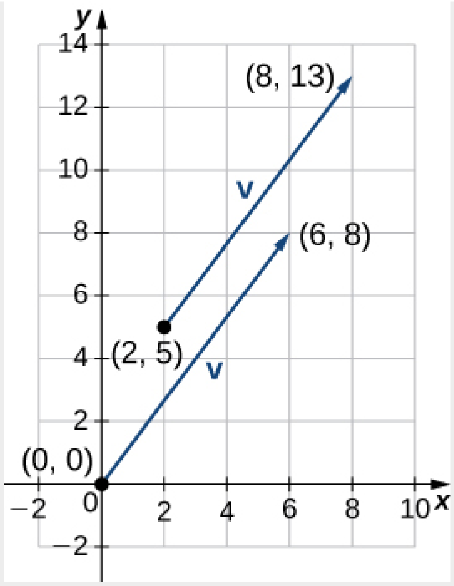
Let $\mathbf{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\mathbf{w} = \langle -2, 4 \rangle$.

a. Express $\mathbf{v}$ in component form and find $\| \mathbf{v} \|$. Then, using algebra, find
b. $\mathbf{v} + \mathbf{w}$,
c. $3\mathbf{v}$, and
d. $\mathbf{v} - 2\mathbf{w}$.

Solution:

a. To place the initial point of $\mathbf{v}$ at the origin, we must translate the vector 2 units to the left and 5 units down ([link]). Using the algebraic method, we can express $\mathbf{v}$ as $\mathbf{v} = \langle 8 - 2, 13 - 5 \rangle = \langle 6, 8 \rangle$:
Equation:

$$\| \mathbf{v} \| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$



In component form,
$\mathbf{v} = \langle 6, 8 \rangle$.

b. To find $\mathbf{v} + \mathbf{w}$, add the *x*-components and the *y*-components separately:
Equation:

$$\mathbf{v} + \mathbf{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\mathbf{v}$, multiply $\mathbf{v}$ by the scalar $k = 3$:
Equation:

$$3\mathbf{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\mathbf{v} - 2\mathbf{w}$, find $-2\mathbf{w}$ and add it to $\mathbf{v}$:
Equation:

$$\mathbf{v} - 2\mathbf{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Note:
Exercise:

Problem:

Let $\mathbf{a} = \langle 7, 1 \rangle$ and let $\mathbf{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\mathbf{a}\|$.
b. Express $\mathbf{b}$ in component form.
c. Find $3\mathbf{a} - 4\mathbf{b}$.

Solution:

a. $\|\mathbf{a}\| = 5\sqrt{2}$, b. $\mathbf{b} = \langle -4, -3 \rangle$, c. $3\mathbf{a} - 4\mathbf{b} = \langle 37, 15 \rangle$

Hint

Use the Pythagorean Theorem to find $\|\mathbf{a}\|$. To find $3\mathbf{a} - 4\mathbf{b}$, start by finding the scalar multiples $3\mathbf{a}$ and $-4\mathbf{b}$.

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Note:
Properties of Vector Operations
Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be vectors in a plane. Let r and s be scalars.
Equation:

i.	$\mathbf{u} + \mathbf{v}$	$=$	$\mathbf{v} + \mathbf{u}$	Commutative property
ii.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$=$	$\mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property
iii.	$\mathbf{u} + 0$	$=$	$\mathbf{u}$	Additive identity property
iv.	$\mathbf{u} + (-\mathbf{u})$	$=$	0	Additive inverse property
v.	$r(s\mathbf{u})$	$=$	$(rs)\mathbf{u}$	Associativity of scalar multiplication
vi.	$(r + s)\mathbf{u}$	$=$	$r\mathbf{u} + s\mathbf{u}$	Distributive property
vii.	$r(\mathbf{u} + \mathbf{v})$	$=$	$r\mathbf{u} + r\mathbf{v}$	Distributive property
viii.	$1\mathbf{u}$	$=$	$\mathbf{u}, 0\mathbf{u} = 0$	Identity and zero properties

Proof of Commutative Property

Let $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$. Apply the commutative property for real numbers:
Equation:

$$\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2 \rangle = \langle x_2 + x_1, y_2 + y_1 \rangle = \mathbf{v} + \mathbf{u}.$$

□

Proof of Distributive Property

Apply the distributive property for real numbers:
Equation:

$$\begin{aligned} r(\mathbf{u} + \mathbf{v}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\mathbf{u} + r\mathbf{v}. \end{aligned}$$

□

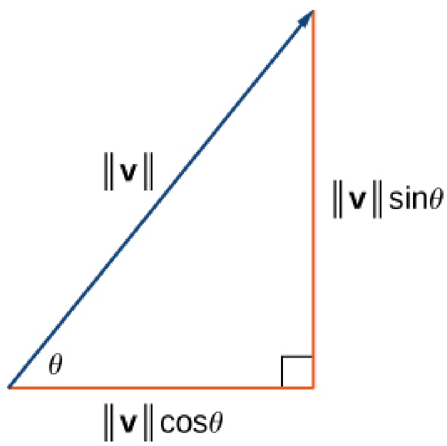
Note:
Exercise:

Problem: Prove the additive inverse property.

Hint

Use the component form of the vectors.

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry ([link]).



The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\mathbf{v}$ and the positive *x*-axis. We can see from the triangle that the components of vector $\mathbf{v}$ are $\langle \| \mathbf{v} \| \cos \theta, \| \mathbf{v} \| \sin \theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

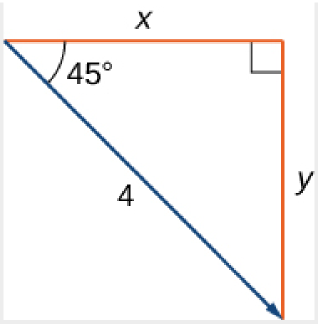
Example:
Exercise:

Problem:
Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of -45° with the *x*-axis.

Solution:

Let x and y represent the components of the vector ([link]). Then $x = 4 \cos \left(-45^{\circ}\right) = 2\sqrt{2}$ and $y = 4 \sin \left(-45^{\circ}\right) = -2\sqrt{2}$. The component form of the vector is $\left\langle 2\sqrt{2}, -2\sqrt{2} \right\rangle$.



Use trigonometric ratios, $x = ||\mathbf{v}||\cos\theta$ and $y = ||\mathbf{v}||\sin\theta$, to identify the components of the vector.

Note:
Exercise:

Problem:

Find the component form of vector **v** with magnitude 10 that forms an angle of 120° with the positive *x*-axis.

Solution:

$$\mathbf{v} = \left\langle -5, 5\sqrt{3} \right\rangle$$

Hint

$x = ||\mathbf{v}||\cos\theta$ and $y = ||\mathbf{v}||\sin\theta$

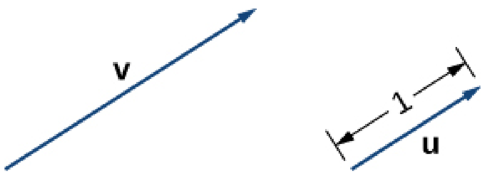
Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector **v**, we can use scalar multiplication to find a unit vector **u** that has the same direction as **v**. To do this, we multiply the vector by the reciprocal of its magnitude:

Equation:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

Recall that when we defined scalar multiplication, we noted that $\|k\mathbf{v}\| = |k| \cdot \|\mathbf{v}\|$. For $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$, it follows that $\|\mathbf{u}\| = \frac{1}{\|\mathbf{v}\|}(\|\mathbf{v}\|) = 1$. We say that $\mathbf{u}$ is the *unit vector in the direction of* $\mathbf{v}$ ([link]). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization**.



The vector $\mathbf{v}$ and associated unit vector $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. In this case, $\|\mathbf{v}\| > 1$.

Example:
Exercise:

Problem:
Finding a Unit Vector

Let $\mathbf{v} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\mathbf{v}$.
b. Find a vector $\mathbf{w}$ with the same direction as $\mathbf{v}$ such that $\|\mathbf{w}\| = 7$.

Solution:

a. First, find the magnitude of $\mathbf{v}$, then divide the components of $\mathbf{v}$ by the magnitude:
Equation:

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

Equation:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

b. The vector $\mathbf{u}$ is in the same direction as $\mathbf{v}$ and $\|\mathbf{u}\| = 1$. Use scalar multiplication to increase the length of $\mathbf{u}$ without changing direction:
Equation:

$$\mathbf{w} = 7\mathbf{u} = 7\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \right\rangle.$$

Note:
Exercise:

Problem:

Let $\mathbf{v} = \langle 9, 2\rangle$. Find a vector with magnitude 5 in the opposite direction as $\mathbf{v}$.

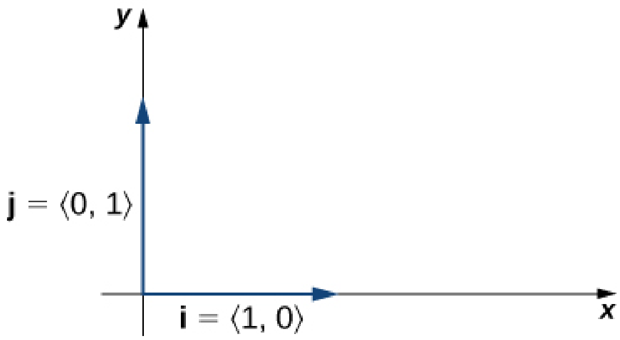
Solution:

$$\left\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}} \right\rangle$$

Hint

First, find a unit vector in the same direction as $\mathbf{v}$.

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ ([link]).



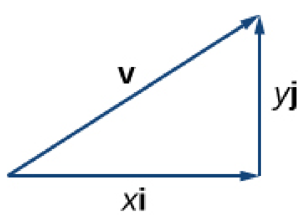
The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

By applying the properties of vectors, it is possible to express any vector in terms of $\mathbf{i}$ and $\mathbf{j}$ in what we call a *linear combination*:

Equation:

$$\mathbf{v} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x \langle 1, 0 \rangle + y \langle 0, 1 \rangle = x\mathbf{i} + y\mathbf{j}.$$

Thus, $\mathbf{v}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in the following figure.



The vector $\mathbf{v}$ is the sum of $x\mathbf{i}$ and $y\mathbf{j}$.

Example:

Exercise:

Problem:

Using Standard Unit Vectors

a. Express the vector $\mathbf{w} = \langle 3, -4 \rangle$ in terms of standard unit vectors.
b. Vector $\mathbf{u}$ is a unit vector that forms an angle of 60° with the positive *x*-axis. Use standard unit vectors to describe $\mathbf{u}$.

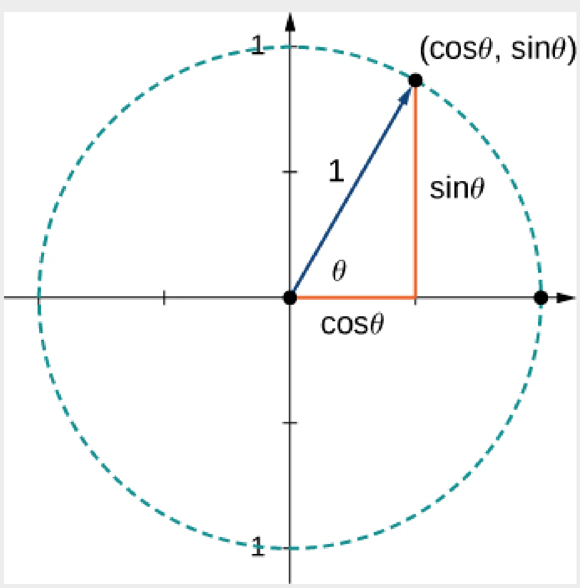
Solution:

a. Resolve vector $\mathbf{w}$ into a vector with a zero *y*-component and a vector with a zero *x*-component:
Equation:

$$\mathbf{w} = \langle 3, -4 \rangle = 3\mathbf{i} - 4\mathbf{j}.$$

b. Because $\mathbf{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position ([link]).
Equation:

$$\begin{aligned} u &= \langle \cos 60^\circ, \sin 60^\circ \rangle \\ &= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\ &= \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}. \end{aligned}$$



The terminal point of $\mathbf{u}$ lies on the unit circle $(\cos\theta, \sin\theta)$.

Note:
Exercise:

Problem:

Let $\mathbf{a} = \langle 16, -11 \rangle$ and let $\mathbf{b}$ be a unit vector that forms an angle of 225° with the positive *x*-axis. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of the standard unit vectors.

Solution:

$\mathbf{a} = 16\mathbf{i} - 11\mathbf{j}$, $\mathbf{b} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

Hint

Use sine and cosine to find the components of $\mathbf{b}$.

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and force. Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example:
Exercise:

Problem:
Finding Resultant Force

Jane’s car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck’s trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of 300 lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of 150 lb on the car. These forces can be represented by vectors, as shown in [link]. The angle between these vectors is 15°. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive *x*-axis.

Two forces acting on a car in different directions.

Solution:

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive *x*-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0\rangle$ or $300\mathbf{i}$.

The second vector has magnitude 150 and makes an angle of 15° with the first, so we can express it as $\langle 150\cos(15^\circ), 150\sin(15^\circ)\rangle$, or
$150\cos(15^\circ)\mathbf{i} + 150\sin(15^\circ)\mathbf{j}$. Then, the sum of the vectors, or resultant vector, is $\mathbf{r} = \langle 300, 0\rangle + \langle 150\cos(15^\circ), 150\sin(15^\circ)\rangle$, and we have
Equation:

$$\begin{aligned}\|\mathbf{r}\| &= \sqrt{(300 + 150\cos(15^\circ))^2 + (150\sin(15^\circ))^2} \\ &\approx 446.6.\end{aligned}$$

The angle θ made by $\mathbf{r}$ and the positive *x*-axis has $\tan\theta = \frac{150\sin 15^\circ}{(300+150\cos 15^\circ)} \approx 0.09$, so $\theta \approx tan^{-1}(0.09) \approx 5^\circ$, which means the resultant force $\mathbf{r}$ has an angle of 5° above the horizontal axis.

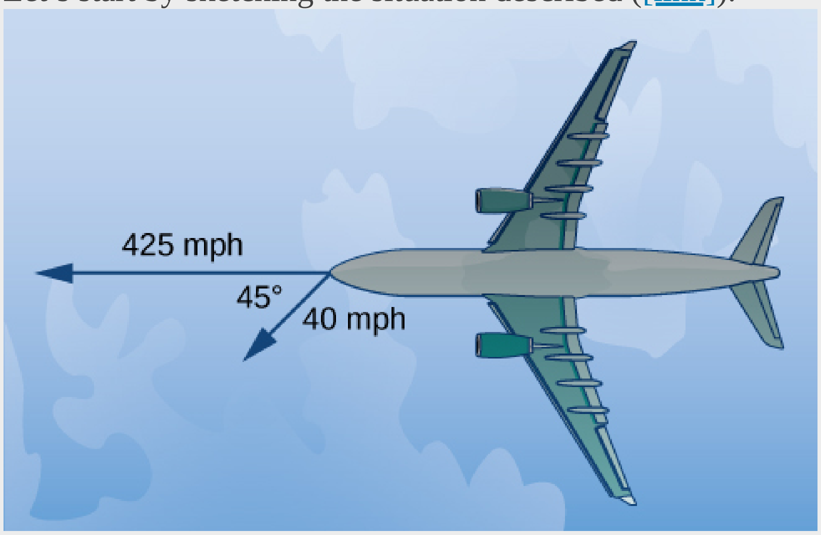
Example:
Exercise:

Problem:
Finding Resultant Velocity

An airplane flies due west at an airspeed of 425 mph. The wind is blowing from the northeast at 40 mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution:

Let's start by sketching the situation described ([link]).



Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. The angle between the plane's course and the wind is 45°. (Figure not drawn to scale.)

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane's velocity vector is $\mathbf{p} = -425\mathbf{i}$. The vector describing the wind makes an angle of 225° with the positive *x*-axis:

Equation:

$$\mathbf{w} = \langle 40\cos(225^\circ), 40\sin(225^\circ)\rangle = \left\langle -\frac{40}{\sqrt{2}}, -\frac{40}{\sqrt{2}}\right\rangle = -\frac{40}{\sqrt{2}}\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}.$$

When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

Equation:

$$\mathbf{p} + \mathbf{w} = -425\mathbf{i} + \left(-\frac{40}{\sqrt{2}}\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}\right) = \left(-425 - \frac{40}{\sqrt{2}}\right)\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:
Equation:

$$\|\mathbf{p}+\mathbf{w}\| = \sqrt{\left(-425-\frac{40}{\sqrt{2}}\right)^2+\left(-\frac{40}{\sqrt{2}}\right)^2} \approx 454.17\,\text{mph}$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\mathbf{p}+\mathbf{w}$:
Equation:

$$\begin{aligned}\tan\theta &= \frac{-\frac{40}{\sqrt{2}}}{\left(-425-\frac{40}{\sqrt{2}}\right)} \approx 0.06\\ \theta &\approx 3.57^\circ.\end{aligned}$$

The overall direction of the plane is 3.57° south of west.

Note:
Exercise:

Problem:

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Solution:

Approximately 516 mph

Hint

Sketch the vectors with the same initial point and find their sum.

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\mathbf{v} = \langle x, y \rangle$.
- The magnitude of a vector is a scalar: $\|\mathbf{v}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\mathbf{u}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. The standard unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. A vector $\mathbf{v} = \langle x, y \rangle$ can be expressed in terms of the standard unit vectors as $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

For the following exercises, consider points $P(-1, 3)$, $Q(1, 5)$, and $R(-3, 7)$. Determine the requested vectors and express each of them a. in component form and b. by using the standard unit vectors.

Exercise:

Problem: $\vec{PQ}$

Solution:

a. $\vec{PQ} = \langle 2, 2 \rangle$; b. $\vec{PQ} = 2\mathbf{i} + 2\mathbf{j}$

Exercise:

Problem: $\vec{PR}$

Exercise:

Problem: $\vec{QP}$

Solution:

a. $\vec{QP} = \langle -2, -2 \rangle$; b. $\vec{QP} = -2\mathbf{i} - 2\mathbf{j}$

Exercise:

Problem: $\vec{RP}$

Exercise:

Problem: $\vec{PQ} + \vec{PR}$

Solution:

a. $\vec{PQ} + \vec{PR} = \langle 0, 6 \rangle$; b. $\vec{PQ} + \vec{PR} = 6\mathbf{j}$

Exercise:

Problem: $\vec{PQ} - \vec{PR}$

Exercise:

Problem: $2\vec{PQ} - 2\vec{PR}$

Solution:

a. $2\vec{PQ} - 2\vec{PR} = \langle 8, -4 \rangle$; b. $2\vec{PQ} - 2\vec{PR} = 8\mathbf{i} - 4\mathbf{j}$

Exercise:

Problem: $2\vec{PQ} + \frac{1}{2}\vec{PR}$

Exercise:

Problem: The unit vector in the direction of $\vec{PQ}$

Solution:

a. $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$; b. $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

Exercise:

Problem: The unit vector in the direction of $\vec{PR}$

Exercise:

Problem:

A vector $\mathbf{v}$ has initial point $(-1, -3)$ and terminal point $(2, 1)$. Find the unit vector in the direction of $\mathbf{v}$. Express the answer in component form.

Solution:

$\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

Exercise:

Problem:

A vector $\mathbf{v}$ has initial point $(-2, 5)$ and terminal point $(3, -1)$. Find the unit vector in the direction of $\mathbf{v}$. Express the answer in component form.

Exercise:

Problem:

The vector $\mathbf{v}$ has initial point $P(1, 0)$ and terminal point Q that is on the *y*-axis and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\mathbf{v}$ is $\sqrt{5}$.

Solution:

$Q(0, 2)$

Exercise:

Problem:

The vector $\mathbf{v}$ has initial point $P(1, 1)$ and terminal point Q that is on the *x*-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\mathbf{v}$ is $\sqrt{10}$.

For the following exercises, use the given vectors $\mathbf{a}$ and $\mathbf{b}$.

a. Determine the vector sum $\mathbf{a} + \mathbf{b}$ and express it in both the component form and by using the standard unit vectors.
b. Find the vector difference $\mathbf{a} - \mathbf{b}$ and express it in both the component form and by using the standard unit vectors.
c. Verify that the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} + \mathbf{b}$, and, respectively, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$ satisfy the triangle inequality.
d. Determine the vectors $2\mathbf{a}$, $-\mathbf{b}$, and $2\mathbf{a} - \mathbf{b}$. Express the vectors in both the component form and by using standard unit vectors.

Exercise:

Problem: $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$

Solution:

a. $\mathbf{a}+\mathbf{b}=3\mathbf{i}+4\mathbf{j}$, $\mathbf{a}+\mathbf{b}=\langle 3,4\rangle$; b. $\mathbf{a}-\mathbf{b}=\mathbf{i}-2\mathbf{j}$, $\mathbf{a}-\mathbf{b}=\langle 1,-2\rangle$; c. Answers will vary; d. $2\mathbf{a}=4\mathbf{i}+2\mathbf{j}$, $2\mathbf{a}=\langle 4,2\rangle$, $-\mathbf{b}=-\mathbf{i}-3\mathbf{j}$, $-\mathbf{b}=\langle -1,-3\rangle$, $2\mathbf{a}-\mathbf{b}=3\mathbf{i}-\mathbf{j}$, $2\mathbf{a}-\mathbf{b}=\langle 3,-1\rangle$

Exercise:

Problem: $\mathbf{a}=2\mathbf{i}$, $\mathbf{b}=-2\mathbf{i}+2\mathbf{j}$

Exercise:

Problem:

Let $\mathbf{a}$ be a standard-position vector with terminal point $(-2,-4)$. Let $\mathbf{b}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of vector $-3\mathbf{a}+\mathbf{b}-4\mathbf{i}+\mathbf{j}$.

Solution:

15

Exercise:

Problem:

Let $\mathbf{a}$ be a standard-position vector with terminal point at $(2,5)$. Let $\mathbf{b}$ be a vector with initial point $(-1,3)$ and terminal point $(1,0)$. Find the magnitude of vector $\mathbf{a}-3\mathbf{b}+14\mathbf{i}-14\mathbf{j}$.

Exercise:

Problem:

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\mathbf{a}=4\mathbf{u}+5\mathbf{v}$ and $\mathbf{b}=\mathbf{u}+2\mathbf{v}$ defined in terms of $\mathbf{u}$ and $\mathbf{v}$. Find the scalar λ such that vectors $\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{u}-\mathbf{v}$ are equivalent.

Solution:

$\lambda=-3$

Exercise:

Problem:

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\mathbf{a}=2\mathbf{u}-4\mathbf{v}$ and $\mathbf{b}=3\mathbf{u}-7\mathbf{v}$ defined in terms of $\mathbf{u}$ and $\mathbf{v}$. Find the scalars α and β such that vectors $\alpha\mathbf{a}+\beta\mathbf{b}$ and $\mathbf{u}-\mathbf{v}$ are equivalent.

Exercise:

Problem:

Consider the vector $\mathbf{a}(t) = \langle \cos t, \sin t \rangle$ with components that depend on a real number t. As the number t varies, the components of $\mathbf{a}(t)$ change as well, depending on the functions that define them.

a. Write the vectors $\mathbf{a}(0)$ and $\mathbf{a}(\pi)$ in component form.
b. Show that the magnitude $\|\mathbf{a}(t)\|$ of vector $\mathbf{a}(t)$ remains constant for any real number t.
c. As t varies, show that the terminal point of vector $\mathbf{a}(t)$ describes a circle centered at the origin of radius 1.

Solution:

a. $\mathbf{a}(0) = \langle 1, 0 \rangle$, $\mathbf{a}(\pi) = \langle -1, 0 \rangle$; b. Answers may vary; c. Answers may vary

Exercise:

Problem:

Consider vector $\mathbf{a}(x) = \left\langle x, \sqrt{1 - x^2} \right\rangle$ with components that depend on a real number $x \in [-1, 1]$. As the number x varies, the components of $\mathbf{a}(x)$ change as well, depending on the functions that define them.

a. Write the vectors $\mathbf{a}(0)$ and $\mathbf{a}(1)$ in component form.
b. Show that the magnitude $\|\mathbf{a}(x)\|$ of vector $\mathbf{a}(x)$ remains constant for any real number x
c. As x varies, show that the terminal point of vector $\mathbf{a}(x)$ describes a circle centered at the origin of radius 1.

Exercise:

Problem:

Show that vectors $\mathbf{a}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{a}(x) = \left\langle x, \sqrt{1 - x^2} \right\rangle$ are equivalent for $x = r$ and $t = 2k\pi$, where k is an integer.

Solution:

Answers may vary

Exercise:

Problem:

Show that vectors $\mathbf{a}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{a}(x) = \left\langle x, \sqrt{1 - x^2} \right\rangle$ are opposite for $x = r$ and $t = \pi + 2k\pi$, where k is an integer.

For the following exercises, find vector $\mathbf{v}$ with the given magnitude and in the same direction as vector $\mathbf{u}$.

Exercise:

Problem: $\|\mathbf{v}\| = 7, \mathbf{u} = \langle 3, 4 \rangle$

Solution:

$\mathbf{v} = \left\langle \frac{21}{5}, \frac{28}{5} \right\rangle$

Exercise:

Problem: $\|\mathbf{v}\| = 3, \mathbf{u} = \langle -2, 5 \rangle$

Exercise:

Problem: $\|\mathbf{v}\| = 7, \mathbf{u} = \langle 3, -5 \rangle$

Solution:

$\mathbf{v} = \left\langle \frac{21\sqrt{34}}{34}, -\frac{35\sqrt{34}}{34} \right\rangle$

Exercise:

Problem: $\|\mathbf{v}\| = 10, \mathbf{u} = \langle 2, -1 \rangle$

For the following exercises, find the component form of vector $\mathbf{u}$, given its magnitude and the angle the vector makes with the positive *x*-axis. Give exact answers when possible.

Exercise:

Problem: $\|\mathbf{u}\| = 2$, $\theta = 30^\circ$

Solution:

$\mathbf{u} = \left\langle \sqrt{3}, 1 \right\rangle$

Exercise:

Problem: $\|\mathbf{u}\| = 6,\, \theta = 60^\circ$

Exercise:

Problem: $\|\mathbf{u}\| = 5,\, \theta = \frac{\pi}{2}$

Solution:

$\mathbf{u} = \langle 0, 5 \rangle$

Exercise:

Problem: $\|\mathbf{u}\| = 8,\, \theta = \pi$

Exercise:

Problem: $\|\mathbf{u}\| = 10,\, \theta = \frac{5\pi}{6}$

Solution:

$\mathbf{u} = \left\langle -5\sqrt{3}, 5 \right\rangle$

Exercise:

Problem: $\|\mathbf{u}\| = 50,\, \theta = \frac{3\pi}{4}$

For the following exercises, vector $\mathbf{u}$ is given. Find the angle $\theta \in [0, 2\pi)$ that vector $\mathbf{u}$ makes with the positive direction of the *x*-axis, in a counter-clockwise direction.

Exercise:

Problem: $\mathbf{u} = 5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$

Solution:

$\theta = \frac{7\pi}{4}$

Exercise:

Problem: $\mathbf{u} = -\sqrt{3}\mathbf{i} - \mathbf{j}$

Exercise:

Problem:

Let $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$ be three nonzero vectors. If $a_1 b_2 - a_2 b_1 \neq 0$, then show there are two scalars, α and β, such that $\mathbf{c} = \alpha \mathbf{a} + \beta \mathbf{b}$.

Solution:

Answers may vary

Exercise:

Problem:

Consider vectors $\mathbf{a} = \langle 2, -4 \rangle$, $\mathbf{b} = \langle -1, 2 \rangle$, and **c** = **0** Determine the scalars α and β such that $\mathbf{c} = \alpha \mathbf{a} + \beta \mathbf{b}$.

Exercise:

Problem:

Let $P\left(x_0, f\left(x_0\right)\right)$ be a fixed point on the graph of the differential function f with a domain that is the set of real numbers.

a. Determine the real number z_0 such that point $Q\left(x_0 + 1, z_0\right)$ is situated on the line tangent to the graph of f at point P.
b. Determine the unit vector $\mathbf{u}$ with initial point P and terminal point Q.

Solution:

a. $z_0 = f(x_0) + f'(x_0)$; b. $\mathbf{u} = \frac{1}{\sqrt{1+[f'(x_0)]^2}} \langle 1, f'(x_0) \rangle$

Exercise:

Problem: Consider the function $f(x) = x^4$, where $x \in \mathbb{R}$.

a. Determine the real number z_0 such that point $Q\left(2, z_0\right)$ s situated on the line tangent to the graph of f at point $P\left(1, 1\right)$.
b. Determine the unit vector $\mathbf{u}$ with initial point P and terminal point Q.

Exercise:

Problem:

Consider f and g two functions defined on the same set of real numbers D. Let $\mathbf{a} = \langle x, f(x) \rangle$ and $\mathbf{b} = \langle x, g(x) \rangle$ be two vectors that describe the graphs of the functions, where $x \in D$. Show that if the graphs of the functions f and g do not intersect, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are not equivalent.

Exercise:

Problem:

Find $x \in \mathbb{R}$ such that vectors $\mathbf{a} = \langle x, \sin x \rangle$ and $\mathbf{b} = \langle x, \cos x \rangle$ are equivalent.

Exercise:

Problem:

Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1)$, $B(2, 4)$, and $C(7, 4)$.

Solution:

$D(6, 1)$

Exercise:

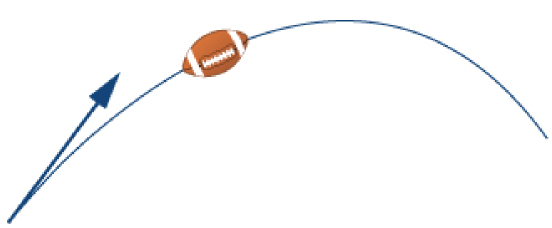
Problem:

Consider the points $A(2, 1)$, $B(10, 6)$, $C(13, 4)$, and $D(16, -2)$. Determine the component form of vector $\vec{AD}$.

Exercise:

Problem:

The speed of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of 30°. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)



Solution:

$\langle 60.62, 35 \rangle$

Exercise:

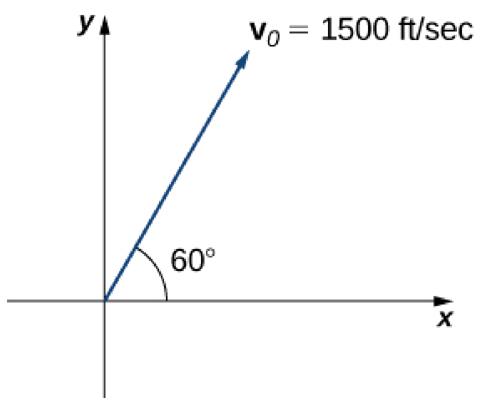
Problem:

A baseball player throws a baseball at an angle of 30° with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)

Exercise:

Problem:

A bullet is fired with an initial velocity of 1500 ft/sec at an angle of 60° with the horizontal. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)



Solution:

The horizontal and vertical components are 750 ft/sec and 1299.04 ft/sec, respectively.

Exercise:

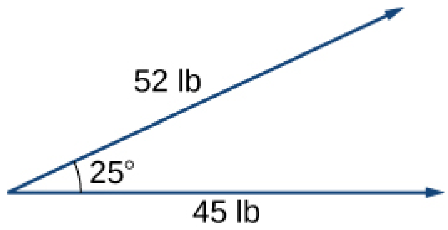
Problem:

[T] A 65-kg sprinter exerts a force of 798 N at a 19° angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)

Exercise:

Problem:

[T] Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is 25°. Find the magnitude and direction angle from the positive *x*-axis of the resultant force that acts on the object. (Round to two decimal places.)



Solution:

The magnitude of resultant force is 94.71 lb; the direction angle is 13.42°.

Exercise:

Problem:

[T] Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is 55°. Find the magnitude and direction angle from the positive *x*-axis of the resultant force that acts on the object. (Round to two decimal places.)

Exercise:

Problem:

[T] Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles 53° and 152°, respectively, with the positive *x*-axis. Find the magnitude and the direction angle from the positive *x*-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

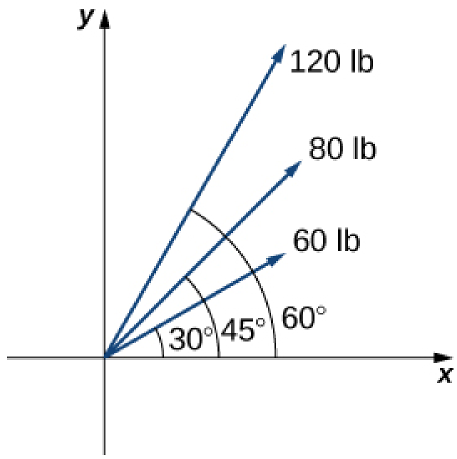
Solution:

The magnitude of the third vector is 60.03 N; the direction angle is 259.38°.

Exercise:

Problem:

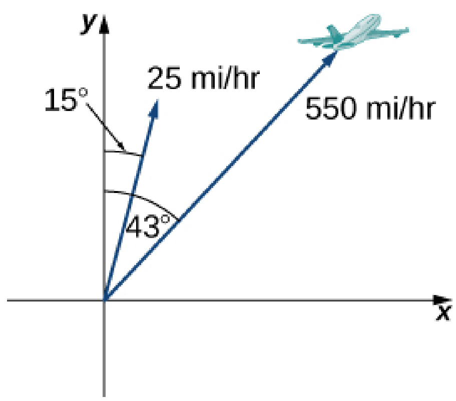
Three forces with magnitudes 80 lb, 120 lb, and 60 lb act on an object at angles of 45°, 60° and 30°, respectively, with the positive *x*-axis. Find the magnitude and direction angle from the positive *x*-axis of the resultant force. (Round to two decimal places.)



Exercise:

Problem:

[T] An airplane is flying in the direction of 43° east of north (also abbreviated as N43E) at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of N15E. What are the ground speed and new direction of the airplane?



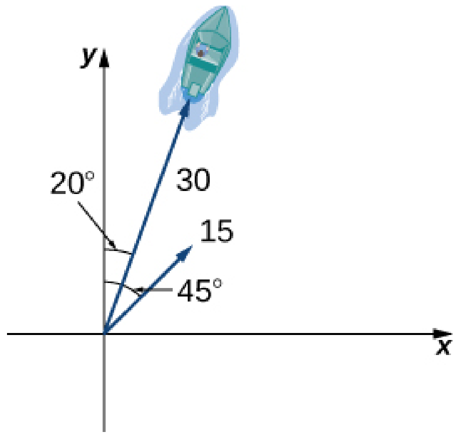
Solution:

The new ground speed of the airplane is 572.19 mph; the new direction is N41.82E.

Exercise:

Problem:

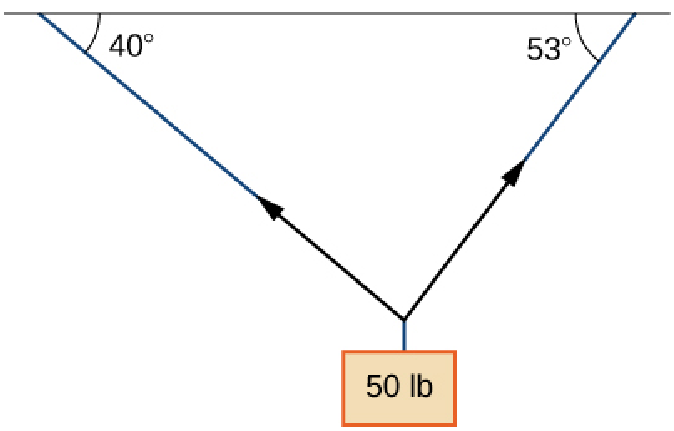
[T] A boat is traveling in the water at 30 mph in a direction of N20E (that is, 20° east of north). A strong current is moving at 15 mph in a direction of N45E. What are the new speed and direction of the boat?



Exercise:

Problem:

[T] A 50-lb weight is hung by a cable so that the two portions of the cable make angles of 40° and 53°, respectively, with the horizontal. Find the magnitudes of the forces of tension T_1 and T_2 in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)



Solution:

$\|\mathbf{T}_1\| = 30.13\ \mathrm{lb}, \|\mathbf{T}_2\| = 38.35\ \mathrm{lb}$

Exercise:

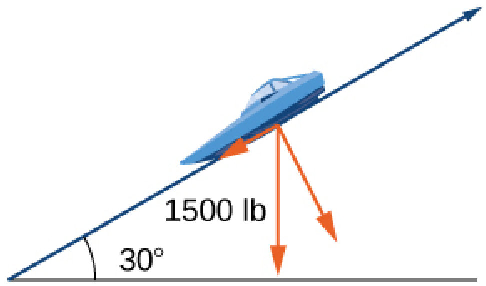
Problem:

[T] A 62-lb weight hangs from a rope that makes the angles of 29° and 61°, respectively, with the horizontal. Find the magnitudes of the forces of tension T_1 and T_2 in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

Exercise:

Problem:

[T] A 1500-lb boat is parked on a ramp that makes an angle of 30° with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\mathbf{v}_1$ that is parallel to the ramp and a vertical vector $\mathbf{v}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\mathbf{v}_1$ and $\mathbf{v}_2$. (Round to the nearest integer.)



Solution:

$\|\mathbf{v}_1\| = 750 \text{ lb}, \|\mathbf{v}_2\| = 1299 \text{ lb}$

Exercise:

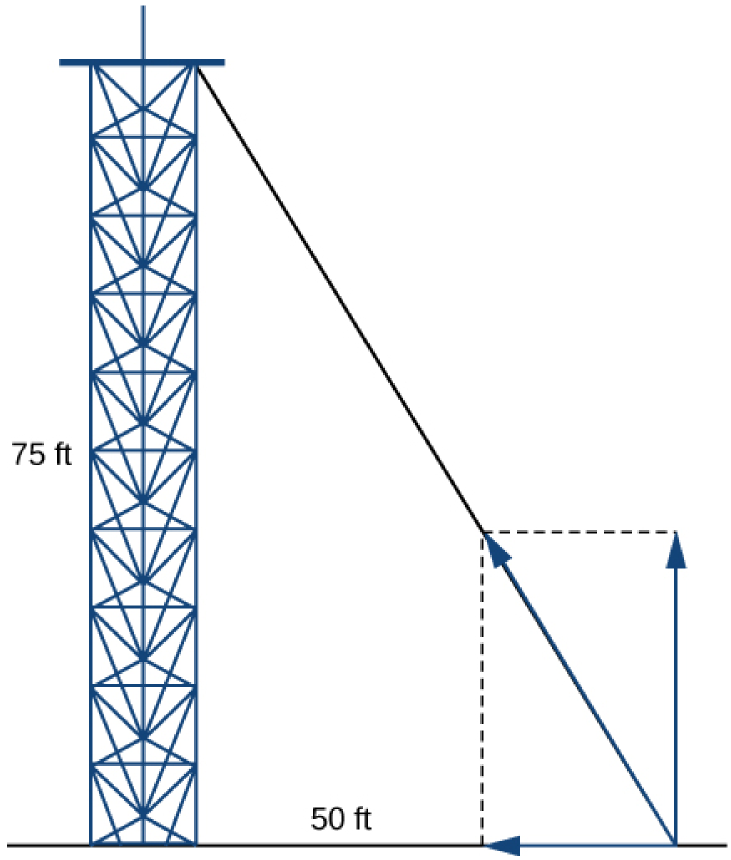
Problem:

[T] An 85-lb box is at rest on a 26° incline. Determine the magnitude of the force parallel to the incline necessary to keep the box from sliding. (Round to the nearest integer.)

Exercise:

Problem:

A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)



Solution:

The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.

Exercise:

Problem:

A telephone pole guy-wire has an angle of elevation of 35° with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

Glossary

component
: a scalar that describes either the vertical or horizontal direction of a vector

equivalent vectors
: vectors that have the same magnitude and the same direction

initial point
: the starting point of a vector

magnitude
: the length of a vector

normalization
: using scalar multiplication to find a unit vector with a given direction

parallelogram method
: a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

scalar
: a real number

scalar multiplication
: a vector operation that defines the product of a scalar and a vector

standard-position vector
: a vector with initial point $(0, 0)$

standard unit vectors
: unit vectors along the coordinate axes: $\mathbf{i} = \langle 1, 0 \rangle, \mathbf{j} = \langle 0, 1 \rangle$

terminal point
: the endpoint of a vector

triangle inequality
: the length of any side of a triangle is less than the sum of the lengths of the other two sides

triangle method
: a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

unit vector
: a vector with margnitude 1

vector
: a mathematical object that has both magnitude and direction

vector addition
: a vector operation that defines the sum of two vectors

vector difference
: the vector difference $\mathbf{v} - \mathbf{w}$ is defined as $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$

vector sum
: the sum of two vectors, $\mathbf{v}$ and $\mathbf{w}$, can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$; then the vector sum $\mathbf{v} + \mathbf{w}$ is the vector with an initial point that coincides with the initial point of $\mathbf{v}$, and with a terminal point that coincides with the terminal point of $\mathbf{w}$

zero vector
: the vector with both initial point and terminal point $(0, 0)$

Vectors in Three Dimensions

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

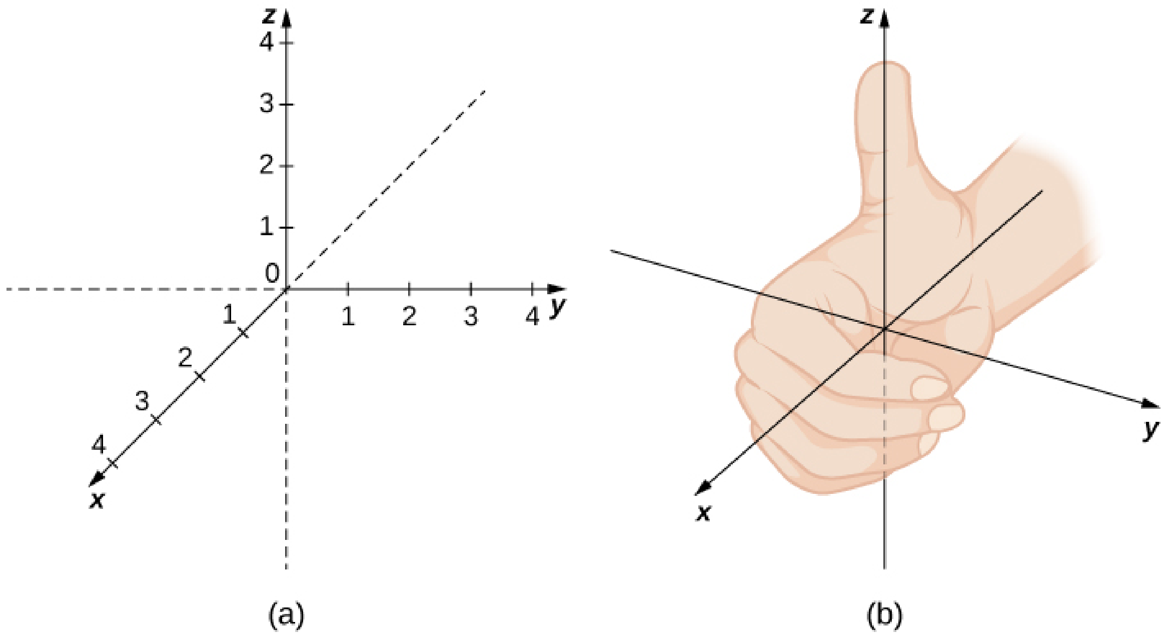
Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal *x*-axis and the vertical *y*-axis. We can add a third dimension, the *z*-axis, which is perpendicular to both the *x*-axis and the *y*-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

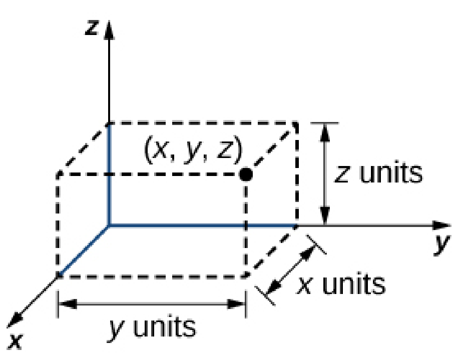
Note:
Definition
The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the *x*-axis, the *y*-axis, and the *z*-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In [link](a), the positive *z*-axis is shown above the plane containing the *x*- and *y*-axes. The positive *x*-axis appears to the left and the positive *y*-axis is to the right. A natural question to ask is: How was arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive *x*-axis, then curl the fingers so they point in the direction of the positive *y*-axis, our thumb points in the direction of the positive *z*-axis. In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.



(a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the *z*-axis, that is perpendicular to both the *x*-axis and the *y*-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the *z*-axis: (x, y, z). A point in space is identified by all three coordinates ([link]). To plot the point (x, y, z), go *x* units along the *x*-axis, then y units in the direction of the *y*-axis, then z units in the direction of the *z*-axis.



To plot the point (x, y, z) go

x units along the *x*-axis, then y units in the direction of the *y*-axis, then z units in the direction of the *z*-axis.

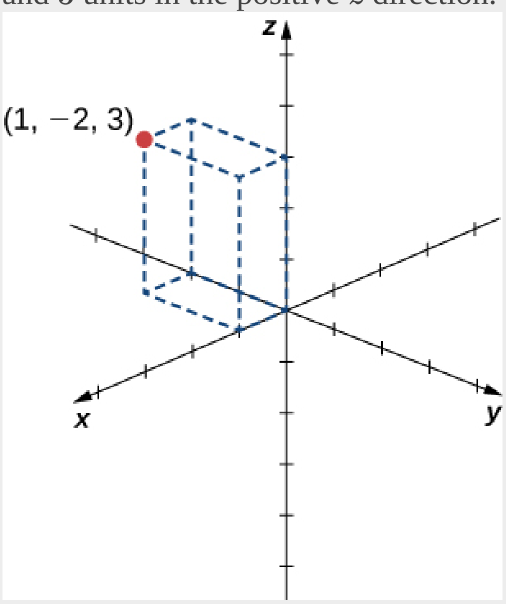
Example:
Exercise:

Problem:
Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution:

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point ([link]).



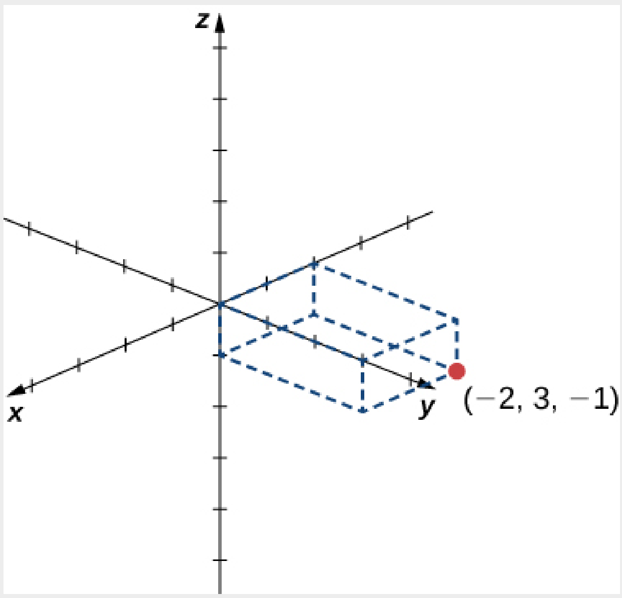
Sketching the point $(1, -2, 3)$.

Note:

Exercise:

Problem: Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Solution:

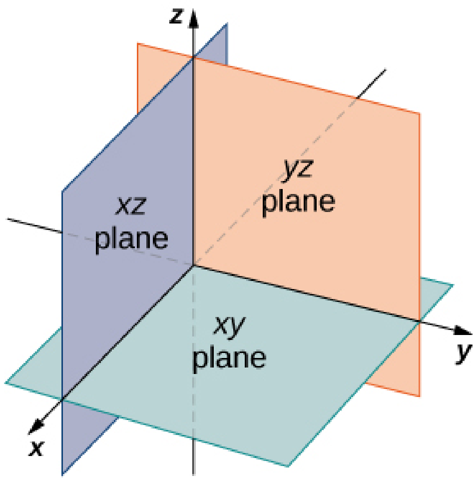


Hint

Start by sketching the coordinate axes. Then sketch a rectangular prism to help find the point in space.

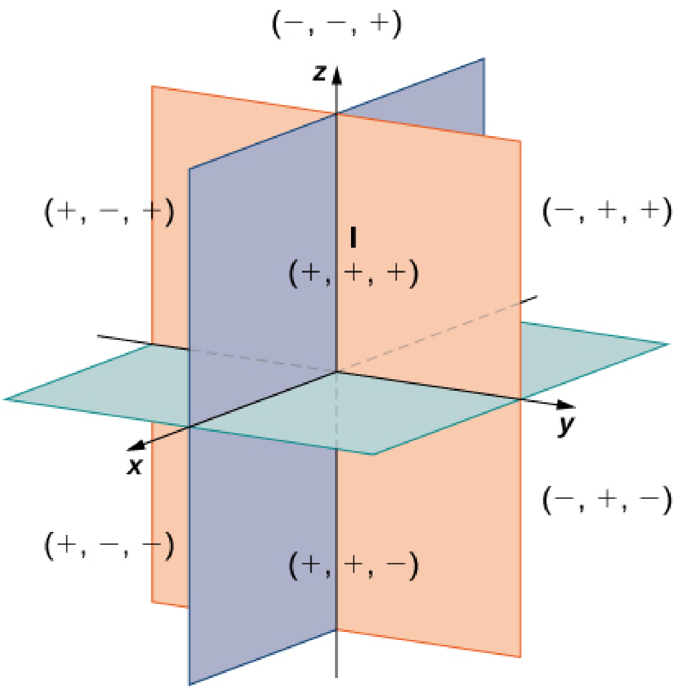
In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the *xy*-plane, the *xz*-plane, and the *yz*-plane ([link]). We define the *xy*-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the *xz*-plane and the *yz*-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the *xy*-plane, the wall to your right is the *xz*-plane, and the wall to your left is the *yz*-plane.



The plane containing the x- and y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants**. The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in [link].



Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We that the distance d between two points (x_1, y_1) and (x_2, y_2) in the *xy*-coordinate plane is given by the formula

Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The formula for the distance between two points in space is a natural extension of this formula.

Note:

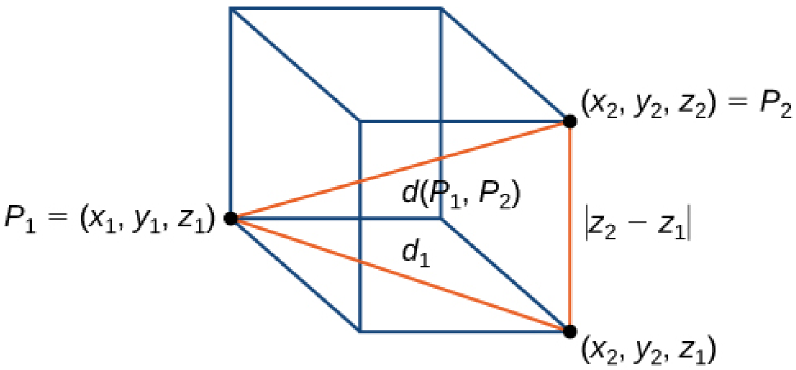
The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The proof of this theorem is left as an exercise. (*Hint:* First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in [link].)

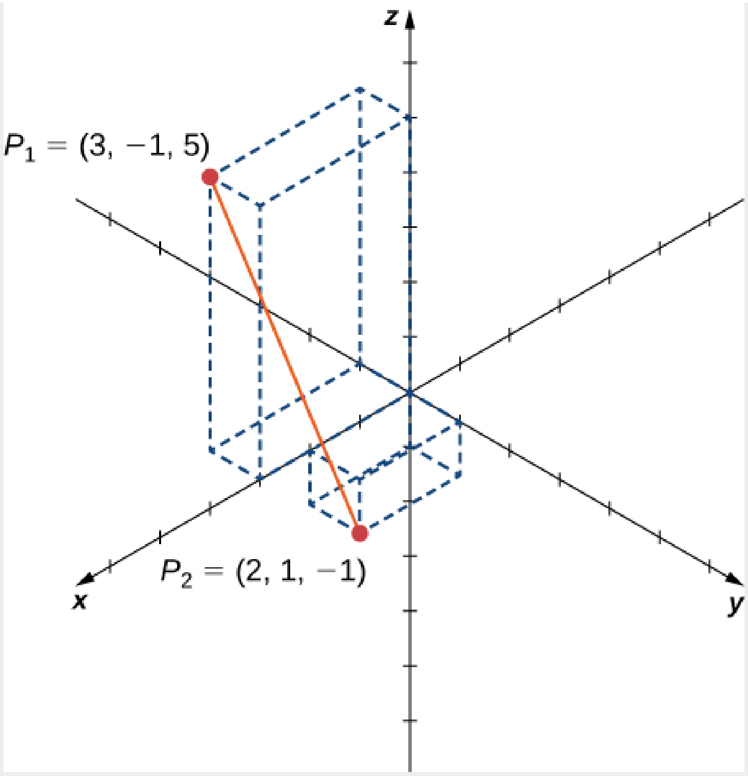


The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example:
Exercise:

Problem:
Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.



Find the distance between the two points.

Solution:

Substitute values directly into the distance formula:

Equation:

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2 - 3)^2 + (1 - (-1))^2 + (-1 - 5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

Note:

Exercise:

Problem: Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

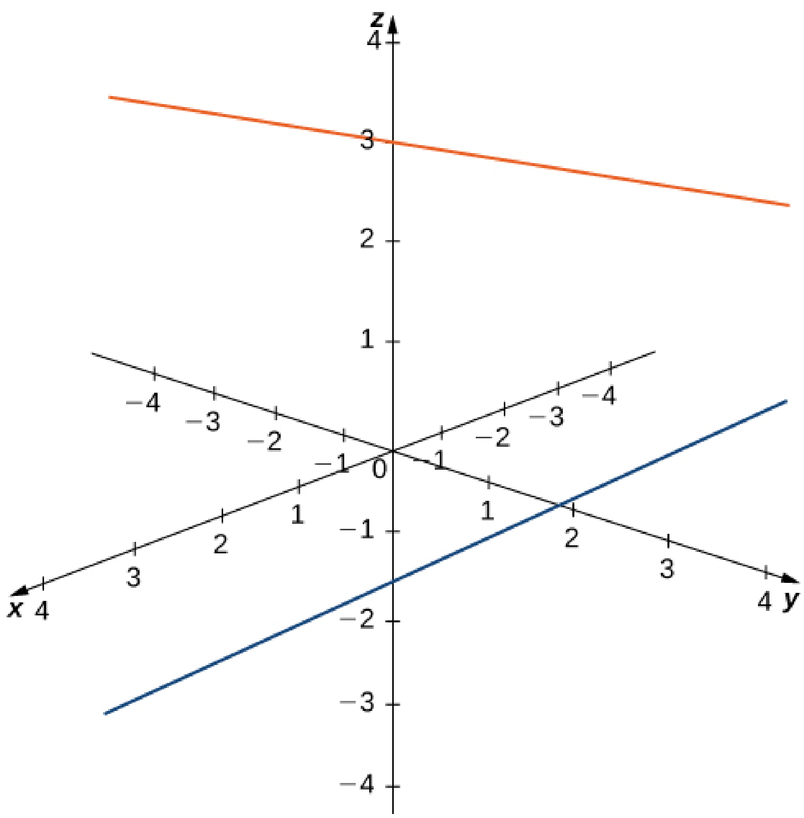
Solution:

$5\sqrt{2}$

Hint

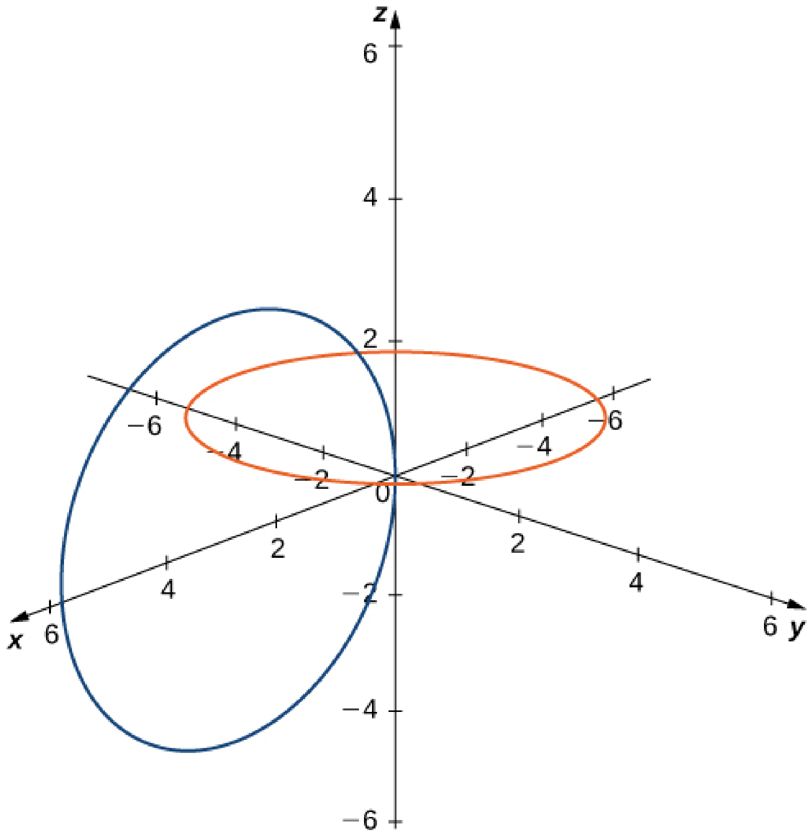
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Before moving on to the next section, let’s get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the line shown in [link]. These two lines are not parallel, nor do they intersect.



These two lines are not parallel, but still do not intersect.

You can also have circles that are interconnected but have no points in common, as in [link].

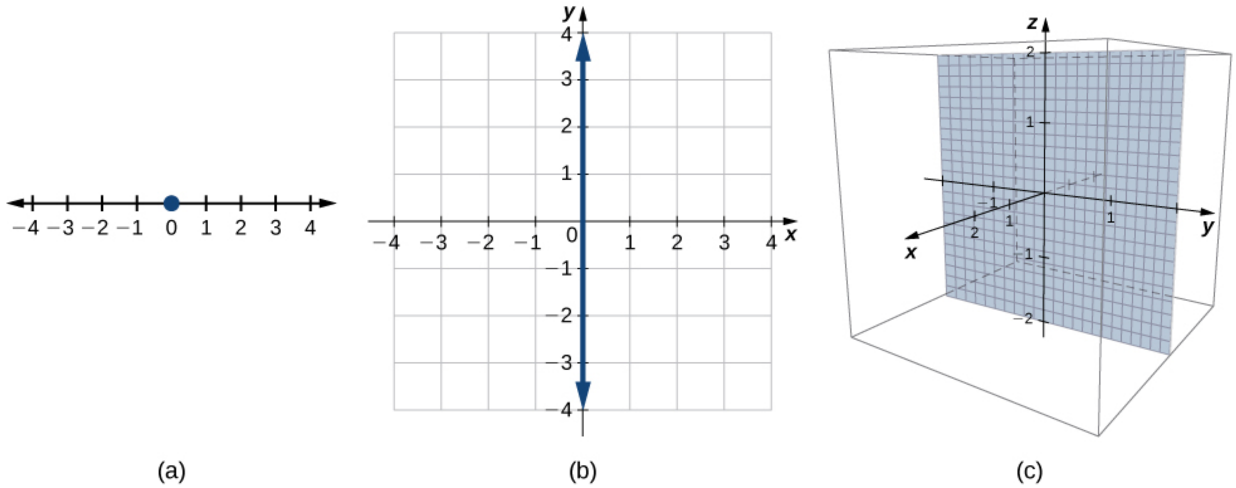


These circles are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

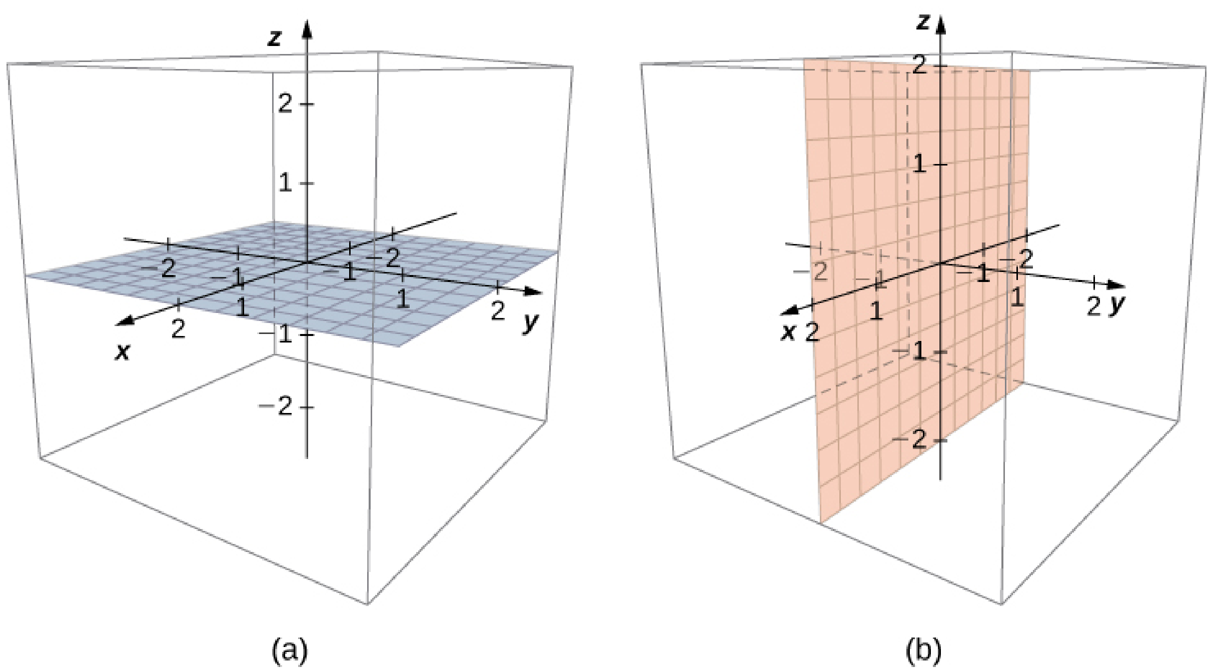
Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}, \mathbb{R}^2, \text{ and } \mathbb{R}^3$ ([link]). From these graphs, we can see the same equation can describe a point, a line, or a plane.



(a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the *y*-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the *yz*-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the *yz*-plane. Similarly, the *xy*-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the *xy*-plane and the equation $y = 0$ describes the *xz*-plane ([link]).



(a) In space, the equation $z = 0$ describes the *xy*-plane. (b) All points in the *xz*-plane

satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the *xy*-plane, for example, the *z*-coordinate of each point in the plane has the same constant value. Only the *x*- and *y*-coordinates of points in that plane vary from point to point.

Note:
Rule: Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the *xy*-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the *xz*-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the *yz*-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example:
Exercise:

Problem:
Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the *yz*-plane.
b. Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$.

Solution:

a. When a plane is parallel to the *yz*-plane, only the *y*- and *z*-coordinates may vary. The *x*-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$ has the same *y*-coordinate. This plane can be represented by the equation $y = -2$.

Note:
Exercise:

Problem:

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the *xy*-plane.

Solution:

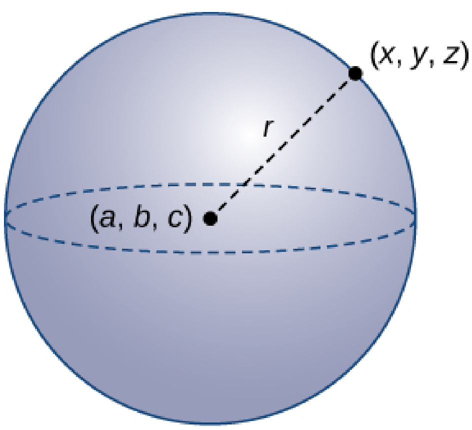
$z = -4$

Hint

If a plane is parallel to the *xy*-plane, the *z*-coordinates of the points in that plane do not vary.

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the *y*-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the *yz*-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Note:
Definition
A **sphere** is the set of all points in space equidistant from a fixed point, the center of the sphere ([link]), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.



Each point (x, y, z) on the

surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Note:
Rule: Equation of a Sphere
The sphere with center (a, b, c) and radius r can be represented by the equation
Equation:

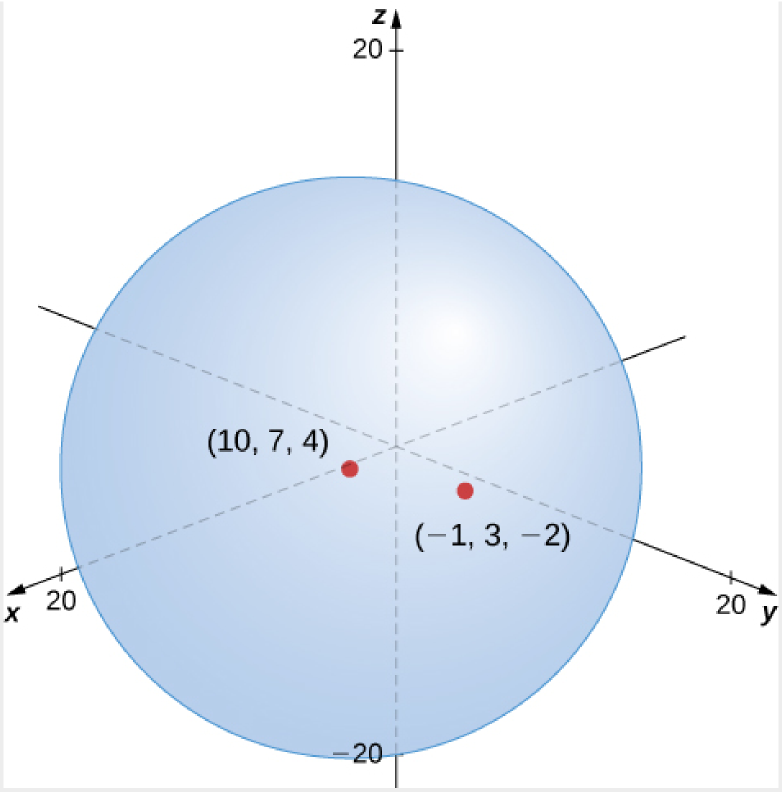
$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

This equation is known as the **standard equation of a sphere**.

Example:
Exercise:

Problem:
Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in [link].



The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

Solution:

Use the distance formula to find the radius r of the sphere:
Equation:

$$\begin{aligned} r &= \sqrt{(-1-10)^2 + (3-7)^2 + (-2-4)^2} \\ &= \sqrt{(-11)^2 + (-4)^2 + (-6)^2} \\ &= \sqrt{173}. \end{aligned}$$

The standard equation of the sphere is
Equation:

$$(x-10)^2 + (y-7)^2 + (z-4)^2 = 173.$$

Note:
Exercise:

Problem:

Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Solution:

$(x+2)^2 + (y-4)^2 + (z+5)^2 = 52$

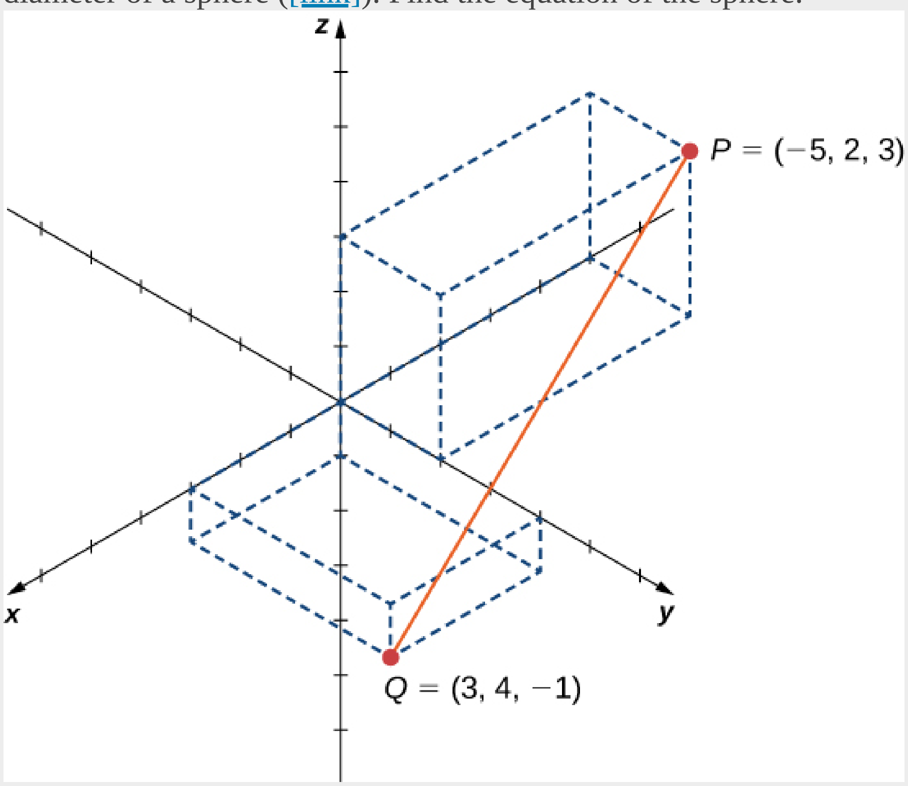
Hint

First use the distance formula to find the radius of the sphere.

Example:
Exercise:

Problem:
Finding the Equation of a Sphere

Let $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$, and suppose line segment PQ forms the diameter of a sphere ([link]). Find the equation of the sphere.



Line segment PQ.

Solution:

Since PQ is a diameter of the sphere, we know the center of the sphere is the midpoint of PQ. Then,

Equation:

$$\begin{aligned} C &= \left(\frac{-5+3}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right) \\ &= (-1, 3, 1). \end{aligned}$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

Equation:

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2+(2-4)^2+(3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21}. \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2+(y-3)^2+(z-1)^2=21$.

Note:
Exercise:

Problem:

Find the equation of the sphere with diameter PQ, where $P=(2,-1,-3)$ and $Q=(-2,5,-1)$.

Solution:

$x^2+(y-2)^2+(z+2)^2=14$

Hint

Find the midpoint of the diameter first.

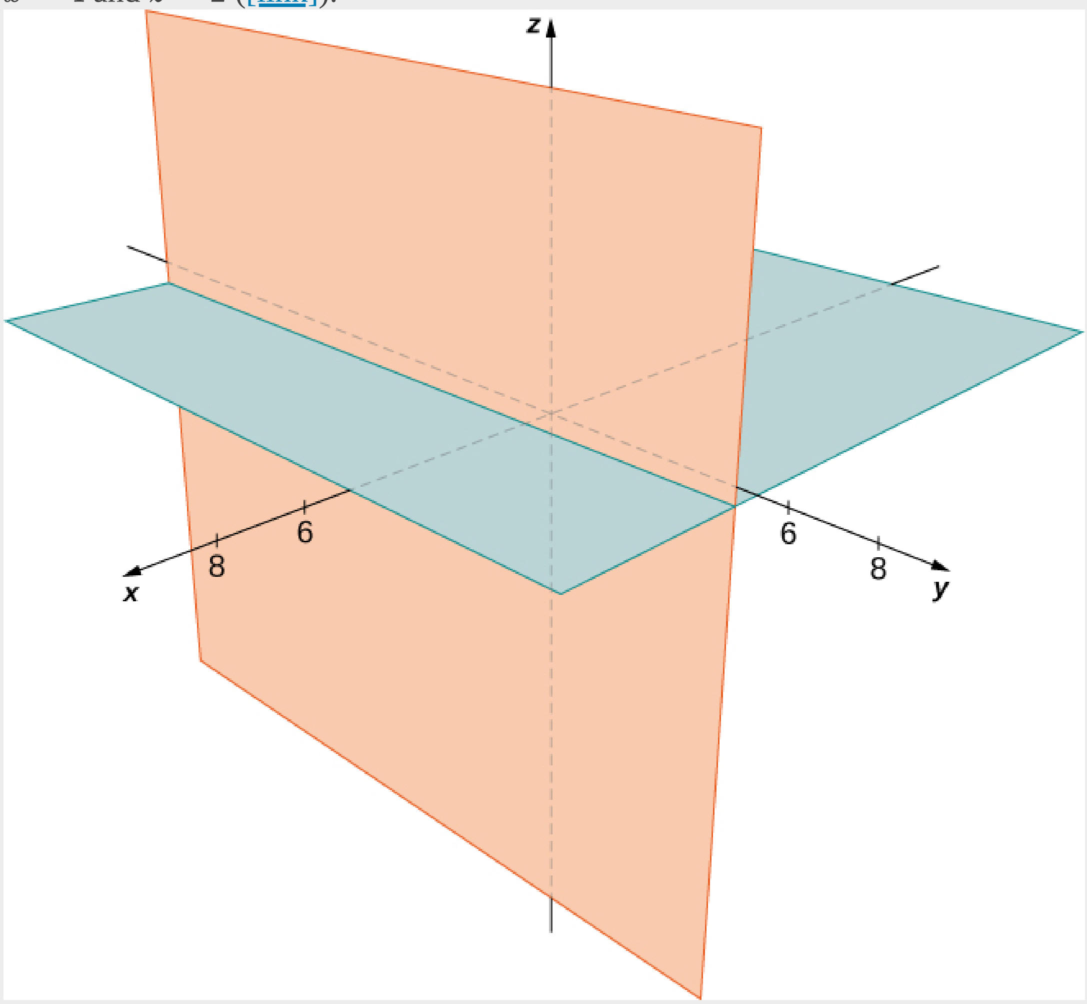
Example:
Exercise:

Problem:
Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2)=0$, and graph the set.

Solution:

We must have either $x-4=0$ or $z-2=0$, so the set of points forms the two planes $x=4$ and $z=2$ ([link]).



The set of points satisfying $(x-4)(z-2)=0$ forms the two planes $x=4$ and $z=2$.

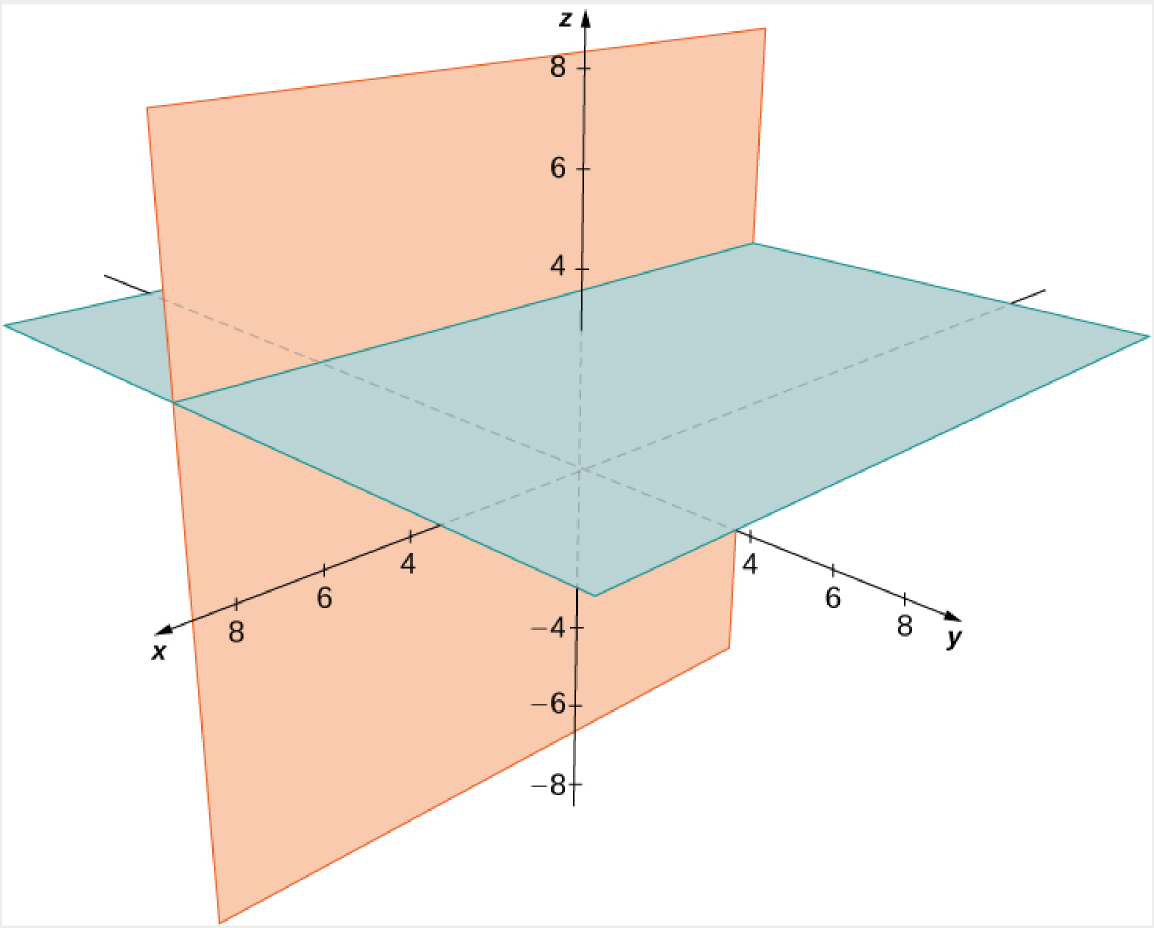
Note:
Exercise:

Problem:

Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.

Solution:

The set of points forms the two planes $y=-2$ and $z=3$.



Hint

One of the factors must be zero.

Example:
Exercise:

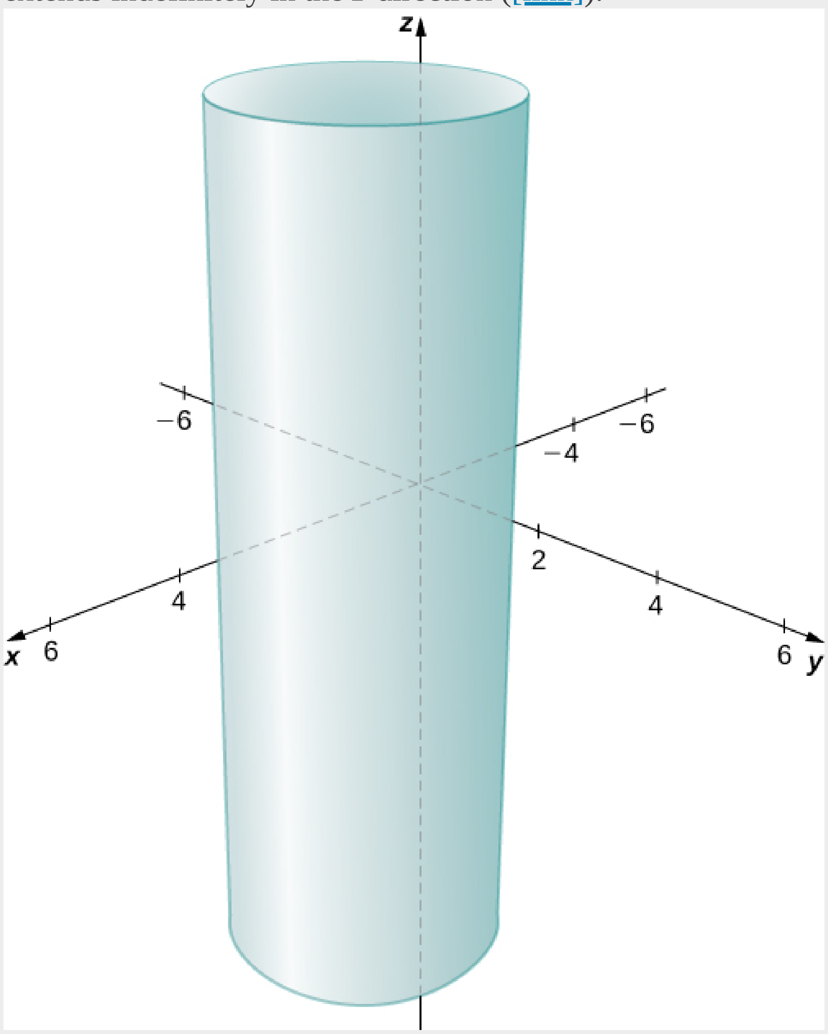
Problem:

Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution:

The *x*- and *y*-coordinates form a circle in the *xy*-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the *z*-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the *z*-direction ([link]).



The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

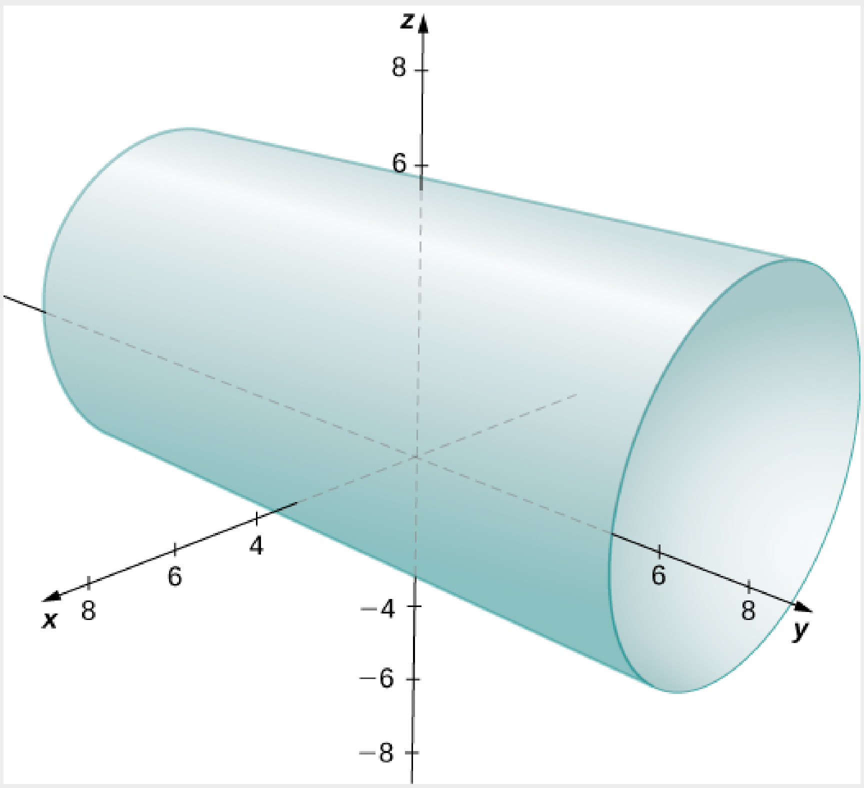
Note:
Exercise:

Problem:

Describe the set of points in three dimensional space that satisfies $x^2 + (z - 2)^2 = 16$, and graph the surface.

Solution:

A cylinder of radius 4 centered on the line with $x = 0$ and $z = 2$.



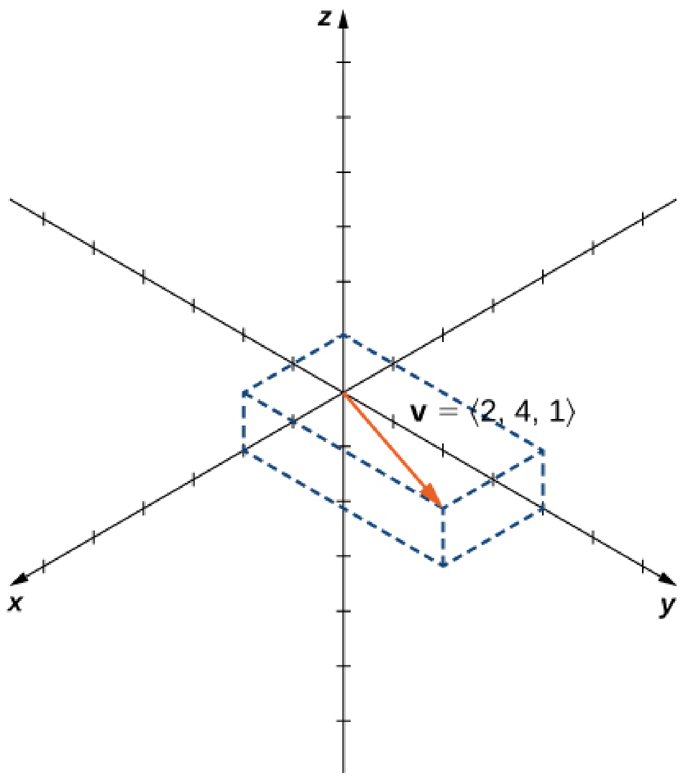
Hint

Think about what happens if you plot this equation in two dimensions in the *xz*-plane.

Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\mathbf{v} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\mathbf{0} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\mathbf{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ ([link]).



Vector $\mathbf{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

Equation:

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \text{ and } k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle.$$

If $k = -1$, then $k\mathbf{v} = (-1)\mathbf{v}$ is written as $-\mathbf{v}$, and vector subtraction is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$.

The standard unit vectors extend easily into three dimensions as well—$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$—and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

Equation:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

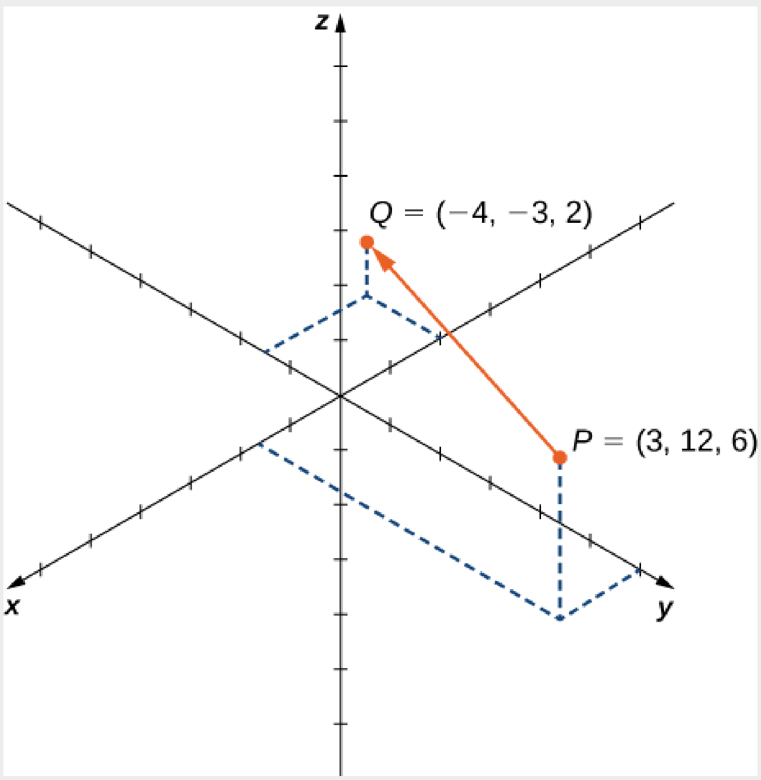
Example:

Exercise:

Problem:

Vector Representations

Let $\vec{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in [link]. Express $\vec{PQ}$ in both component form and using standard unit vectors.



The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$.

Solution:

In component form,
Equation:

$$\begin{aligned}\vec{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= \langle -4 - 3, -3 - 12, 2 - 6 \rangle = \langle -7, -15, -4 \rangle.\end{aligned}$$

In standard unit form,
Equation:

$$\vec{PQ} = -7\mathbf{i} - 15\mathbf{j} - 4\mathbf{k}.$$

Note:
Exercise:

Problem:

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\vec{ST}$ in component form and in standard unit form.

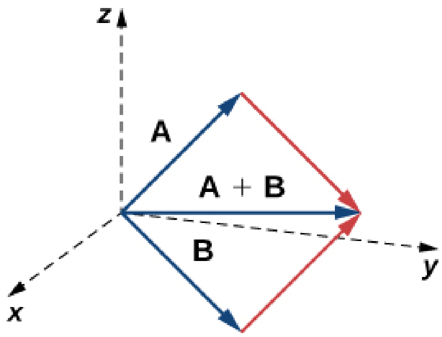
Solution:

$\vec{ST} = \langle -1, -9, 1 \rangle = -\mathbf{i} - 9\mathbf{j} + \mathbf{k}$

Hint

Write $\vec{ST}$ in component form first. T is the terminal point of $\vec{ST}$.

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space ([link]).



To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Note:
Rule: Properties of Vectors in Space
Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.
Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$
Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$
Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$
Vector magnitude: $\| \mathbf{v} \| = \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2}$
Unit vector in the direction of v: $\frac{1}{\| \mathbf{v} \|}\mathbf{v} = \frac{1}{\| \mathbf{v} \|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\| \mathbf{v} \|}, \frac{y_1}{\| \mathbf{v} \|}, \frac{z_1}{\| \mathbf{v} \|} \right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

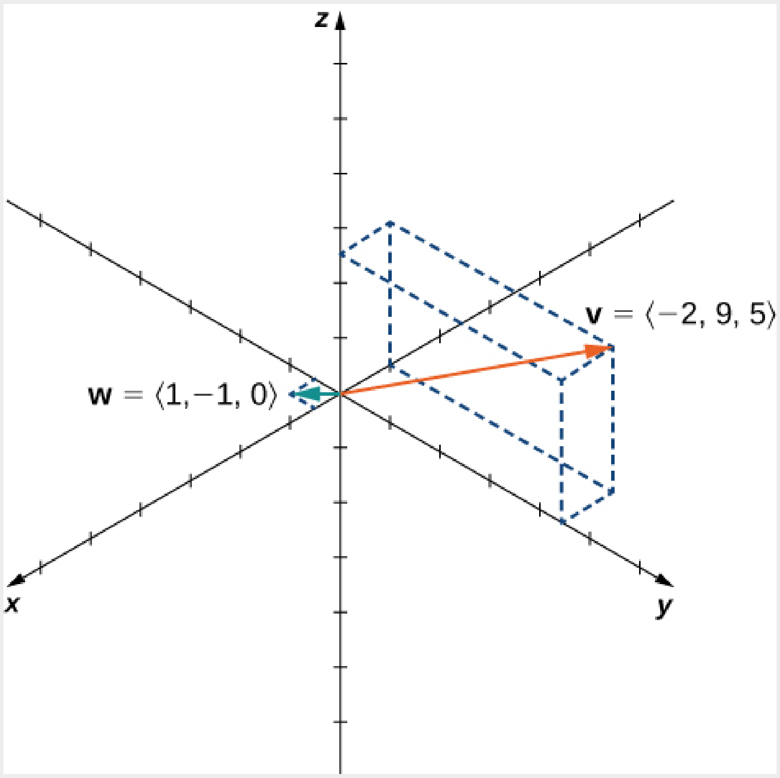
We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example:
Exercise:

Problem:
Vector Operations in Three Dimensions

Let $\mathbf{v} = \langle -2, 9, 5 \rangle$ and $\mathbf{w} = \langle 1, -1, 0 \rangle$ ([link]). Find the following vectors.

a. $3\mathbf{v} - 2\mathbf{w}$
b. $5 \,\|\, \mathbf{w} \,\|$
c. $\|5\mathbf{w}\|$
d. A unit vector in the direction of $\mathbf{v}$



The vectors $\mathbf{v} = \langle -2, 9, 5 \rangle$ and $\mathbf{w} = \langle 1, -1, 0 \rangle$.

Solution:

a. First, use scalar multiplication of each vector, then subtract:
Equation:

$$\begin{aligned} 3\mathbf{v} - 2\mathbf{w} &= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle \\ &= \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle \\ &= \langle -6 - 2, 27 - (-2), 15 - 0 \rangle \\ &= \langle -8, 29, 15 \rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:
Equation:

$$5\left\|\mathbf{w}\right\|=5\sqrt{1^2+(-1)^2+0^2}=5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:
Equation:

$$||5\mathbf{w}||=||\langle 5,-5,0\rangle||=\sqrt{5^2+(-5)^2+0^2}=\sqrt{50}=5\sqrt{2}.$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:
Equation:

$$\begin{aligned}\frac{\mathbf{v}}{||\mathbf{v}||} &= \frac{1}{||\mathbf{v}||}\langle -2,9,5\rangle \\ &= \frac{1}{\sqrt{(-2)^2+9^2+5^2}}\langle -2,9,5\rangle \\ &= \frac{1}{\sqrt{110}}\langle -2,9,5\rangle \\ &= \left\langle \frac{-2}{\sqrt{110}},\frac{9}{\sqrt{110}},\frac{5}{\sqrt{110}}\right\rangle.\end{aligned}$$

Note:
Exercise:

Problem:

Let $\mathbf{v}=\langle -1,-1,1\rangle$ and $\mathbf{w}=\langle 2,0,1\rangle$. Find a unit vector in the direction of $5\mathbf{v}+3\mathbf{w}$.

Solution:

$$\left\langle \frac{1}{3\sqrt{10}},-\frac{5}{3\sqrt{10}},\frac{8}{3\sqrt{10}}\right\rangle$$

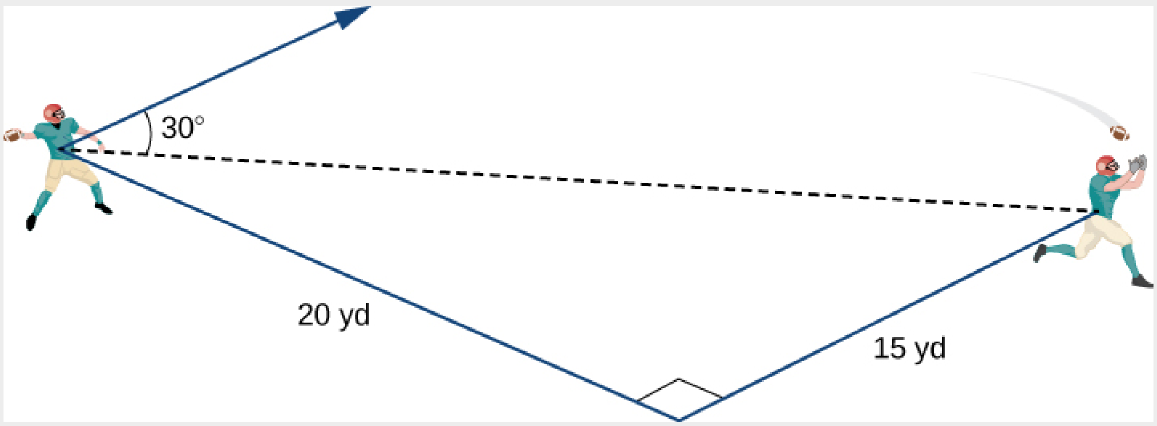
Hint

Start by writing $5\mathbf{v}+3\mathbf{w}$ in component form.

Example:
Exercise:

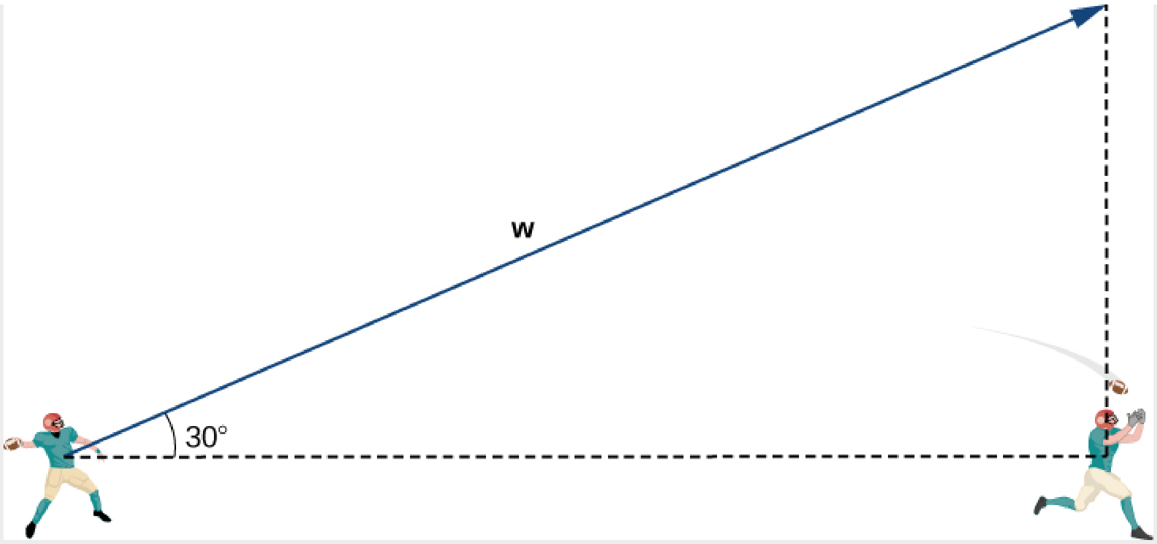
Problem:
Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of 30° (see the following figure). Write the initial velocity vector of the ball, $\mathbf{v}$, in component form.



Solution:

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\mathbf{w}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of 30° (see the following figure). This vector would have the same direction as $\mathbf{v}$, but it may not have the right magnitude.



The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

Equation:

$$\text{Dist from QB to receiver} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25 \text{ yd.}$$

We have $\frac{25}{\|\mathbf{w}\|} = \cos 30^\circ$. Then the magnitude of $\mathbf{w}$ is given by

Equation:

$$\|\mathbf{w}\| = \frac{25}{\cos 30^\circ} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\mathbf{w}$ is

Equation:

$$\text{Vert dist from receiver to terminal point of } \mathbf{w} = \|\mathbf{w}\| \sin 30^\circ = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd.}$$

Then $\mathbf{w} = \left\langle 20, 15, \frac{25}{\sqrt{3}} \right\rangle$, and has the same direction as $\mathbf{v}$.

Recall, though, that we calculated the magnitude of $\mathbf{w}$ to be $\|\mathbf{w}\| = \frac{50}{\sqrt{3}}$, and $\mathbf{v}$ has magnitude 60 mph. So, we need to multiply vector $\mathbf{w}$ by an appropriate constant, k. We want to find a value of k so that $\|k\mathbf{w}\| = 60$ mph. We have

Equation:

$$\|k\mathbf{w}\| = k\|\mathbf{w}\| = k\frac{50}{\sqrt{3}} \text{ mph},$$

so we want

Equation:

$$\begin{aligned} k\tfrac{50}{\sqrt{3}} &= 60 \\ k &= \tfrac{60\sqrt{3}}{50} \\ k &= \tfrac{6\sqrt{3}}{5}. \end{aligned}$$

Then

Equation:

$$\mathbf{v} = k\mathbf{w} = k\left\langle 20, 15, \frac{25}{\sqrt{3}} \right\rangle = \frac{6\sqrt{3}}{5}\left\langle 20, 15, \frac{25}{\sqrt{3}} \right\rangle = \left\langle 24\sqrt{3}, 18\sqrt{3}, 30 \right\rangle.$$

Let’s double-check that $\|\mathbf{v}\| = 60$. We have

Equation:

$$\|\mathbf{v}\| = \sqrt{\left(24\sqrt{3}\right)^2 + \left(18\sqrt{3}\right)^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph}.$$

So, we have found the correct components for $\mathbf{v}$.

Note:
Exercise:

Problem:

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of 45°. Write the initial velocity vector of the ball, $\mathbf{v}$, in component form.

Solution:

$\mathbf{v} = \left\langle 16\sqrt{2}, 12\sqrt{2}, 20\sqrt{2} \right\rangle$

Hint

Follow the process used in the previous example.

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula
  Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- In three dimensions, the equations $x = a, y = b,$ and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is
  Equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $\mathbf{v} = \langle x, y, z \rangle$, or in terms of the standard unit vectors, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

  - **Scalar multiplication:** $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$
  - **Vector addition:**
    $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$
  - **Vector subtraction:**
    $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$
  - **Vector magnitude:** $\|\mathbf{v}\| = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2}$
  - **Unit vector in the direction of v:** $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \right\rangle,$ $\mathbf{v} \neq \mathbf{0}$

Key Equations

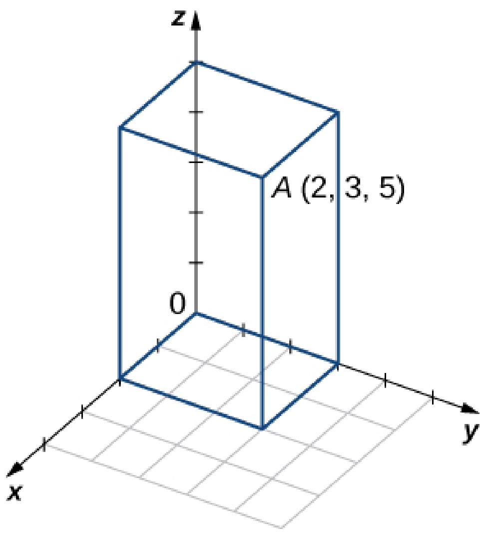
- **Distance between two points in space:**
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Sphere with center (a, b, c) and radius r:**
  $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

Exercise:

Problem:

Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find

a. the coordinates of the other six vertices of the box and
b. the length of the diagonal of the box determined by the vertices O and A.

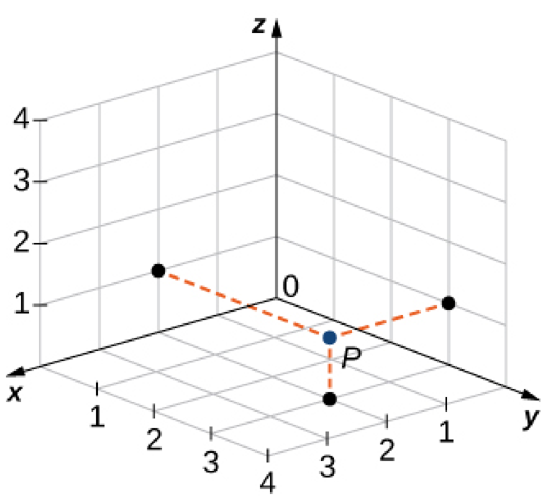


Solution:

a. $(2,0,5), (2,0,0), (2,3,0), (0,3,0), (0,3,5), (0,0,5)$; b. $\sqrt{38}$

Exercise:

Problem: Find the coordinates of point P and determine its distance to the origin.



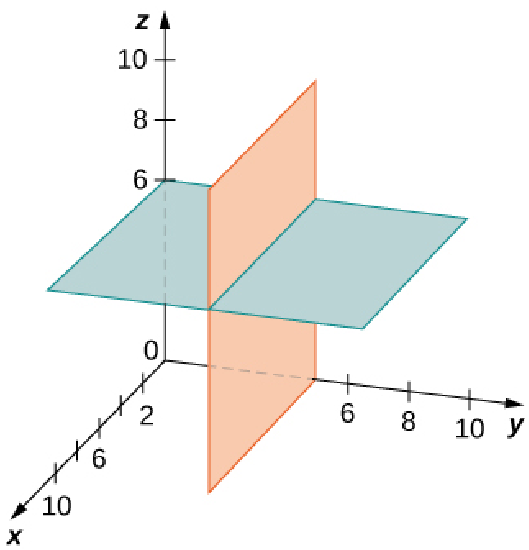
For the following exercises, describe and graph the set of points that satisfies the given equation.

Exercise:

Problem: $(y-5)(z-6)=0$

Solution:

A union of two planes: $y=5$ (a plane parallel to the *xz*-plane) and $z=6$ (a plane parallel to the *xy*-plane)



Exercise:

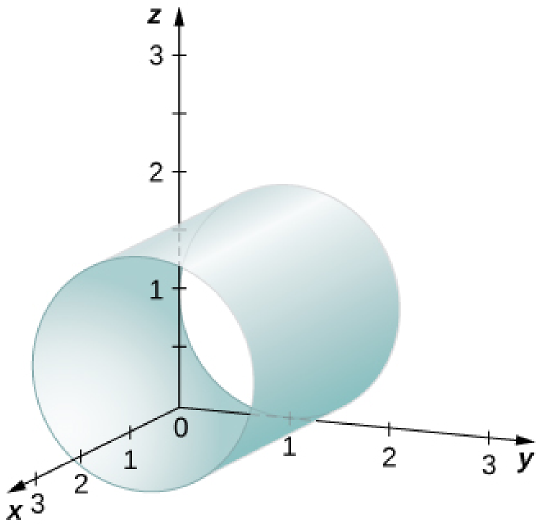
Problem: $(z-2)(z-5)=0$

Exercise:

Problem: $(y-1)^2+(z-1)^2=1$

Solution:

A cylinder of radius 1 centered on the line $y=1, z=1$



Exercise:

Problem: $(x-2)^2 + (z-5)^2 = 4$

Exercise:

Problem:

Write the equation of the plane passing through point $(1, 1, 1)$ that is parallel to the *xy*-plane.

Solution:

$z = 1$

Exercise:

Problem:

Write the equation of the plane passing through point $(1, -3, 2)$ that is parallel to the *xz*-plane.

Exercise:

Problem:

Find an equation of the plane passing through points $(1, -3, -2)$, $(0, 3, -2)$, and $(1, 0, -2)$.

Solution:

$z = -2$

Exercise:

Problem:

Find an equation of the plane passing through points $(1, 9, 2)$, $(1, 3, 6)$, and $(1, -7, 8)$.

For the following exercises, find the equation of the sphere in standard form that satisfies the given conditions.

Exercise:

Problem: Center $C\,(-1, 7, 4)$ and radius 4

Solution:

$(x + 1)^2 + (y - 7)^2 + (z - 4)^2 = 16$

Exercise:

Problem: Center $C\,(-4, 7, 2)$ and radius 6

Exercise:

Problem: Diameter PQ, where $P\,(-1, 5, 7)$ and $Q\,(-5, 2, 9)$

Solution:

$(x + 3)^2 + (y - 3.5)^2 + (z - 8)^2 = \frac{29}{4}$

Exercise:

Problem: Diameter PQ, where $P\,(-16, -3, 9)$ and $Q\,(-2, 3, 5)$

For the following exercises, find the center and radius of the sphere with an equation in general form that is given.

Exercise:

Problem: $P(1, 2, 3)$ $x^2 + y^2 + z^2 - 4z + 3 = 0$

Solution:

Center $C\,(0, 0, 2)$ and radius 1

Exercise:

Problem: $x^2 + y^2 + z^2 - 6x + 8y - 10z + 25 = 0$

For the following exercises, express vector $\vec{PQ}$ with the initial point at P and the terminal point at Q

a. in component form and
b. by using standard unit vectors.

Exercise:

Problem: $P\,(3, 0, 2)$ and $Q\,(-1, -1, 4)$

Solution:

a. $\vec{PQ} = \langle -4, -1, 2\rangle$; b. $\vec{PQ} = -4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Exercise:

Problem: $P\,(0, 10, 5)$ and $Q\,(1, 1, -3)$

Exercise:

Problem:

$P\,(-2, 5, -8)$ and $M\,(1, -7, 4)$, where M is the midpoint of the line segment PQ

Solution:

a. $\vec{PQ} = \langle 6, -24, 24\rangle$; b. $\vec{PQ} = 6\mathbf{i} - 24\mathbf{j} + 24\mathbf{k}$

Exercise:

Problem:

$Q\,(0, 7, -6)$ and $M\,(-1, 3, 2)$, where M is the midpoint of the line segment PQ

Exercise:

Problem:

Find terminal point Q of vector $\vec{PQ} = \langle 7, -1, 3\rangle$ with the initial point at $P\,(-2, 3, 5)$.

Solution:

$Q(5, 2, 8)$

Exercise:

Problem:

Find initial point P of vector $\vec{PQ} = \langle -9, 1, 2\rangle$ with the terminal point at $Q\,(10, 0, -1)$.

For the following exercises, use the given vectors $\mathbf{a}$ and $\mathbf{b}$ to find and express the vectors $\mathbf{a}+\mathbf{b}$, $4\mathbf{a}$, and $-5\mathbf{a}+3\mathbf{b}$ in component form.

Exercise:

Problem: $\mathbf{a}=\langle -1,-2,4\rangle$, $\mathbf{b}=\langle -5,6,-7\rangle$

Solution:

$\mathbf{a}+\mathbf{b}=\langle -6,4,-3\rangle$, $4\mathbf{a}=\langle -4,-8,16\rangle$, $-5\mathbf{a}+3\mathbf{b}=\langle -10,28,-41\rangle$

Exercise:

Problem: $\mathbf{a}=\langle 3,-2,4\rangle$, $\mathbf{b}=\langle -5,6,-9\rangle$

Exercise:

Problem: $\mathbf{a}=-\mathbf{k}$, $\mathbf{b}=-\mathbf{i}$

Solution:

$\mathbf{a}+\mathbf{b}=\langle -1,0,-1\rangle$, $4\mathbf{a}=\langle 0,0,-4\rangle$, $-5\mathbf{a}+3\mathbf{b}=\langle -3,0,5\rangle$

Exercise:

Problem: $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$

For the following exercises, vectors $\mathbf{u}$ and $\mathbf{v}$ are given. Find the magnitudes of vectors $\mathbf{u}-\mathbf{v}$ and $-2\mathbf{u}$.

Exercise:

Problem: $\mathbf{u}=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, $\mathbf{v}=-\mathbf{i}+5\mathbf{j}-\mathbf{k}$

Solution:

$\|\mathbf{u}-\mathbf{v}\|=\sqrt{38}$, $\|-2\mathbf{u}\|=2\sqrt{29}$

Exercise:

Problem: $\mathbf{u}=\mathbf{i}+\mathbf{j}$, $\mathbf{v}=\mathbf{j}-\mathbf{k}$

Exercise:

Problem: $\mathbf{u}=\langle 2\cos t,-2\sin t,3\rangle$, $\mathbf{v}=\langle 0,0,3\rangle$, where t is a real number.

Solution:

$\|\mathbf{u}-\mathbf{v}\|=2$, $\|-2\mathbf{u}\|=2\sqrt{13}$

Exercise:

Problem: $\mathbf{u}=\langle 0,1,\sinh t\rangle$, $\mathbf{v}=\langle 1,1,0\rangle$, where t is a real number.

For the following exercises, find the unit vector in the direction of the given vector $\mathbf{a}$ and express it using standard unit vectors.

Exercise:

Problem: $\mathbf{a}=3\mathbf{i}-4\mathbf{j}$

Solution:

$\mathbf{a}=\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$

Exercise:

Problem: $\mathbf{a}=\langle 4,-3,6\rangle$

Exercise:

Problem: $\mathbf{a}=\vec{PQ}$, where $P(-2,3,1)$ and $Q(0,-4,4)$

Solution:

$\left\langle \frac{2}{\sqrt{62}}\mathbf{i}-\frac{7}{\sqrt{62}}\mathbf{j}+\frac{3}{\sqrt{62}}\mathbf{k}\right\rangle$

Exercise:

Problem: $\mathbf{a}=\vec{OP}$, where $P(-1,-1,1)$

Exercise:

Problem:

$\mathbf{a}=\mathbf{u}-\mathbf{v}+\mathbf{w}$, where $\mathbf{u}=\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{v}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, and $\mathbf{w}=-\mathbf{i}+\mathbf{j}+3\mathbf{k}$

Solution:

$\left\langle -\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$

Exercise:

Problem: $\mathbf{a}=2\mathbf{u}+\mathbf{v}-\mathbf{w}$, where $\mathbf{u}=\mathbf{i}-\mathbf{k}$, $\mathbf{v}=2\mathbf{j}$, and $\mathbf{w}=\mathbf{i}-\mathbf{j}$

Exercise:

Problem:

Determine whether $\vec{AB}$ and $\vec{PQ}$ are equivalent vectors, where $A\,(1,1,1)$, $B\,(3,3,3)$, $P\,(1,4,5)$, and $Q\,(3,6,7)$.

Solution:

Equivalent vectors

Exercise:

Problem:

Determine whether the vectors $\vec{AB}$ and $\vec{PQ}$ are equivalent, where $A\,(1,4,1)$, $B\,(-2,2,0)$, $P\,(2,5,7)$, and $Q\,(-3,2,1)$.

For the following exercises, find vector **u** with a magnitude that is given and satisfies the given conditions.

Exercise:

Problem: $\mathbf{v} = \langle 7,-1,3\rangle$, $\|\mathbf{u}\| = 10$, $\mathbf{u}$ and $\mathbf{v}$ have the same direction

---

Solution:

$$\mathbf{u} = \left\langle \frac{70}{\sqrt{59}}, -\frac{10}{\sqrt{59}}, \frac{30}{\sqrt{59}} \right\rangle$$

Exercise:

Problem: $\mathbf{v} = \langle 2,4,1\rangle$, $\|\mathbf{u}\| = 15$, $\mathbf{u}$ and $\mathbf{v}$ have the same direction

Exercise:

Problem:

$\mathbf{v} = \langle 2\sin t, 2\cos t, 1\rangle$, $\|\mathbf{u}\| = 2$, $\mathbf{u}$ and $\mathbf{v}$ have opposite directions for any t, where t is a real number

---

Solution:

$$\mathbf{u} = \left\langle -\frac{4}{\sqrt{5}}\sin t, -\frac{4}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}} \right\rangle$$

Exercise:

Problem:

$\mathbf{v} = \langle 3\sinh t, 0, 3\rangle$, $\|\mathbf{u}\| = 5$, $\mathbf{u}$ and $\mathbf{v}$ have opposite directions for any t, where t is a real number

Exercise:

Problem:

Determine a vector of magnitude 5 in the direction of vector $\vec{AB}$, where $A(2, 1, 5)$ and $B(3, 4, -7)$.

Solution:

$\left\langle \frac{5}{\sqrt{154}}, \frac{15}{\sqrt{154}}, -\frac{60}{\sqrt{154}} \right\rangle$

Exercise:

Problem:

Find a vector of magnitude 2 that points in the opposite direction than vector $\vec{AB}$, where $A(-1, -1, 1)$ and $B(0, 1, 1)$. Express the answer in component form.

Exercise:

Problem:

Consider the points $A\,(2, \alpha, 0)$, $B\,(0, 1, \beta)$, and $C\,(1, 1, \beta)$, where α and β are negative real numbers. Find α and β such that $\vec{OA} - \vec{OB} + \vec{OC} = \vec{OB} = 4$.

Solution:

$\alpha = -\sqrt{7}$, $\beta = -\sqrt{15}$

Exercise:

Problem:

Consider points $A\,(\alpha, 0, 0)$, $B\,(0, \beta, 0)$, and $C\,(\alpha, \beta, \beta)$, where α and β are positive real numbers. Find α and β such that $OA + OB = \sqrt{2}$ and $\|OC\| = \sqrt{3}$.

Exercise:

Problem:

Let $P\,(x, y, z)$ be a point situated at an equal distance from points $A\,(1, -1, 0)$ and $B\,(-1, 2, 1)$. Show that point P lies on the plane of equation $-2x + 3y + z = 2$.

Exercise:

Problem:

Let $P\left(x, y, z\right)$ be a point situated at an equal distance from the origin and point $A\left(4, 1, 2\right)$. Show that the coordinates of point P satisfy the equation $8x + 2y + 4z = 21$.

Exercise:

Problem:

The points A, B, and C are collinear (in this order) if the relation $\vec{AB} + \vec{BC} = \vec{AC}$ is satisfied. Show that $A(5, 3, -1)$, $B(-5, -3, 1)$, and $C(-15, -9, 3)$ are collinear points.

Exercise:

Problem: Show that points $A(1, 0, 1)$, $B(0, 1, 1)$, and $C(1, 1, 1)$ are not collinear.

Exercise:

Problem:

[T] A force $\mathbf{F}$ of 50 N acts on a particle in the direction of the vector $\vec{OP}$, where $P(3, 4, 0)$.

a. Express the force as a vector in component form.
b. Find the angle between force $\mathbf{F}$ and the positive direction of the *x*-axis. Express the answer in degrees rounded to the nearest integer.

Solution:

a. $\mathbf{F} = \langle 30, 40, 0 \rangle$; b. 53°

Exercise:

Problem:

[T] A force $\mathbf{F}$ of 40 N acts on a box in the direction of the vector $\vec{OP}$, where $P(1, 0, 2)$.

a. Express the force as a vector by using standard unit vectors.
b. Find the angle between force $\mathbf{F}$ and the positive direction of the *x*-axis.

Exercise:

Problem:

If $\mathbf{F}$ is a force that moves an object from point $P_1\,(x_1, y_1, z_1)$ to another point $P_2\,(x_2, y_2, z_2)$, then the displacement vector is defined as $\mathbf{D} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$. A metal container is lifted 10 m vertically by a constant force $\mathbf{F}$. Express the displacement vector $\mathbf{D}$ by using standard unit vectors.

Solution:

$\mathbf{D} = 10\mathbf{k}$

Exercise:

Problem:

A box is pulled 4 yd horizontally in the *x*-direction by a constant force $\mathbf{F}$. Find the displacement vector in component form.

Exercise:

Problem:

The sum of the forces acting on an object is called the *resultant* or *net force*. An object is said to be in static equilibrium if the resultant force of the forces that act on it is zero. Let $\mathbf{F}_1 = \langle 10, 6, 3 \rangle$, $\mathbf{F}_2 = \langle 0, 4, 9 \rangle$, and $\mathbf{F}_3 = \langle 10, -3, -9 \rangle$ be three forces acting on a box. Find the force $\mathbf{F}_4$ acting on the box such that the box is in static equilibrium. Express the answer in component form.

Solution:

$\mathbf{F}_4 = \langle -20, -7, -3 \rangle$

Exercise:

Problem:

[T] Let $\mathbf{F}_k = \left\langle 1, k, k^2 \right\rangle$, $k = 1, \ldots, n$ be n forces acting on a particle, with $n \geq 2$.

a. Find the net force $\mathbf{F} = \sum_{k=1}^{n} F_k$. Express the answer using standard unit vectors.

b. Use a computer algebra system (CAS) to find *n* such that $\|\mathbf{F}\| < 100$.

Exercise:

Problem:

The force of gravity $\mathbf{F}$ acting on an object is given by $\mathbf{F} = m\mathbf{g}$, where *m* is the mass of the object (expressed in kilograms) and $\mathbf{g}$ is acceleration resulting from gravity, with $\|\mathbf{g}\| = 9.8\ \mathrm{N/kg}$. A 2-kg disco ball hangs by a chain from the ceiling of a room.

a. Find the force of gravity $\mathbf{F}$ acting on the disco ball and find its magnitude.
b. Find the force of tension $\mathbf{T}$ in the chain and its magnitude.
   Express the answers using standard unit vectors.

(credit: modification of work by Kenneth Lu, Flickr)

Solution:

a. $\mathbf{F} = -19.6\mathbf{k}$, $\|\mathbf{F}\| = 19.6\ \mathrm{N}$; b. $\mathbf{T} = 19.6\mathbf{k}$, $\|\mathbf{T}\| = 19.6\ \mathrm{N}$

Exercise:

Problem:

A 5-kg pendant chandelier is designed such that the alabaster bowl is held by four chains of equal length, as shown in the following figure.

a. Find the magnitude of the force of gravity acting on the chandelier.

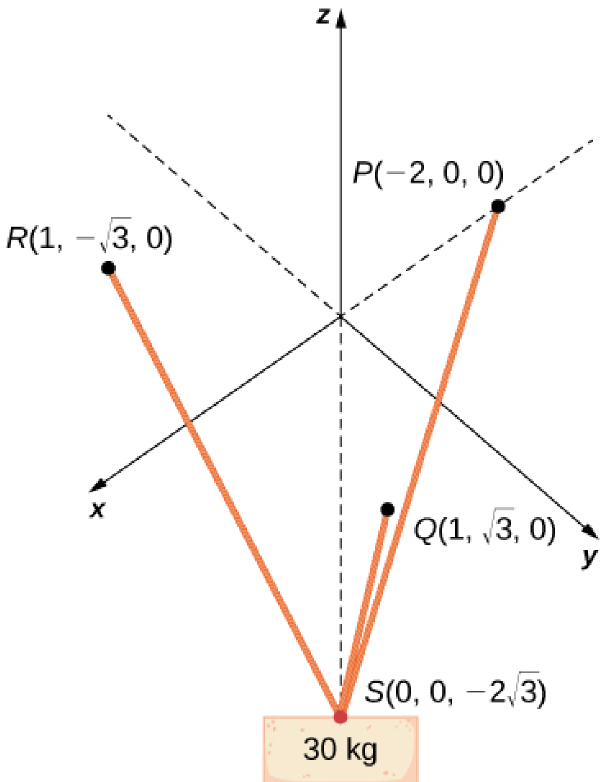
b. Find the magnitudes of the forces of tension for each of the four chains (assume chains are essentially vertical).

Exercise:

Problem:

[T] A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2, 0, 0)$, $Q\left(1, \sqrt{3}, 0\right)$, and $R\left(1, -\sqrt{3}, 0\right)$. The load is located at $S\left(0, 0, -2\sqrt{3}\right)$, as shown in the following figure. Let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ be the forces of tension resulting from the load in cables RS, QS, and PS, respectively.

a. Find the gravitational force $\mathbf{F}$ acting on the block of cement that counterbalances the sum $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ of the forces of tension in the cables.
b. Find forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. Express the answer in component form.



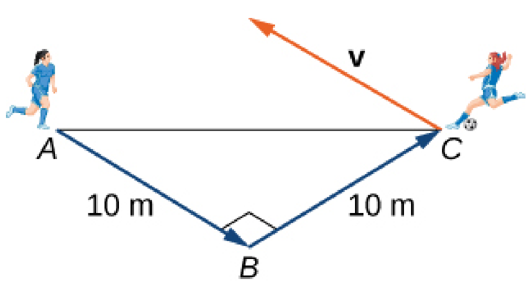
Solution:

a. $\mathbf{F} = -294\mathbf{k}$ N; b. $\mathbf{F}_1 = \left\langle -\frac{49\sqrt{3}}{3}, 49, -98 \right\rangle$, $\mathbf{F}_2 = \left\langle -\frac{49\sqrt{3}}{3}, -49, -98 \right\rangle$, and $\mathbf{F}_3 = \left\langle \frac{98\sqrt{3}}{3}, 0, -98 \right\rangle$ (each component is expressed in newtons)

Exercise:

Problem:

Two soccer players are practicing for an upcoming game. One of them runs 10 m from point A to point B. She then turns left at 90° and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/sec at an upward angle of 45° to her teammate, who is located at point A. Write the velocity of the ball in component form.



Exercise:

Problem:

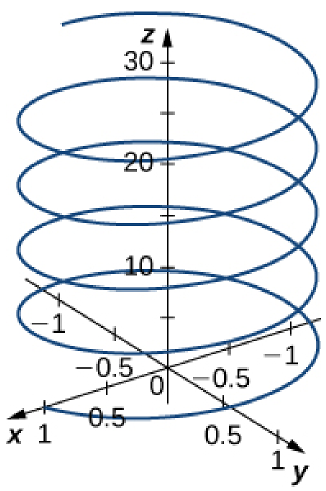
Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be the position vector of a particle at the time $t \in [0, T]$, where x, y, and z are smooth functions on $[0, T]$. The instantaneous velocity of the particle at time t is defined by vector $\mathbf{v}(t) = \langle x\prime(t), y\prime(t), z\prime(t)\rangle$, with components that are the derivatives with respect to t, of the functions *x*, *y*, and *z*, respectively. The magnitude $\|\mathbf{v}(t)\|$ of the instantaneous velocity vector is called the *speed of the particle at time* t. Vector $\mathbf{a}(t) = \langle x\prime\prime(t), y\prime\prime(t), z\prime\prime(t)\rangle$, with components that are the second derivatives with respect to t, of the functions x, y, and z, respectively, gives the acceleration of the particle at time t. Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\mathbf{r}$ are expressed in centimeters and time is expressed in seconds.

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first second. Round your answer to two decimal places.
b. Use a CAS to visualize the path of the particle—that is, the set of all points of coordinates $(\cos t, \sin t, 2t)$, where $t \in [0, 30]$.

Solution:

a. $\mathbf{v}(1) = \langle -0.84, 0.54, 2\rangle$ (each component is expressed in centimeters per second); $\|\mathbf{v}(1)\| = 2.24$ (expressed in centimeters per second); $\mathbf{a}(1) = \langle -0.54, -0.84, 0\rangle$ (each component expressed in centimeters per second squared);

b.



Exercise:

Problem:

[T] Let $\mathbf{r}(t) = \langle t, 2t^2, 4t^2 \rangle$ be the position vector of a particle at time t (in seconds), where $t \in [0, 10]$ (here the components of $\mathbf{r}$ are expressed in centimeters).

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first two seconds. Round your answer to two decimal places.
b. Use a CAS to visualize the path of the particle defined by the points $(t, 2t^2, 4t^2)$, where $t \in [0, 60]$.

Glossary

coordinate plane
: a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the *xy*-plane, *xz*-plane, or the *yz*-plane

right-hand rule
: a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the *z*-axis in such a way that the fingers curl from the positive *x*-axis to the positive *y*-axis, the thumb points in the direction of the positive *z*-axis

octants
: the eight regions of space created by the coordinate planes

sphere
: the set of all points equidistant from a given point known as the *center*

standard equation of a sphere
: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ describes a sphere with center (a, b, c) and radius r

three-dimensional rectangular coordinate system
: a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x, y, z) that plots its location relative to the defining axes

The Dot Product

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that *work* is done by the force. In Introduction to Applications of Integration on integration applications, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the *dot product*, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Note:
Definition
The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components
Equation:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note that if **u** and **v** are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then

Equation:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example:

Exercise:

Problem:

Calculating Dot Products

a. Find the dot product of $\mathbf{u} = \langle 3, 5, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 0 \rangle$.
b. Find the scalar product of $\mathbf{p} = 10\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$ and $\mathbf{q} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

Solution:

a. Substitute the vector components into the formula for the dot product:
   Equation:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) = -3 + 15 + 0 = 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:
   Equation:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 10(-2) + (-4)(1) + (7)(6) = -20 - 4 + 42 = 18. \end{aligned}$$

Note:
Exercise:

Problem: Find $\mathbf{u} \cdot \mathbf{v}$, where $\mathbf{u} = \langle 2, 9, -1 \rangle$ and $\mathbf{v} = \langle -3, 1, -4 \rangle$.

Solution:

7

Hint

Multiply corresponding components and then add their products.

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Note:
Properties of the Dot Product
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.
Equation:

$$\begin{array}{lrcll} \text{i.} & \mathbf{u} \cdot \mathbf{v} & = & \mathbf{v} \cdot \mathbf{u} & \text{Commutative property} \\ \text{ii.} & \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) & = & \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} & \text{Distributive property} \\ \text{iii.} & c(\mathbf{u} \cdot \mathbf{v}) & = & (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) & \text{Associative property} \\ \text{iv.} & \mathbf{v} \cdot \mathbf{v} & = & \| \mathbf{v} \|^2 & \text{Property of magnitude} \end{array}$$

Proof

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then
Equation:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= u_1v_1+u_2v_2+u_3v_3 \\ &= v_1u_1+v_2u_2+v_3u_3 \\ &= \langle v_1,v_2,v_3\rangle\cdot\langle u_1,u_2,u_3\rangle \\ &= \mathbf{v}\cdot\mathbf{u}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

Equation:

$$\begin{aligned}c\left(\mathbf{u}\cdot\mathbf{v}\right) &= c\left(u_1v_1+u_2v_2+u_3v_3\right) \\ &= c\left(u_1v_1\right)+c\left(u_2v_2\right)+c\left(u_3v_3\right) \\ &= (cu_1)v_1+(cu_2)v_2+(cu_3)v_3 \\ &= \langle cu_1,cu_2,cu_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= c\langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= (c\mathbf{u})\cdot\mathbf{v}.\end{aligned}$$

The proof that $c\left(\mathbf{u}\cdot\mathbf{v}\right)=\mathbf{u}\cdot\left(c\mathbf{v}\right)$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

Equation:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v} &= \langle v_1,v_2,v_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= (v_1)^2+(v_2)^2+(v_3)^2 \\ &= \left[\sqrt{(v_1)^2+(v_2)^2+(v_3)^2}\right]^2 \\ &= \|\mathbf{v}\|^2.\end{aligned}$$

□

Note that the definition of the dot product yields $\mathbf{0}\cdot\mathbf{v}=\mathbf{0}$. By property iv., if $\mathbf{v}\cdot\mathbf{v}=0$, then $\mathbf{v}=\mathbf{0}$.

Example:
Exercise:

Problem:
Using Properties of the Dot Product

Let $\mathbf{a} = \langle 1, 2, -3 \rangle$, $\mathbf{b} = \langle 0, 2, 4 \rangle$, and $\mathbf{c} = \langle 5, -1, 3 \rangle$. Find each of the following products.

a. $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
b. $\mathbf{a} \cdot (2\mathbf{c})$
c. $\|\mathbf{b}\|^2$

Solution:

a. Note that this expression asks for the scalar multiple of $\mathbf{c}$ by $\mathbf{a} \cdot \mathbf{b}$:
Equation:

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{b})\mathbf{c} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle) \langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4)) \langle 5, -1, 3 \rangle \\ &= -8 \langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle.\end{aligned}$$

b. This expression is a dot product of vector $\mathbf{a}$ and scalar multiple $2\mathbf{c}$:
Equation:

$$\begin{aligned}\mathbf{a} \cdot (2\mathbf{c}) &= 2(\mathbf{a} \cdot \mathbf{c}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12.\end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:
Equation:

$$\|\mathbf{b}\|^2 = \mathbf{b} \cdot \mathbf{b} = \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle = 0^2 + 2^2 + 4^2 = 0 + 4 + 16 = 20.$$

Note:
Exercise:

Problem:

Find the following products for $\mathbf{p} = \langle 7, 0, 2 \rangle$, $\mathbf{q} = \langle -2, 2, -2 \rangle$, and $\mathbf{r} = \langle 0, 2, -3 \rangle$.

a. $(\mathbf{r} \cdot \mathbf{p})\mathbf{q}$
b. $\|\mathbf{p}\|^2$

Solution:

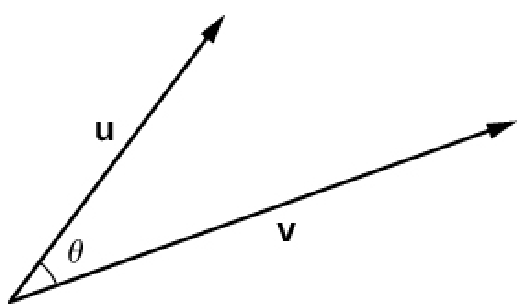
a. $(\mathbf{r} \cdot \mathbf{p})\mathbf{q} = \langle 12, -12, 12 \rangle$; b. $\|\mathbf{p}\|^2 = 53$

Hint

$\mathbf{r} \cdot \mathbf{p}$ is a scalar.

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them ([link]). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.



Let θ be the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$

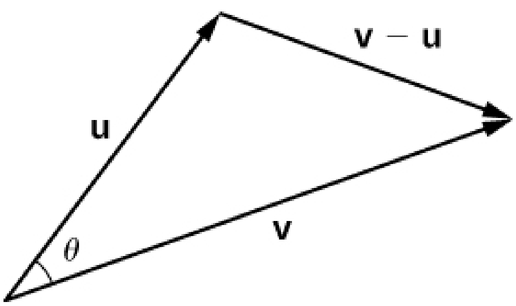
such that $0 \leq \theta \leq \pi$.

Note:
Evaluating a Dot Product
The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:
Equation:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta.$$

Proof

Place vectors $\mathbf{u}$ and $\mathbf{v}$ in standard position and consider the vector $\mathbf{v} - \mathbf{u}$ ([link]). These three vectors form a triangle with side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|,$ and $\|\mathbf{v} - \mathbf{u}\|$.



The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

Equation:

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

The dot product provides a way to rewrite the left side of this equation:
Equation:

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\\ &= (\mathbf{v}-\mathbf{u})\cdot\mathbf{v}-(\mathbf{v}-\mathbf{u})\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{u}\\ &= \|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2.\end{aligned}$$

Substituting into the law of cosines yields
Equation:

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ \|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ -2\mathbf{u}\cdot\mathbf{v} &= -2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ \mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.\end{aligned}$$

□

We can use this form of the dot product to find the measure of the angle between two nonzero vectors. The following equation rearranges [link] to solve for the cosine of the angle:
Equation:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0^\circ \le \theta \le 180^\circ$ (or $0 \le \theta \le \pi$ if we are working in radians). The inverse

cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example:
Exercise:

Problem:
Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$
b. $\langle 2, 5, 6 \rangle$ and $\langle -2, -4, 4 \rangle$

Solution:

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into [link]:
Equation:

$$\begin{aligned} \cos\theta &= \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}-3\mathbf{k})}{\|\mathbf{i}+\mathbf{j}+\mathbf{k}\|\cdot\|2\mathbf{i}-\mathbf{j}-3\mathbf{k}\|} \\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}} \\ &= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}. \end{aligned}$$

Therefore, $\theta = \arccos\frac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:
Equation:

$$\begin{aligned} \cos\theta &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\langle 2,5,6\rangle\|\cdot\|\langle -2,-4,4\rangle\|} \\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}} \\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0. \end{aligned}$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.

Note:
Exercise:

Problem:

Find the measure of the angle, in radians, formed by vectors $\mathbf{a} = \langle 1, 2, 0 \rangle$ and $\mathbf{b} = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

Solution:

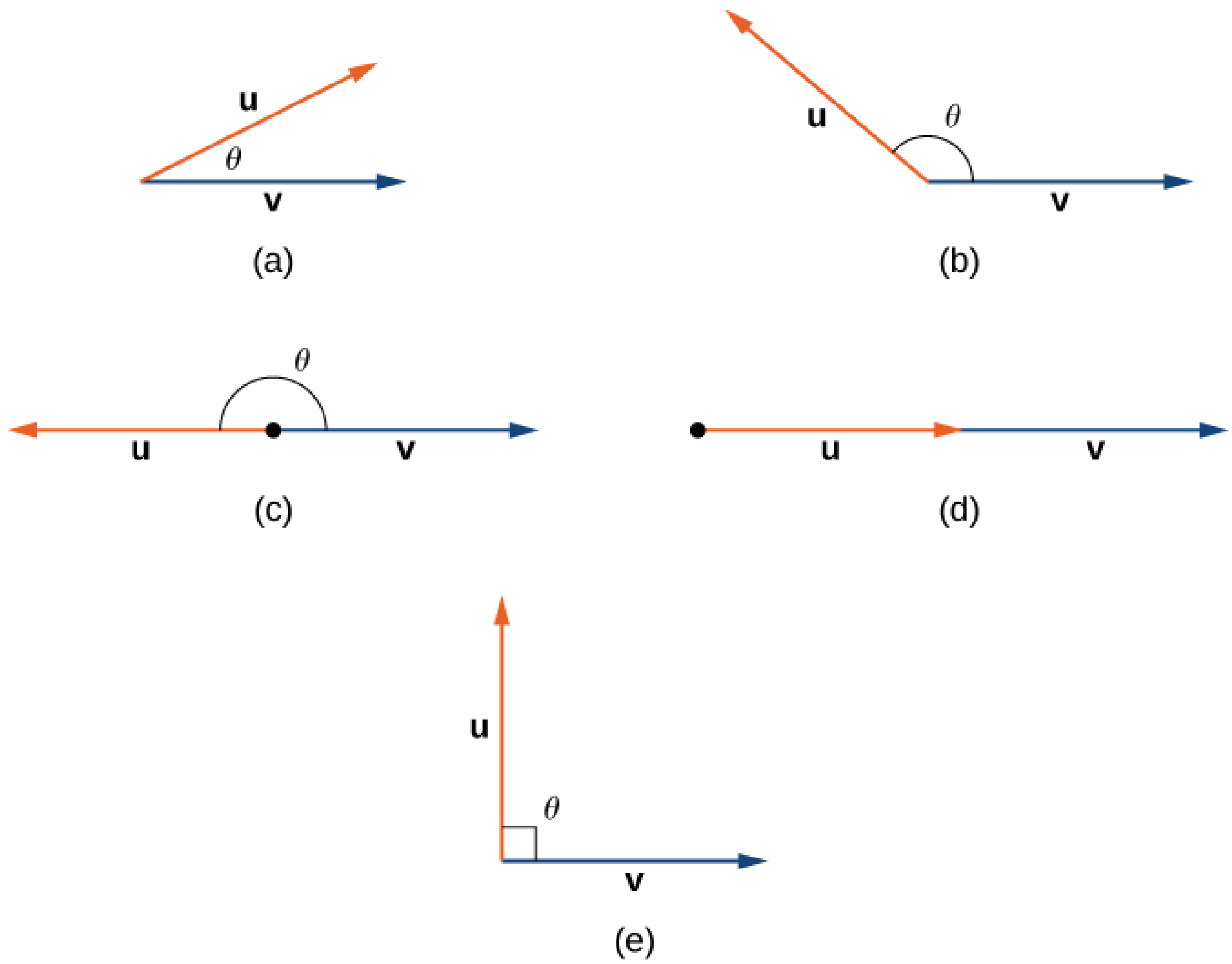
$\theta \approx 0.22$ rad

Hint

Use the equation $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$.

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle ([link]). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

Proof



(a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Note:
Orthogonal Vectors
The nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal vectors** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Proof

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\mathbf{u} \cdot \mathbf{v} = 0$. Then
Equation:

$$\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta = 0.$$

However, $\|\mathbf{u}\| \neq 0$ and $\|\mathbf{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90^\circ$, and the vectors are orthogonal.

Now assume $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then $\theta = 90^\circ$ and we have
Equation:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos 90^\circ = \|\mathbf{u}\| \, \|\mathbf{v}\| \,(0) = 0.$$

□

The terms *orthogonal*, *perpendicular*, and *normal* each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example:
Exercise:

Problem:
Identifying Orthogonal Vectors

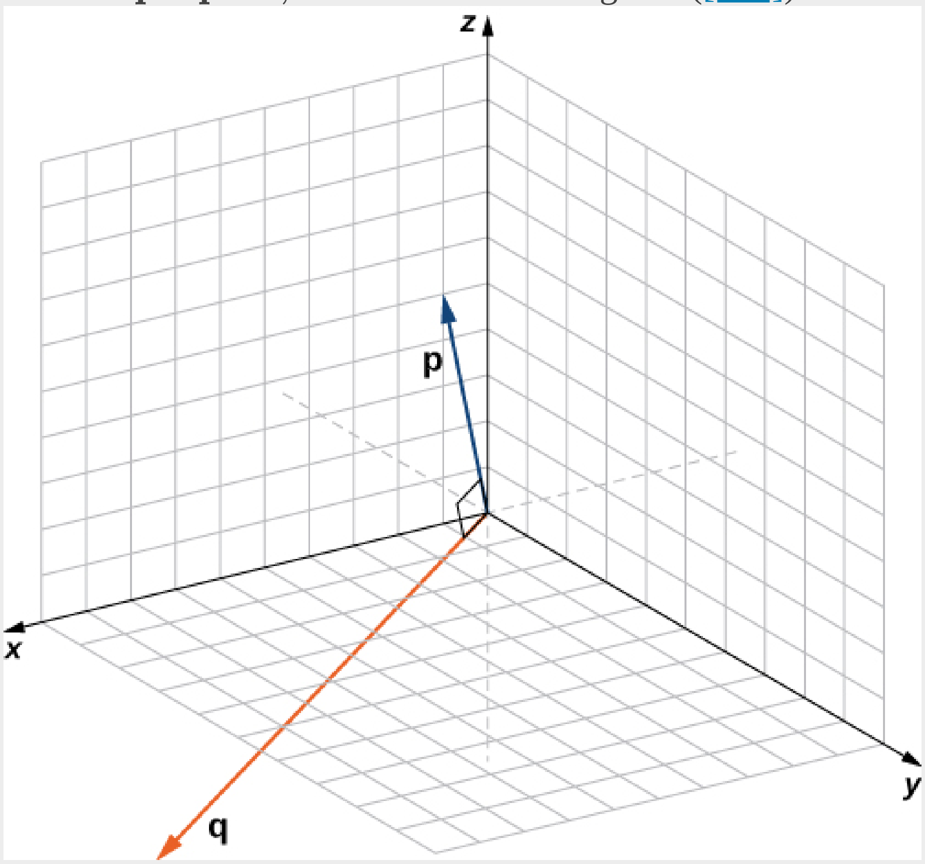
Determine whether $\mathbf{p} = \langle 1, 0, 5 \rangle$ and $\mathbf{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution:

Using the definition, we need only check the dot product of the vectors:
Equation:

$$\mathbf{p} \cdot \mathbf{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\mathbf{p} \cdot \mathbf{q} = 0$, the vectors are orthogonal ([link]).



Vectors **p** and **q** form a right angle when their initial points are aligned.

Note:
Exercise:

Problem:

For which value of *x* is $\mathbf{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\mathbf{q} = \langle x, -1, 2 \rangle$?

Solution:

$x = 5$

Hint

Vectors $\mathbf{p}$ and $\mathbf{q}$ are orthogonal if and only if $\mathbf{p} \cdot \mathbf{q} = 0$.

Example:
Exercise:

Problem:
Measuring the Angle Formed by Two Vectors

Let $\mathbf{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\mathbf{v}$ and $\mathbf{i}$
b. $\mathbf{v}$ and $\mathbf{j}$
c. $\mathbf{v}$ and $\mathbf{k}$

Solution:

a. Let α be the angle formed by $\mathbf{v}$ and $\mathbf{i}$:
Equation:

$$\begin{aligned} \cos \alpha &= \frac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{v}\| \cdot \|\mathbf{i}\|} \\ &= \frac{\langle 2,3,3 \rangle \cdot \langle 1,0,0 \rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} \\ &= \frac{2}{\sqrt{22}}. \end{aligned}$$

Equation:

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \text{ rad.}$$

b. Let β represent the angle formed by $\mathbf{v}$ and $\mathbf{j}$:
Equation:

$$\begin{aligned} \cos\beta &= \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\cdot\|\mathbf{j}\|} \\ &= \frac{\langle 2,3,3\rangle\cdot\langle 0,1,0\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} \\ &= \frac{3}{\sqrt{22}}. \end{aligned}$$

Equation:

$$\beta = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad.}$$

c. Let γ represent the angle formed by **v** and **k**:
Equation:

$$\begin{aligned} \cos\gamma &= \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\cdot\|\mathbf{k}\|} \\ &= \frac{\langle 2,3,3\rangle\cdot\langle 0,0,1\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} \\ &= \frac{3}{\sqrt{22}}. \end{aligned}$$

Equation:

$$\gamma = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad.}$$

Note:
Exercise:

Problem:

Let $\mathbf{v} = \langle 3,-5,1\rangle$. Find the measure of the angles formed by each pair of vectors.

a. **v** and **i**
b. **v** and **j**
c. **v** and **k**

Solution:

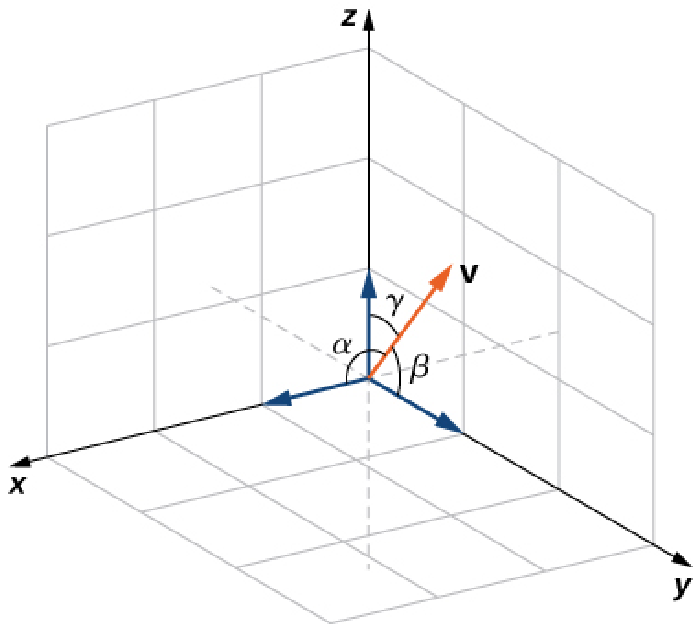
a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

Hint

$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Note:
Definition
The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector ([link]). The cosines for these angles are called the **direction cosines**.



Angle α is formed by vector **v** and unit vector **i**. Angle β is formed by vector **v** and unit vector **j**. Angle γ is formed by vector **v** and unit vector **k**.

In [link], the direction cosines of $\mathbf{v} = \langle 2, 3, 3 \rangle$ are $\cos \alpha = \frac{2}{\sqrt{22}}$, $\cos \beta = \frac{3}{\sqrt{22}}$, and $\cos \gamma = \frac{3}{\sqrt{22}}$. The direction angles of **v** are $\alpha = 1.130\,\text{rad}$, $\beta = 0.877\,\text{rad}$, and $\gamma = 0.877\,\text{rad}$.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\mathbf{q} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\mathbf{p} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\mathbf{q} \cdot \mathbf{p}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example:
Exercise:

Problem:
Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution:

The cost, price, and quantity vectors are
Equation:

$$\begin{aligned}\mathbf{c} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \\ \mathbf{p} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \\ \mathbf{q} &= \langle 1258, 342, 2426, 1354\rangle.\end{aligned}$$

AAA sales for the month of May can be calculated using the dot product $\mathbf{p} \cdot \mathbf{q}$. We have

Equation:

$$\begin{aligned}\mathbf{p} \cdot \mathbf{q} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5.\end{aligned}$$

So, AAA took in \$16,267.50 during the month of May.

To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $\mathbf{c} \cdot \mathbf{q}$ to get

Equation:

$$\begin{aligned}\mathbf{c} \cdot \mathbf{q} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8.\end{aligned}$$

So, AAA paid \$1,883.80 for the items they sold. Their profit, then, is given by

Equation:

$$\begin{aligned}\mathbf{p} \cdot \mathbf{q} - \mathbf{c} \cdot \mathbf{q} &= 16267.5 - 1883.8 \\ &= 14383.7.\end{aligned}$$

Therefore, AAA Party Supply Store made \$14,383.70 in May.

Note:

Exercise:

Problem:

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to $2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Solution:

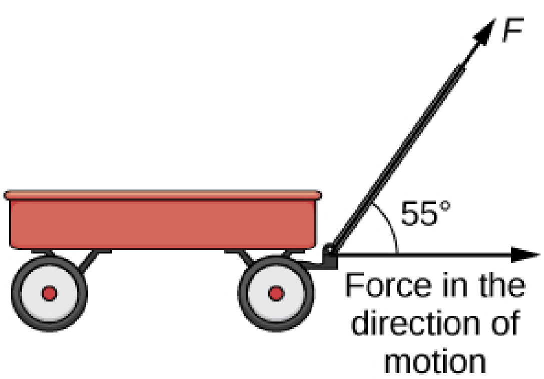
Sales = $15,685.50; profit = $14,073.15

Hint

Use four-dimensional vectors for cost, price, and quantity sold.

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward ([link]). We return to this example and learn how to solve it after we see how to calculate projections.



When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Note:
Definition
The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is the vector labeled $\text{proj}_{\mathbf{u}}\mathbf{v}$ in [link]. It has the same initial point as $\mathbf{u}$ and $\mathbf{v}$ and the same direction as $\mathbf{u}$, and represents the component of $\mathbf{v}$ that acts in the direction of $\mathbf{u}$. If θ represents the angle between $\mathbf{u}$ and $\mathbf{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\mathbf{u}}\mathbf{v}$ is $\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \|\mathbf{v}\|\cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes
Equation:

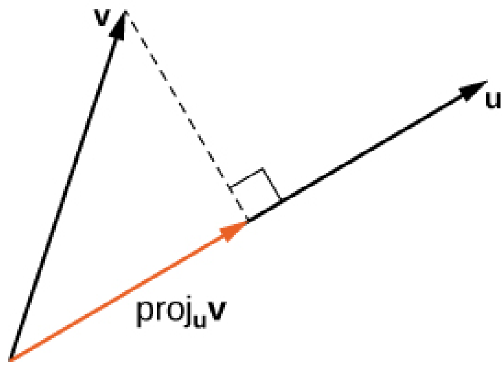
$$\begin{aligned} \|\text{proj}_{\mathbf{u}}\mathbf{v}\| &= \|\mathbf{v}\|\cos\theta \\ &= \|\mathbf{v}\|\left(\frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \\ &= \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|}. \end{aligned}$$

We now multiply by a unit vector in the direction of $\mathbf{u}$ to get $\text{proj}_{\mathbf{u}}\mathbf{v}$:
Equation:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

The length of this vector is also known as the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$ and is denoted by
Equation:

$$\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \text{comp}_{\mathbf{u}}\mathbf{v} = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|}.$$



The projection of **v** onto **u** shows the component of vector **v** in the direction of **u**.

Example:
Exercise:

Problem:
Finding Projections

Find the projection of **v** onto **u.**

a. $\mathbf{v} = \langle 3, 5, 1 \rangle$ and $\mathbf{u} = \langle -1, 4, 3 \rangle$
b. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$

Solution:

a. Substitute the components of **v** and **u** into the formula for the projection:
Equation:

$$\begin{aligned}\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle \\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle \\ &= \frac{20}{26}\langle -1,4,3\rangle \\ &= \left\langle -\frac{10}{13},\frac{40}{13},\frac{30}{13}\right\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:
Equation:

$$\begin{aligned}\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= \frac{(\mathbf{i}+6\mathbf{j})\cdot(3\mathbf{i}-2\mathbf{j})}{\|\mathbf{i}+6\mathbf{j}\|^2}(\mathbf{i}+6\mathbf{j}) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\mathbf{i}+6\mathbf{j}) \\ &= -\frac{9}{37}(\mathbf{i}+6\mathbf{j}) \\ &= -\frac{9}{37}\mathbf{i}-\frac{54}{37}\mathbf{j}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the *resolution of a vector into components*. Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\mathbf{v}=\langle 6,-4\rangle$ and let $\mathbf{u}=\langle 3,1\rangle$. We want to decompose the vector $\mathbf{v}$ into orthogonal components such that one of the component vectors has the same direction as $\mathbf{u}$.

We first find the component that has the same direction as $\mathbf{u}$ by projecting $\mathbf{v}$ onto $\mathbf{u}$. Let $\mathbf{p}=\text{proj}_{\mathbf{u}}\mathbf{v}$. Then, we have
Equation:

$$\begin{aligned}\mathbf{p} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= \frac{18-4}{9+1}\mathbf{u} \\ &= \frac{7}{5}\mathbf{u}=\frac{7}{5}\langle 3,1\rangle=\left\langle\frac{21}{5},\frac{7}{5}\right\rangle.\end{aligned}$$

Now consider the vector $\mathbf{q} = \mathbf{v} - \mathbf{p}$. We have
Equation:

$$\begin{aligned} \mathbf{q} &= \mathbf{v} - \mathbf{p} \\ &= \langle 6, -4 \rangle - \left\langle \frac{21}{5}, \frac{7}{5} \right\rangle \\ &= \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle. \end{aligned}$$

Clearly, by the way we defined $\mathbf{q}$, we have $\mathbf{v} = \mathbf{q} + \mathbf{p}$, and
Equation:

$$\begin{aligned} \mathbf{q} \cdot \mathbf{p} &= \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle \cdot \left\langle \frac{21}{5}, \frac{7}{5} \right\rangle \\ &= \frac{9(21)}{25} + \frac{-27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0. \end{aligned}$$

Therefore, $\mathbf{q}$ and $\mathbf{p}$ are orthogonal.

Example:
Exercise:

Problem:
Resolving Vectors into Components

Express $\mathbf{v} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = \langle 2, 3, 2 \rangle$.

Solution:

Let $\mathbf{p}$ represent the projection of $\mathbf{v}$ onto $\mathbf{u}$:
Equation:

$$
\begin{aligned}
\mathbf{p} &= \text{proj}_{\mathbf{u}}\mathbf{v} \\
&= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{\langle 2,3,2\rangle\cdot\langle 8,-3,-3\rangle}{\|\langle 2,3,2\rangle\|^2}\langle 2,3,2\rangle \\
&= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle \\
&= \frac{1}{17}\langle 2,3,2\rangle \\
&= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle.
\end{aligned}
$$

Then,

Equation:

$$\mathbf{q} = \mathbf{v} - \mathbf{p} = \langle 8, -3, -3\rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle = \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.$$

To check our work, we can use the dot product to verify that **p** and **q** are orthogonal vectors:

Equation:

$$\mathbf{p}\cdot\mathbf{q} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \cdot \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle = \frac{268}{17} - \frac{162}{17} - \frac{106}{17} = 0.$$

Then,

Equation:

$$\mathbf{v} = \mathbf{p} + \mathbf{q} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.$$

Note:

Exercise:

Problem:

Express $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$.

Solution:

$\mathbf{v} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p} = \frac{18}{5}\mathbf{i} + \frac{9}{5}\mathbf{j}$ and $\mathbf{q} = \frac{7}{5}\mathbf{i} - \frac{14}{5}\mathbf{j}$

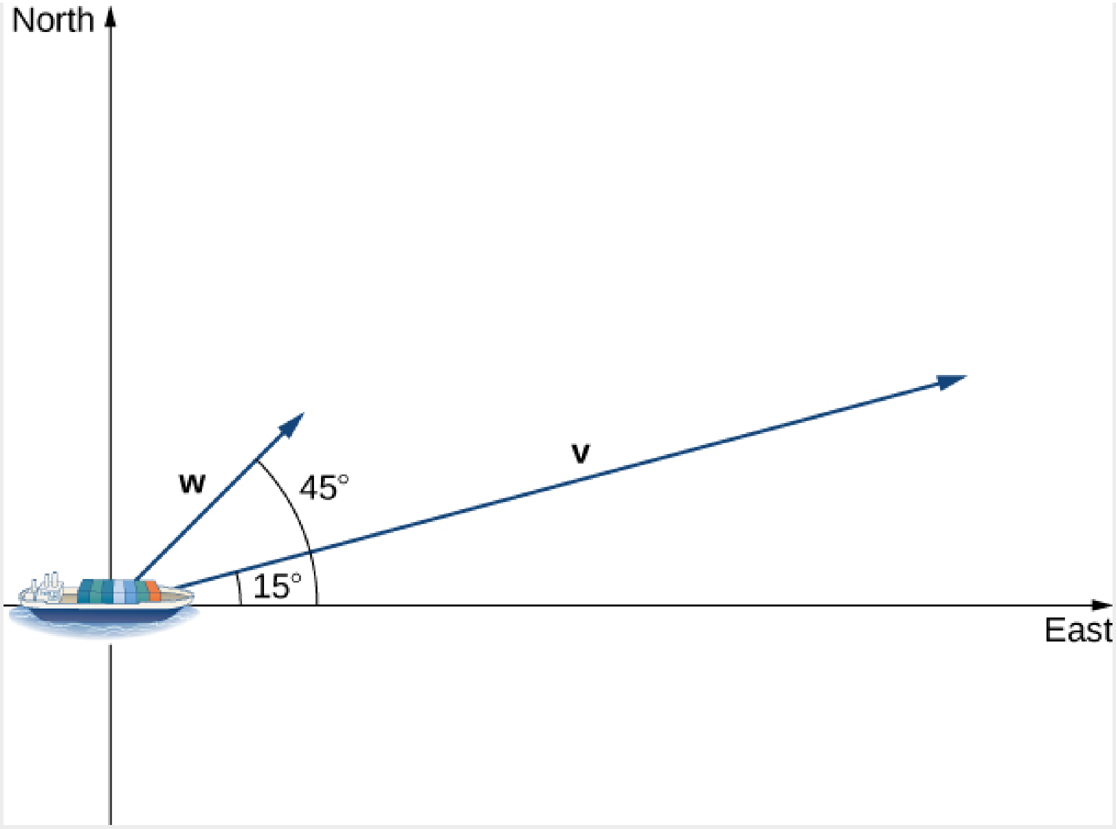
Hint

Start by finding the projection of **v** onto **u**.

Example:
Exercise:

Problem:
Scalar Projection of Velocity

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution:

Let **v** be the velocity vector generated by the engine, and let **w** be the velocity vector of the current. We already know $\|\mathbf{v}\| = 20$ along the desired route. We just need to add in the scalar projection of **w** onto **v**. We get

Equation:

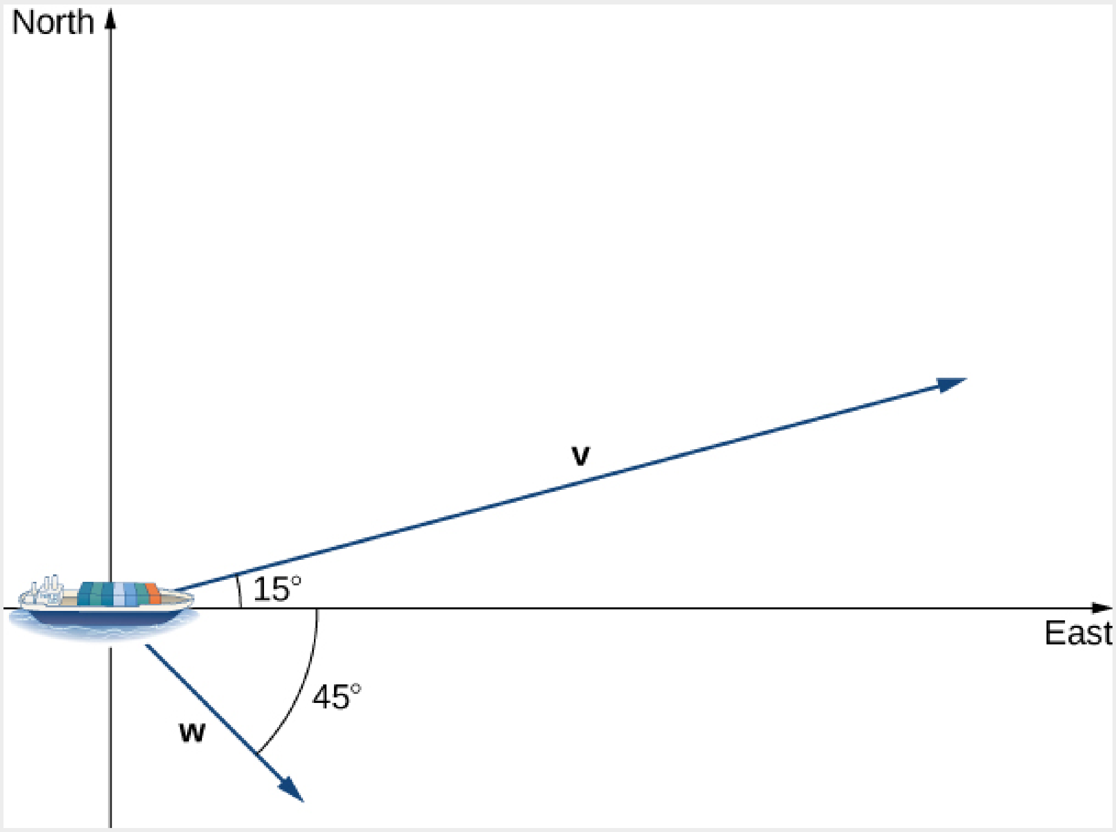
$$\begin{aligned} \text{comp}_{\mathbf{v}}\mathbf{w} &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|} \\ &= \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(30^\circ)}{\|\mathbf{v}\|} \\ &= \|\mathbf{w}\|\cos(30^\circ) \\ &= 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.} \end{aligned}$$

The ship is moving at 21.73 knots in the direction 15° north of east.

Note:
Exercise:

Problem:

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Solution:

21 knots

Hint

Compute the scalar projection of **w** onto **v**.

Work

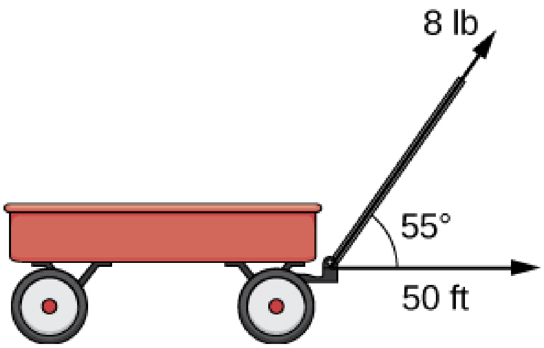
Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector **F** and the displacement of an object by a vector **s**, then the **work done by the force** is the dot product of **F** and **s**.

Note:
Definition
When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force **F**, acting at an angle θ from the line of motion, is given by
Equation:

$$W = \mathbf{F} \cdot \vec{PQ} = \|\mathbf{F}\| \, \vec{PQ} \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force ([link]).



The horizontal component of the force is the projection of **F** onto the positive *x*-axis.

We have

Equation:

$$W = \|\mathbf{F}\| \, \vec{PQ} \cos\theta = 8\,(50)\,(\cos(55^\circ)) \approx 229 \text{ ft} \cdot \text{lb}.$$

In U.S. standard units, we measure the magnitude of force $\|\mathbf{F}\|$ in pounds. The magnitude of the displacement vector $\vec{PQ}$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example:

Exercise:

Problem:

Calculating Work

A conveyor belt generates a force $\mathbf{F} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution:

The displacement vector $\vec{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

Equation:

$$\vec{PQ} = \langle 9 - 1, 4 - 1, 7 - 1 \rangle = \langle 8, 3, 6 \rangle = 8\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}.$$

Work is the dot product of force and displacement:

Equation:

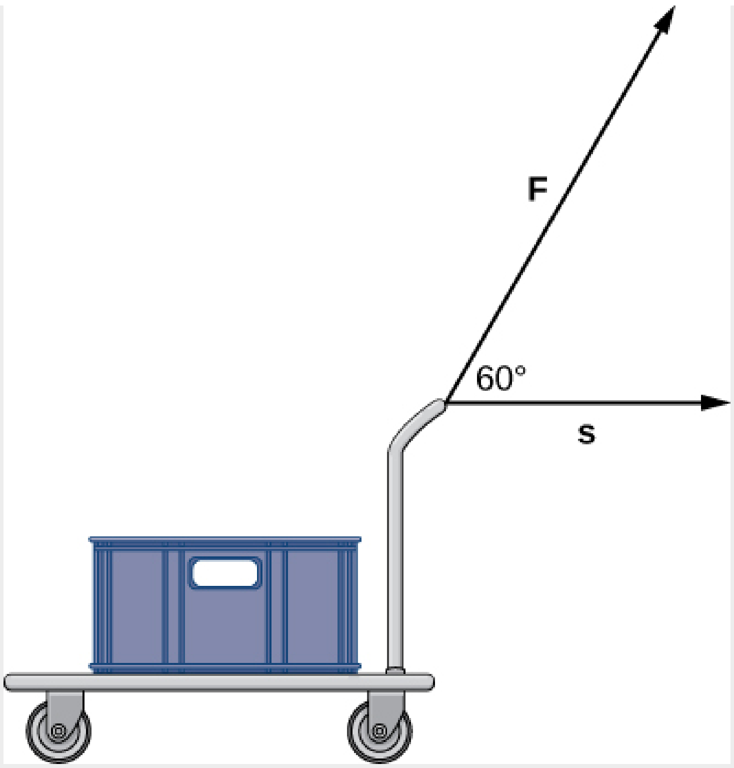
$$\begin{aligned} W &= \mathbf{F} \cdot \vec{PQ} \\ &= (5\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (8\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37 \mathrm{N} \cdot \mathrm{m} \\ &= 37 \, \mathrm{J}. \end{aligned}$$

Note:

Exercise:

Problem:

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground ([link]). What is the work done by this force?



Solution:

150 ft-lb

Hint

Use the definition of work as the dot product of force and distance.

Key Concepts

- The dot product, or scalar product, of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.
- The dot product satisfies the following properties:
    - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    - $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
    - $c\,(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
    - $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

- The dot product of two vectors can be expressed, alternatively, as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors **u** and **v** are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines*.
- The vector projection of **v** onto **u** is the vector $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$. The magnitude of this vector is known as the *scalar projection* of **v** onto **u**, given by $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector **F** and the displacement is represented by the vector **s**, then the work done *W* is given by the formula $W = \mathbf{F} \cdot s = \|\mathbf{F}\| \, \|\mathbf{s}\| \cos\theta$.

Key Equations

- **Dot product of u and v**
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \end{aligned}$$
- **Cosine of the angle formed by u and v**
$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$
- **Vector projection of v onto u**
$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$
- **Scalar projection of v onto u**
$\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}$
- **Work done by a force F to move an object through displacement vector $\vec{PQ}$**
$W = \mathbf{F} \cdot \vec{PQ} = \|\mathbf{F}\| \, \vec{PQ} \cos\theta$

For the following exercises, the vectors **u** and **v** are given. Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$.

Exercise:

Problem: $\mathbf{u} = \langle 3, 0 \rangle$, $\mathbf{v} = \langle 2, 2 \rangle$

Solution:

6

Exercise:

Problem: $\mathbf{u} = \langle 3, -4 \rangle$, $\mathbf{v} = \langle 4, 3 \rangle$

Exercise:

Problem: $\mathbf{u} = \langle 2, 2, -1 \rangle$, $\mathbf{v} = \langle -1, 2, 2 \rangle$

Solution:

0

Exercise:

Problem: $\mathbf{u} = \langle 4, 5, -6 \rangle$, $\mathbf{v} = \langle 0, -2, -3 \rangle$

For the following exercises, the vectors **a**, **b**, and **c** are given. Determine the vectors $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ and $(\mathbf{a} \cdot \mathbf{c})\mathbf{b}$. Express the vectors in component form.

Exercise:

Problem: $\mathbf{a} = \langle 2, 0, -3 \rangle$, $\mathbf{b} = \langle -4, -7, 1 \rangle$, $\mathbf{c} = \langle 1, 1, -1 \rangle$

Solution:

$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \langle -11, -11, 11 \rangle$; $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \langle -20, -35, 5 \rangle$

Exercise:

Problem: $\mathbf{a} = \langle 0, 1, 2 \rangle$, $\mathbf{b} = \langle -1, 0, 1 \rangle$, $\mathbf{c} = \langle 1, 0, -1 \rangle$

Exercise:

Problem: $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} - 2\mathbf{k}$

Solution:

$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \langle 1, 0, -2 \rangle$; $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \langle 1, 0, -1 \rangle$

Exercise:

Problem: $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

For the following exercises, the two-dimensional vectors **a** and **b** are given.

a. Find the measure of the angle θ between **a** and **b**. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.
b. Is θ an acute angle?

Exercise:

Problem: [T] $\mathbf{a} = \langle 3, -1 \rangle$, $\mathbf{b} = \langle -4, 0 \rangle$

Solution:

a. $\theta = 2.82$ rad; b. θ is not acute.

Exercise:

Problem: [T] $\mathbf{a} = \langle 2, 1 \rangle$, $\mathbf{b} = \langle -1, 3 \rangle$

Exercise:

Problem: $\mathbf{u} = 3\mathbf{i}$, $\mathbf{v} = 4\mathbf{i} + 4\mathbf{j}$

Solution:

a. $\theta = \frac{\pi}{4}$ rad; b. θ is acute.

Exercise:

Problem: $\mathbf{u} = 5\mathbf{i}$, $\mathbf{v} = -6\mathbf{i} + 6\mathbf{j}$

For the following exercises, find the measure of the angle between the three-dimensional vectors **a** and **b**. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

Exercise:

Problem: $\mathbf{a} = \langle 3, -1, 2 \rangle$, $\mathbf{b} = \langle 1, -1, -2 \rangle$

Solution:

$\theta = \frac{\pi}{2}$

Exercise:

Problem: $\mathbf{a} = \langle 0, -1, -3 \rangle$, $\mathbf{b} = \langle 2, 3, -1 \rangle$

Exercise:

Problem: $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} - \mathbf{k}$

Solution:

$\theta = \frac{\pi}{3}$

Exercise:

Problem: $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Exercise:

Problem:

[T] $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$

Solution:

$\theta = 2$ rad

Exercise:

Problem:

[T] $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{v} - \mathbf{w}$, where $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\mathbf{w} = 6\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

For the following exercises determine whether the given vectors are orthogonal.

Exercise:

Problem: $\mathbf{a} = \langle x, y \rangle$, $\mathbf{b} = \langle -y, x \rangle$, where *x* and *y* are nonzero real numbers

Solution:

Orthogonal

Exercise:

Problem: $\mathbf{a} = \langle x, x \rangle$, $\mathbf{b} = \langle -y, y \rangle$, where *x* and *y* are nonzero real numbers

Exercise:

Problem: $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Solution:

Not orthogonal

Exercise:

Problem: $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Exercise:

Problem:

Find all two-dimensional vectors **a** orthogonal to vector $\mathbf{b} = \langle 3, 4 \rangle$. Express the answer in component form.

Solution:

$\mathbf{a} = \left\langle -\frac{4\alpha}{3}, \alpha \right\rangle$, where $\alpha \neq 0$ is a real number

Exercise:

Problem:

Find all two-dimensional vectors **a** orthogonal to vector $\mathbf{b} = \langle 5, -6 \rangle$. Express the answer by using standard unit vectors.

Exercise:

Problem:

Determine all three-dimensional vectors **u** orthogonal to vector $\mathbf{v} = \langle 1, 1, 0 \rangle$. Express the answer by using standard unit vectors.

Solution:

$\mathbf{u} = -\alpha\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$, where α and β are real numbers such that $\alpha^2 + \beta^2 \neq 0$

Exercise:

Problem:

Determine all three-dimensional vectors $\mathbf{u}$ orthogonal to vector $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Express the answer in component form.

Exercise:

Problem:

Determine the real number α such that vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 9\mathbf{i} + \alpha\mathbf{j}$ are orthogonal.

Solution:

$\alpha = -6$

Exercise:

Problem:

Determine the real number α such that vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + \alpha\mathbf{j}$ are orthogonal.

Exercise:

Problem: **[T]** Consider the points $P(4, 5)$ and $Q(5, -7)$.

a. Determine vectors $\vec{OP}$ and $\vec{OQ}$. Express the answer by using standard unit vectors.
b. Determine the measure of angle *O* in triangle *OPQ*. Express the answer in degrees rounded to two decimal places.

Solution:

a. $\vec{OP} = 4\mathbf{i} + 5\mathbf{j}$, $\vec{OQ} = 5\mathbf{i} - 7\mathbf{j}$; b. 105.8°

Exercise:

Problem: **[T]** Consider points $A(1,1)$, $B(2,-7)$, and $C(6,3)$.

a. Determine vectors $\vec{BA}$ and $\vec{BC}$. Express the answer in component form.
b. Determine the measure of angle *B* in triangle *ABC*. Express the answer in degrees rounded to two decimal places.

Exercise:

Problem:

Determine the measure of angle *A* in triangle *ABC,* where $A(1,1,8)$, $B(4,-3,-4)$, and $C(-3,1,5)$. Express your answer in degrees rounded to two decimal places.

Solution:

68.33°

Exercise:

Problem:

Consider points $P(3,7,-2)$ and $Q(1,1,-3)$. Determine the angle between vectors $\vec{OP}$ and $\vec{OQ}$. Express the answer in degrees rounded to two decimal places.

For the following exercises, determine which (if any) pairs of the following vectors are orthogonal.

Exercise:

Problem: $\mathbf{u}=\langle 3,7,-2\rangle$, $\mathbf{v}=\langle 5,-3,-3\rangle$, $\mathbf{w}=\langle 0,1,-1\rangle$

Solution:

u and **v** are orthogonal; **v** and **w** are orthogonal.

Exercise:

Problem: $\mathbf{u}=\mathbf{i}-\mathbf{k}$, $\mathbf{v}=5\mathbf{j}-5\mathbf{k}$, $\mathbf{w}=10\mathbf{j}$

Exercise:

Problem:

Use vectors to show that a parallelogram with equal diagonals is a square.

Exercise:

Problem:

Use vectors to show that the diagonals of a rhombus are perpendicular.

Exercise:

Problem:

Show that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ is true for any vectors **u**, **v**, and **w**.

Exercise:

Problem:

Verify the identity $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ for vectors $\mathbf{u} = \langle 1, 0, 4 \rangle$, $\mathbf{v} = \langle -2, 3, 5 \rangle$, and $\mathbf{w} = \langle 4, -2, 6 \rangle$.

For the following problems, the vector **u** is given.

a. Find the direction cosines for the vector u.
b. Find the direction angles for the vector u expressed in degrees. (Round the answer to the nearest integer.)

Exercise:

Problem: $\mathbf{u} = \langle 2, 2, 1 \rangle$

Solution:

a. $\cos\alpha = \frac{2}{3}$, $\cos\beta = \frac{2}{3}$, and $\cos\gamma = \frac{1}{3}$; b. $\alpha = 48^\circ$, $\beta = 48^\circ$, and $\gamma = 71^\circ$

Exercise:

Problem: $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Exercise:

Problem: $\mathbf{u} = \langle -1, 5, 2 \rangle$

Solution:

a. $\cos\alpha = -\frac{1}{\sqrt{30}}$, $\cos\beta = \frac{5}{\sqrt{30}}$, and $\cos\gamma = \frac{2}{\sqrt{30}}$; b. $\alpha = 101^\circ$, $\beta = 24^\circ$, and $\gamma = 69^\circ$

Exercise:

Problem: $\mathbf{u} = \langle 2, 3, 4 \rangle$

Exercise:

Problem:

Consider $\mathbf{u} = \langle a, b, c \rangle$ a nonzero three-dimensional vector. Let $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ be the directions of the cosines of $\mathbf{u}$. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Exercise:

Problem:

Determine the direction cosines of vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and show they satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

For the following exercises, the vectors **u** and **v** are given.

a. Find the vector projection $\mathrm{w} = \mathrm{proj}_{\mathbf{u}}\mathbf{v}$ of vector **v** onto vector **u**. Express your answer in component form.
b. Find the scalar projection $\mathrm{comp}_{\mathrm{u}}\mathrm{v}$ of vector **v** onto vector **u.**

Exercise:

Problem: $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

Solution:

a. $\mathbf{w} = \left\langle \frac{80}{29}, \frac{32}{29} \right\rangle$; b. $\mathrm{comp}_{\mathrm{u}}\mathbf{v} = \frac{16}{\sqrt{29}}$

Exercise:

Problem: $\mathbf{u} = \langle -4, 7 \rangle$, $\mathbf{v} = \langle 3, 5 \rangle$

Exercise:

Problem: $\mathbf{u} = 3\mathbf{i} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$

Solution:

a. $\mathbf{w} = \left\langle \frac{24}{13}, 0, \frac{16}{13} \right\rangle$; b. $\text{comp}_{\mathrm{u}}\mathbf{v} = \frac{8}{\sqrt{13}}$

Exercise:

Problem: $\mathbf{u} = \langle 4, 4, 0 \rangle$, $\mathbf{v} = \langle 0, 4, 1 \rangle$

Exercise:

Problem: Consider the vectors $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$.

a. Find the component form of vector $\mathbf{w} = \text{proj}_{\mathbf{u}}\mathbf{v}$ that represents the projection of $\mathbf{v}$ onto $\mathbf{u}$.
b. Write the decomposition $\mathbf{v} = \mathbf{w} + \mathbf{q}$ of vector $\mathbf{v}$ into the orthogonal components $\mathbf{w}$ and $\mathbf{q}$, where $\mathbf{w}$ is the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{q}$ is a vector orthogonal to the direction of $\mathbf{u}$.

Solution:

a. $\mathbf{w} = \left\langle \frac{24}{25}, -\frac{18}{25} \right\rangle$; b. $\mathbf{q} = \left\langle \frac{51}{25}, \frac{68}{25} \right\rangle$,
$\mathbf{v} = \mathbf{w} + \mathbf{q} = \left\langle \frac{24}{25}, -\frac{18}{25} \right\rangle + \left\langle \frac{51}{25}, \frac{68}{25} \right\rangle$

Exercise:

Problem: Consider vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 4\mathbf{j} + 2\mathbf{k}$.

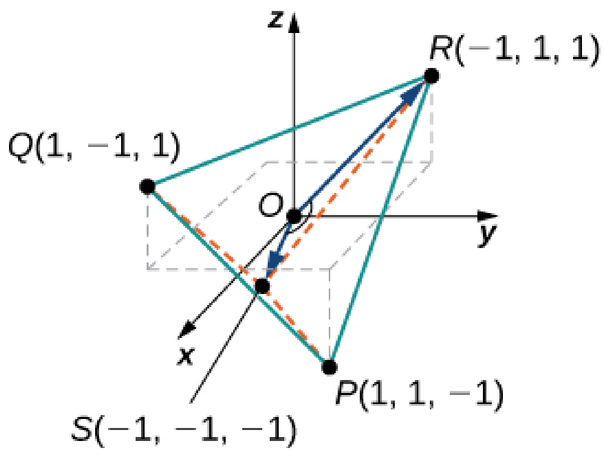
a. Find the component form of vector $\mathbf{w} = \text{proj}_{\mathrm{u}}\mathbf{v}$ that represents the projection of $\mathbf{v}$ onto $\mathbf{u}$.
b. Write the decomposition $\mathbf{v} = \mathbf{w} + \mathbf{q}$ of vector $\mathbf{v}$ into the orthogonal components $\mathbf{w}$ and $\mathbf{q}$, where $\mathbf{w}$ is the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{q}$ is a vector orthogonal to the direction of $\mathbf{u}$.

Exercise:

Problem:

A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P\,(1,1,-1), Q\,(1,-1,1), R\,(-1,1,1),$ and $S\,(-1,-1,-1)$ (see figure).

a. Find the distance between the hydrogen atoms located at *P* and *R*.
b. Find the angle between vectors $\vec{OS}$ and $\vec{OR}$ that connect the carbon atom with the hydrogen atoms located at *S* and *R*, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.



Solution:

a. $2\sqrt{2}$; b. 109.47°

Exercise:

Problem:

[T] Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.

Exercise:

Problem:

Find the work done by force $\mathbf{F} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

Solution:

$17\text{N} \cdot \text{m}$

Exercise:

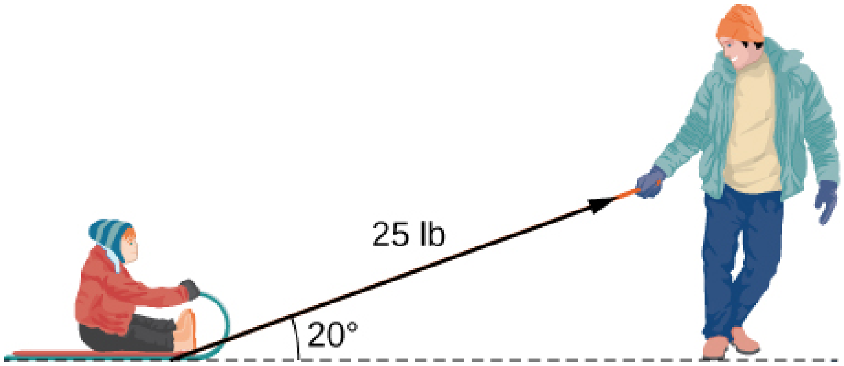
Problem:

[T] A sled is pulled by exerting a force of 100 N on a rope that makes an angle of 25° with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)

Exercise:

Problem:

[T] A father is pulling his son on a sled at an angle of 20° with the horizontal with a force of 25 lb (see the following image). He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)



Solution:

$1175 \text{ ft} \cdot \text{lb}$

Exercise:

Problem:

[T] A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules $(1\text{J} = 1\text{N} \cdot \text{m})$ rounded to the nearest integer.

Exercise:

Problem:

[T] A boat sails north aided by a wind blowing in a direction of $\text{N}30^\circ\text{E}$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

Solution:

4330.13 ft-lb

Exercise:

Problem:

Vector $\mathbf{p} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\mathbf{n} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\mathbf{p} \cdot \mathbf{n}$ and state its meaning.

Exercise:

Problem:

[T] Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and the terminal point of the vector is point $P(1, 1, 0)$. The second force has a magnitude of 40 lb and the terminal point of its vector is point $Q(0, 1, 1)$. Let $\mathbf{F}$ be the resultant force of forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

a. Find the magnitude of $\mathbf{F}$. (Round the answer to one decimal place.)
b. Find the direction angles of $\mathbf{F}$. (Express the answer in degrees rounded to one decimal place.)

Solution:

a. $\|\mathbf{F}_1 + \mathbf{F}_2\| = 52.9$ lb; b. The direction angles are $\alpha = 74.5^\circ$, $\beta = 36.7^\circ$, and $\gamma = 57.7^\circ$.

Exercise:

Problem:

[T] Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of **r** are expressed in centimeters and time in seconds. Let $\vec{OP}$ be the position vector of the particle after 1 sec.

a. Show that all vectors $\vec{PQ}$, where $Q(x, y, z)$ is an arbitrary point, orthogonal to the instantaneous velocity vector $\mathbf{v}(1)$ of the particle after 1 sec, can be expressed as $\vec{PQ} = \langle x - \cos 1, y - \sin 1, z - 2 \rangle$, where $x \sin 1 - y \cos 1 - 2z + 4 = 0$. The set of point Q describes a plane called the *normal plane* to the path of the particle at point *P*.
b. Use a CAS to visualize the instantaneous velocity vector and the normal plane at point *P* along with the path of the particle.

Glossary

direction angles
: the angles formed by a nonzero vector and the coordinate axes

direction cosines
: the cosines of the angles formed by a nonzero vector and the coordinate axes

dot product or scalar product
: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

scalar projection
: the magnitude of the vector projection of a vector

orthogonal vectors
: vectors that form a right angle when placed in standard position

vector projection
: the component of a vector that follows a given direction

work done by a force

work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector **F** and the displacement of an object by a vector **s**, then the work done by the force is the dot product of **F** and **s**.

The Cross Product

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the *cross product*, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\mathbf{u}$ and $\mathbf{v}$—that is, we want to find $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$. Therefore, w_1, w_2, and w_3 must satisfy
Equation:

$$\begin{aligned} u_1w_1 + u_2w_2 + u_3w_3 &= 0 \\ v_1w_1 + v_2w_2 + v_3w_3 &= 0. \end{aligned}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives
Equation:

$$(u_1v_3 - v_1u_3)w_1 + (u_2v_3 - v_2u_3)w_2 = 0.$$

If we select
Equation:

$$\begin{aligned} w_1 &= u_2v_3 - u_3v_2 \\ w_2 &= -(u_1v_3 - u_3v_1), \end{aligned}$$

we get a possible solution vector. Substituting these values back into the original equations gives
Equation:

$$w_3 = u_1v_2 - u_2v_1.$$

That is, vector
Equation:

$$\mathbf{w} = \langle u_2v_3 - u_3v_2, -\left(u_1v_3 - u_3v_1\right), u_1v_2 - u_2v_1\rangle$$

is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, which leads us to define the following operation, called the cross product.

Note:
Definition
Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then, the **cross product $\mathbf{u} \times \mathbf{v}$** is vector
Equation:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\ &= \langle u_2v_3 - u_3v_2, -\left(u_1v_3 - u_3v_1\right), u_1v_2 - u_2v_1\rangle.\end{aligned}$$

From the way we have developed $\mathbf{u} \times \mathbf{v}$, it should be clear that the cross product is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. However, it never hurts to check. To show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$, we calculate the dot product of $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$.
Equation:

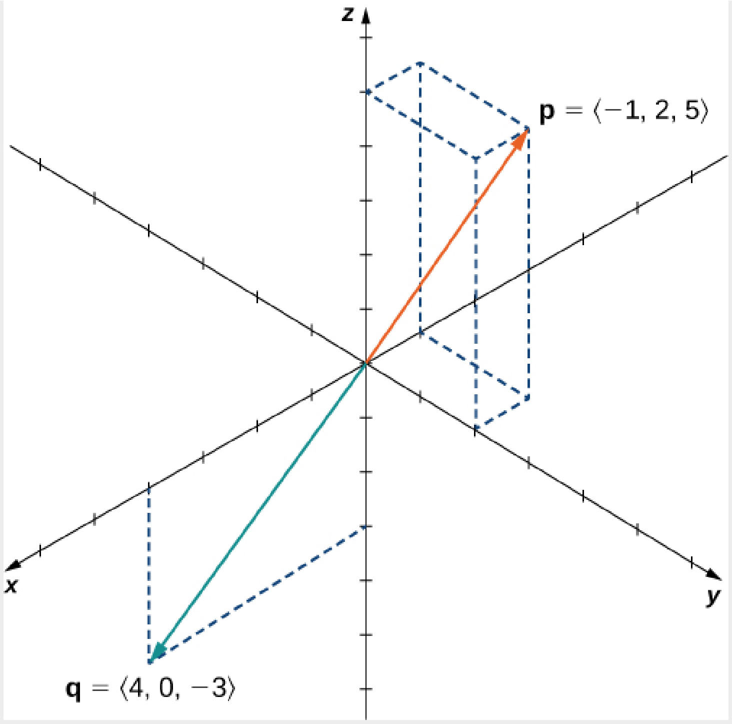
$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= \langle u_1, u_2, u_3\rangle \cdot \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1\rangle \\ &= u_1\left(u_2v_3 - u_3v_2\right) + u_2\left(-u_1v_3 + u_3v_1\right) + u_3\left(u_1v_2 - u_2v_1\right) \\ &= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_2u_3v_1 \\ &= \left(u_1u_2v_3 - u_1u_2v_3\right) + \left(-u_1u_3v_2 + u_1u_3v_2\right) + \left(u_2u_3v_1 - u_2u_3v_1\right) \\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\mathbf{v}$.

Example:
Exercise:

Problem:
Finding a Cross Product

Let $\mathbf{p} = \langle -1, 2, 5\rangle$ and $\mathbf{q} = \langle 4, 0, -3\rangle$ ([link]). Find $\mathbf{p} \times \mathbf{q}$.



Finding a cross product to two given vectors.

Solution:

Substitute the components of the vectors into [link]:
Equation:

$$\begin{aligned}\mathbf{p} \times \mathbf{q} &= \langle -1,2,5\rangle \times \langle 4,0,-3\rangle \\ &= \langle p_2q_3 - p_3q_2, p_1q_3 - p_3q_1, p_1q_2 - p_2q_1\rangle \\ &= \langle 2\,(-3) - 5\,(0), -(-1)\,(-3) + 5\,(4), (-1)\,(0) - 2\,(4)\rangle \\ &= \langle -6,17,-8\rangle.\end{aligned}$$

Note:
Exercise:

Problem:

Find $\mathbf{p} \times \mathbf{q}$ for $\mathbf{p} = \langle 5,1,2\rangle$ and $\mathbf{q} = \langle -2,0,1\rangle$. Express the answer using standard unit vectors.

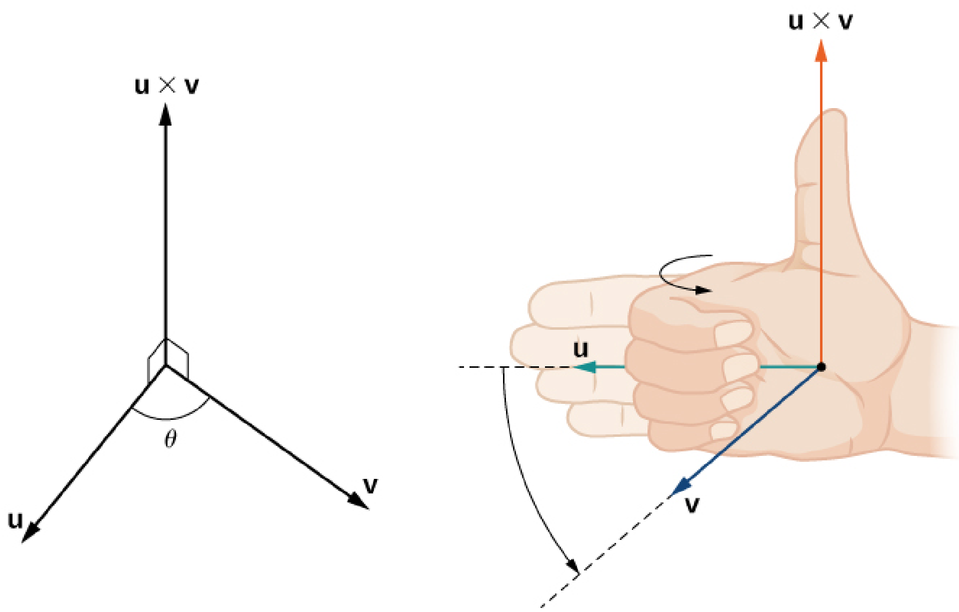
Solution:

$\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$

Hint

Use the formula $\mathbf{u}\ \times\ \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.

Although it may not be obvious from [link], the direction of $\mathbf{u}\ \times\ \mathbf{v}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\mathbf{u}$, then curl the fingers toward vector $\mathbf{v}$, the thumb points in the direction of the cross product, as shown.



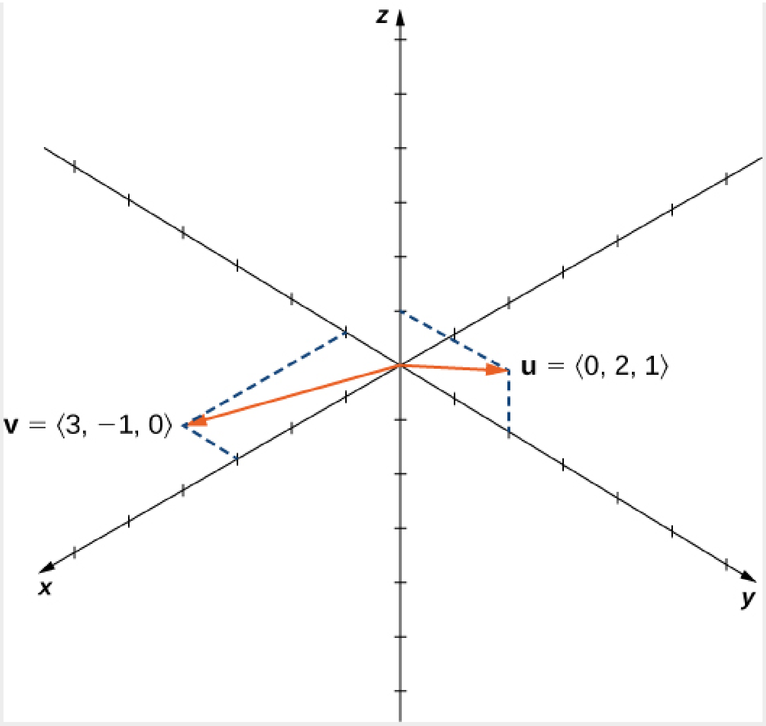
The direction of $\mathbf{u}\ \times\ \mathbf{v}$ is determined by the right-hand rule.

Notice what this means for the direction of $\mathbf{v}\ \times\ \mathbf{u}$. If we apply the right-hand rule to $\mathbf{v}\ \times\ \mathbf{u}$, we start with our fingers pointed in the direction of $\mathbf{v}$, then curl our fingers toward the vector $\mathbf{u}$. In this case, the thumb points in the opposite direction of $\mathbf{u}\ \times\ \mathbf{v}$. (Try it!)

Example:
Exercise:

Problem:
Anticommutativity of the Cross Product

Let $\mathbf{u} = \langle 0, 2, 1\rangle$ and $\mathbf{v} = \langle 3, -1, 0\rangle$. Calculate $\mathbf{u}\ \times\ \mathbf{v}$ and $\mathbf{v}\ \times\ \mathbf{u}$ and graph them.



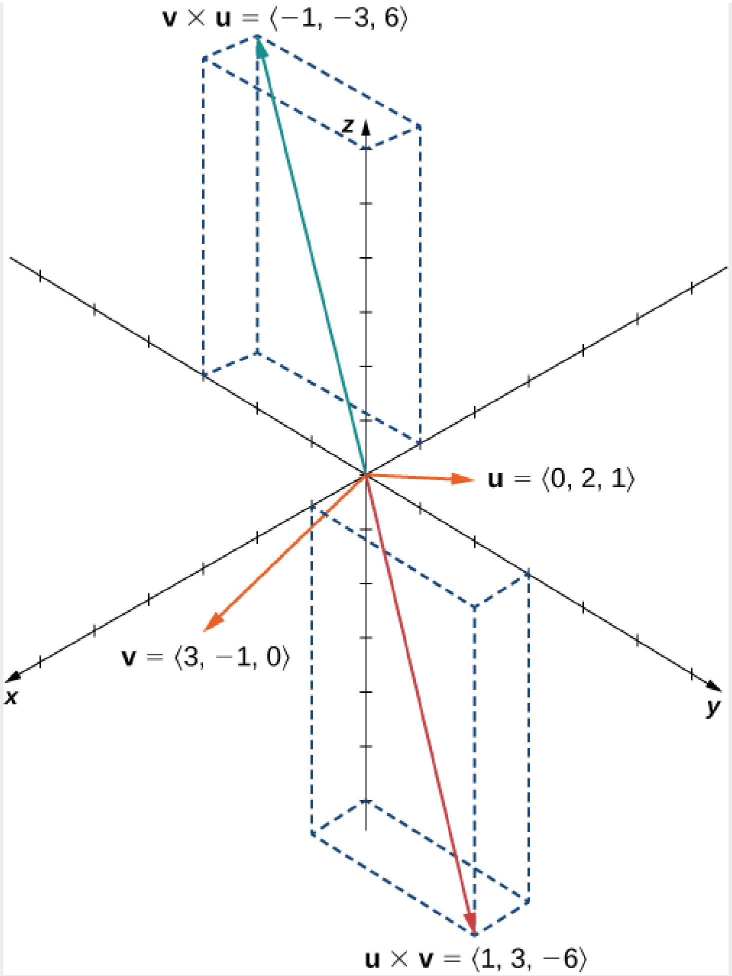
Are the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ in the same direction?

Solution:

We have
Equation:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle \\ \mathbf{v} \times \mathbf{u} &= \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle. \end{aligned}$$

We see that, in this case, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ ([link]). We prove this in general later in this section.



The cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are both orthogonal to $\mathbf{u}$ and $\mathbf{v}$, but in opposite directions.

Note:
Exercise:

Problem:

Suppose vectors $\mathbf{u}$ and $\mathbf{v}$ lie in the *xy*-plane (the *z*-component of each vector is zero). Now suppose the *x*- and *y*-components of $\mathbf{u}$ and the *y*-component of $\mathbf{v}$ are all positive, whereas the *x*-component of $\mathbf{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\mathbf{u} \times \mathbf{v}$ point?

Solution:

Up (the positive *z*-direction)

Hint

Remember the right-hand rule.

The cross products of the standard unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that
Equation:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar $0.$) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\mathbf{i}$ and $\mathbf{j}$ is parallel to $\mathbf{k}$. Similarly, the vector product of $\mathbf{i}$ and $\mathbf{k}$ is parallel to $\mathbf{j}$, and the vector product of $\mathbf{j}$ and $\mathbf{k}$ is parallel to $\mathbf{i}$. We can use the right-hand rule to determine the direction of each product. Then we have
Equation:

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

These formulas come in handy later.

Example:
Exercise:

Problem:
Cross Product of Standard Unit Vectors

Find $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$.

Solution:

We know that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$. Therefore, $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i} = \mathbf{0}$.

Note:
Exercise:

Problem: Find $(\mathbf{i} \times \mathbf{j}) \times (\mathbf{k} \times \mathbf{i})$.

Solution:

$-\mathbf{i}$

Hint

Remember the right-hand rule.

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product**. These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Note:
Properties of the Cross Product
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.
Equation:

$$\begin{array}{llcll} \text{i.} & \mathbf{u}\times\mathbf{v} & = & -(\mathbf{v}\times\mathbf{u}) & \text{Anticommutative property} \\ \text{ii.} & \mathbf{u}\times(\mathbf{v}+\mathbf{w}) & = & \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w} & \text{Distributive property} \\ \text{iii.} & c(\mathbf{u}\times\mathbf{v}) & = & (c\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\mathbf{v}) & \text{Multiplication by a constant} \\ \text{iv.} & \mathbf{u}\times\mathbf{0} & = & \mathbf{0}\times\mathbf{u}=\mathbf{0} & \text{Cross product of the zero vector} \\ \text{v.} & \mathbf{v}\times\mathbf{v} & = & \mathbf{0} & \text{Cross product of a vector with itself} \\ \text{vi.} & \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) & = & (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} & \text{Scalar triple product} \end{array}$$

Proof

For property i ., we want to show $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$. We have
Equation:

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \langle u_1,u_2,u_3\rangle \times \langle v_1,v_2,v_3\rangle \\ &= \langle u_2v_3-u_3v_2,-u_1v_3+u_3v_1,u_1v_2-u_2v_1\rangle \\ &= -\langle u_3v_2-u_2v_3,-u_3v_1+u_1v_3,u_2v_1-u_1v_2\rangle \\ &= -\langle v_1,v_2,v_3\rangle \times \langle u_1,u_2,u_3\rangle \\ &= -(\mathbf{v}\times\mathbf{u}). \end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property iv ., this follows directly from the definition of the cross product. We have
Equation:

$$\begin{aligned} \mathbf{u}\times\mathbf{0} &= \langle u_2(0)-u_3(0),-(u_2(0)-u_3(0)),u_1(0)-u_2(0)\rangle \\ &= \langle 0,0,0\rangle=\mathbf{0}. \end{aligned}$$

Then, by property i., $\mathbf{0}\times\mathbf{u}=\mathbf{0}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\mathbf{0}$.

Property vi. looks like the associative property, but note the change in operations:
Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{u}\cdot\langle v_2w_3-v_3w_2,-v_1w_3+v_3w_1,v_1w_2-v_2w_1\rangle\\ &= u_1\left(v_2w_3-v_3w_2\right)+u_2\left(-v_1w_3+v_3w_1\right)+u_3\left(v_1w_2-v_2w_1\right)\\ &= u_1v_2w_3-u_1v_3w_2-u_2v_1w_3+u_2v_3w_1+u_3v_1w_2-u_3v_2w_1\\ &= (u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3\\ &= \langle u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1\rangle\cdot\langle w_1,w_2,w_3\rangle\\ &= (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}.\end{aligned}$$

□

Example:
Exercise:

Problem:
Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\mathbf{i}\times 3\mathbf{j})\times\mathbf{j}$.

Solution:
Equation:

$$\begin{aligned}(2\mathbf{i}\times 3\mathbf{j})\times\mathbf{j} &= 2\left(\mathbf{i}\times 3\mathbf{j}\right)\times\mathbf{j}\\ &= 2\left(3\right)\left(\mathbf{i}\times\mathbf{j}\right)\times\mathbf{j}\\ &= \left(6\mathbf{k}\right)\times\mathbf{j}\\ &= 6\left(\mathbf{k}\times\mathbf{j}\right)\\ &= 6\left(-\mathbf{i}\right)=-6\mathbf{i}.\end{aligned}$$

Note:
Exercise:

Problem: Use the properties of the cross product to calculate $(\mathbf{i}\times\mathbf{k})\times(\mathbf{k}\times\mathbf{j})$.

Solution:

$-\mathbf{k}$

Hint

$\mathbf{u}\times\mathbf{v}=-\left(\mathbf{v}\times\mathbf{u}\right)$

So far in this section, we have been concerned with the direction of the vector $\mathbf{u} \times \mathbf{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\mathbf{u} \times \mathbf{v}$ involving the magnitudes of $\mathbf{u}$ and $\mathbf{v}$, and the sine of the angle between them.

Note:
Magnitude of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then, $\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \cdot \| \mathbf{v} \| \cdot \sin \theta$.

Proof

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then
Equation:

$$\begin{aligned}
\| \mathbf{u} \times \mathbf{v} \|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\
&\quad - \left(u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3\right) \\
&= \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\
&= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \cos^2\theta \\
&= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \left(1 - \cos^2\theta\right) \\
&= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \left(\sin^2\theta\right).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \le \theta \le 180^\circ$, we have the desired result:
Equation:

$$\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \ \| \mathbf{v} \| \sin \theta.$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example:
Exercise:

Problem:
Calculating the Cross Product

Use [link] to find the magnitude of the cross product of $\mathbf{u} = \langle 0, 4, 0 \rangle$ and $\mathbf{v} = \langle 0, 0, -3 \rangle$.

Solution:

We have
Equation:

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\| &= \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta \\ &= \sqrt{0^2 + 4^2 + 0^2} \cdot \sqrt{0^2 + 0^2 + (-3)^2} \cdot \sin\tfrac{\pi}{2} \\ &= 4(3)(1) = 12. \end{aligned}$$

Note:
Exercise:

Problem: Use [link] to find the magnitude of $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle -8, 0, 0 \rangle$ and $\mathbf{v} = \langle 0, 2, 0 \rangle$.

Solution:

16

Hint

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Determinants and the Cross Product

Using [link] to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using **determinant** notation.

A 2×2 determinant is defined by
Equation:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2.$$

For example,
Equation:

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3\,(1) - 5\,(-2) = 3 + 10 = 13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:
Equation:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

[link] is referred to as the *expansion of the determinant along the first row*. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example:
Exercise:

Problem:
Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution:

We have
Equation:

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2\,(4 - 9) - 5\,(-4 + 6) - 1\,(-3 + 2) \\ &= 2\,(-5) - 5\,(2) - 1\,(-1) = -10 - 10 + 1 \\ &= -19. \end{aligned}$$

Note:
Exercise:

Problem: Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Solution:

40

Hint

Expand along the first row. Don’t forget the second term is negative!

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Note:
Rule: Cross Product Calculated by a Determinant
Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by
Equation:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Example:
Exercise:

Problem:
Using Determinant Notation to find $\mathbf{p} \times \mathbf{q}$

Let $\mathbf{p} = \langle -1, 2, 5 \rangle$ and $\mathbf{q} = \langle 4, 0, -3 \rangle$. Find $\mathbf{p} \times \mathbf{q}$.

Solution:

We set up our determinant by putting the standard unit vectors across the first row, the components of $\mathbf{u}$ in the second row, and the components of $\mathbf{v}$ in the third row. Then, we have
Equation:

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \mathbf{k} \\ &= (-6 - 0)\mathbf{i} - (3 - 20)\mathbf{j} + (0 - 8)\mathbf{k} \\ &= -6\mathbf{i} + 17\mathbf{j} - 8\mathbf{k}. \end{aligned}$$

Notice that this answer confirms the calculation of the cross product in [link].

Note:
Exercise:

Problem: Use determinant notation to find $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 8, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 0, 4\rangle$.

Solution:

$8\mathbf{i} - 35\mathbf{j} + 2\mathbf{k}$

Hint

Calculate the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example:
Exercise:

Problem:
Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\mathbf{a} = \langle 5, 2, -1\rangle$ and $\mathbf{b} = \langle 0, -1, 4\rangle$. Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution:

The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$. We can calculate it with a determinant:
Equation:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{k} \\ &= (8-1)\mathbf{i} - (20-0)\mathbf{j} + (-5-0)\mathbf{k} \\ &= 7\mathbf{i} - 20\mathbf{j} - 5\mathbf{k}. \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:
Equation:

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

Note:
Exercise:

Problem: Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle 1, 1, 4 \rangle$.

Solution:

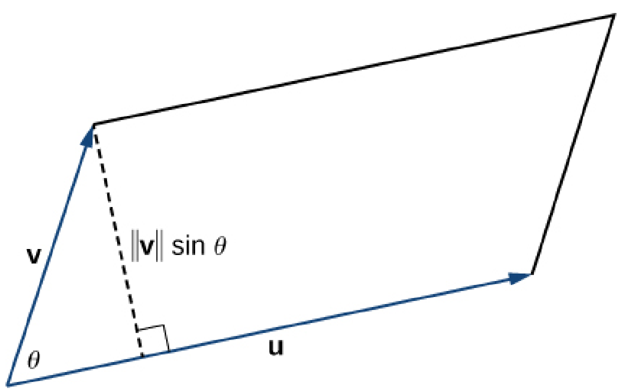
$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$

Hint

Normalize the cross product.

To use the cross product for calculating areas, we state and prove the following theorem.

Note:
Area of a Parallelogram
If we locate vectors $\mathbf{u}$ and $\mathbf{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$ ([link]).



The parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\| \sin \theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.
Equation:

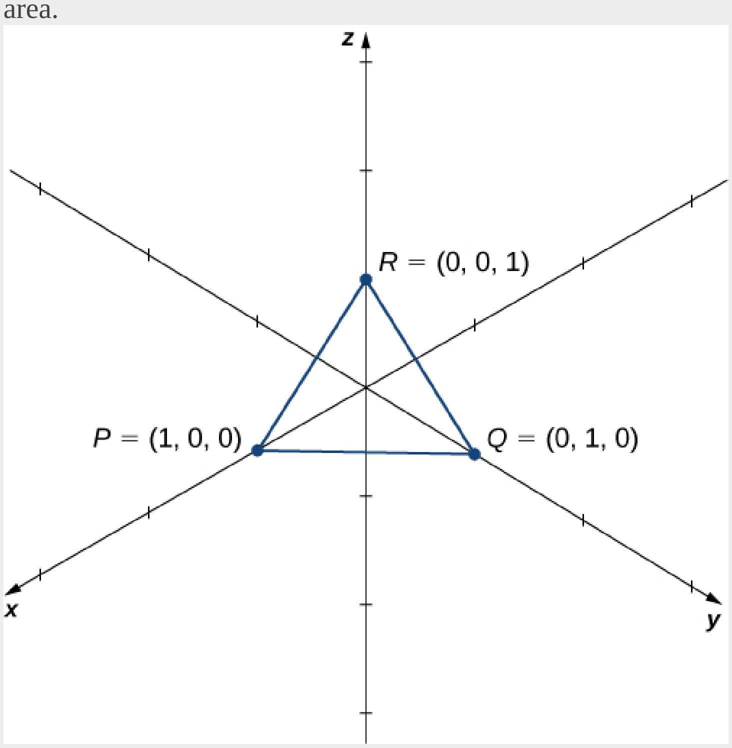
$$\begin{aligned}\text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\mathbf{u}\| \left(\|\mathbf{v}\| \sin \theta\right) \\ &= \|\mathbf{u} \times \mathbf{v}\|\end{aligned}$$

□

Example:
Exercise:

Problem:
Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0),$ and $R = (0, 0, 1)$ be the vertices of a triangle ([link]). Find its area.



Finding the area of a triangle by using the cross product.

Solution:

We have $\vec{PQ} = \langle 0-1, 1-0, 0-0 \rangle = \langle -1, 1, 0 \rangle$ and $\vec{PR} = \langle 0-1, 0-0, 1-0 \rangle = \langle -1, 0, 1 \rangle$.
The area of the parallelogram with adjacent sides $\vec{PQ}$ and $\vec{PR}$ is given by $\vec{PQ} \times \vec{PR}$:
Equation:

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1-0)\mathbf{i} - (-1-0)\mathbf{j} + (0-(-1))\mathbf{k} = \mathbf{i}+\mathbf{j}+\mathbf{k}$$

$$\vec{PQ} \times \vec{PR} = \| \langle 1,1,1 \rangle \| = \sqrt{1^2+1^2+1^2} = \sqrt{3}.$$

The area of ΔPQR is half the area of the parallelogram, or $\sqrt{3}/2$.

Note:
Exercise:

Problem:

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

Solution:

$6\sqrt{13}$

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the triple scalar product because the result is a scalar.

Note:
Definition
The **triple scalar product** of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Note:
Calculating a Triple Scalar Product
The triple scalar product of vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ is the determinant of the 3×3 matrix formed by the components of the vectors:
Equation:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Proof

The calculation is straightforward.

Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \langle u_1,u_2,u_3\rangle\cdot\langle v_2w_3-v_3w_2,-v_1w_3+v_3w_1,v_1w_2-v_2w_1\rangle\\ &= u_1\left(v_2w_3-v_3w_2\right)+u_2\left(-v_1w_3+v_3w_1\right)+u_3\left(v_1w_2-v_2w_1\right)\\ &= u_1\left(v_2w_3-v_3w_2\right)-u_2\left(v_1w_3-v_3w_1\right)+u_3\left(v_1w_2-v_2w_1\right)\\ &= \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\end{aligned}$$

□

Example:

Exercise:

Problem:

Calculating the Triple Scalar Product

Let $\mathbf{u}=\langle 1,3,5\rangle$, $\mathbf{v}=\langle 2,-1,0\rangle$ and $\mathbf{w}=\langle -3,0,-1\rangle$. Calculate the triple scalar product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

Solution:

Apply [link] directly:

Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}1 & 3 & 5\\ 2 & -1 & 0\\ -3 & 0 & -1\end{vmatrix}\\ &= 1\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}-3\begin{vmatrix}2 & 0\\ -3 & -1\end{vmatrix}+5\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix}\\ &= (1-0)-3\left(-2-0\right)+5\left(0-3\right)\\ &= 1+6-15=-8.\end{aligned}$$

Note:

Exercise:

Problem:

Calculate the triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a} = \langle 2, -4, 1 \rangle$, $\mathbf{b} = \langle 0, 3, -1 \rangle$, and $\mathbf{c} = \langle 5, -3, 3 \rangle$.

Solution:

17

Hint

Place the vectors as the rows of a 3×3 matrix, then calculate the determinant.

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

Equation:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = d \qquad \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -d.$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

Equation:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ 4 & 1 & -1 \end{vmatrix} &= \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 4 & 1 \end{vmatrix} \\ &= (0-3) - 2\,(2-12) + (-2-0) = -3 + 20 - 2 = 15. \end{aligned}$$

Switching the top two rows we have

Equation:

$$\begin{vmatrix} -2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = -2\,(-2-1) + 3\,(1-8) = 6 - 21 = -15.$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Applying [link], we have

Equation:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \text{and} \quad \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

We can obtain the determinant for calculating $\mathbf{u}\cdot(\mathbf{w}\ \times\ \mathbf{v})$ by switching the bottom two rows of $\mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w})$. Therefore, $\mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w})=-\mathbf{u}\cdot(\mathbf{w}\ \times\ \mathbf{v})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:
Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w}) &= -\mathbf{u}\cdot(\mathbf{w}\ \times\ \mathbf{v}) \\ \mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w}) &= \mathbf{v}\cdot(\mathbf{w}\ \times\ \mathbf{u})=\mathbf{w}\cdot(\mathbf{u}\ \times\ \mathbf{v}).\end{aligned}$$

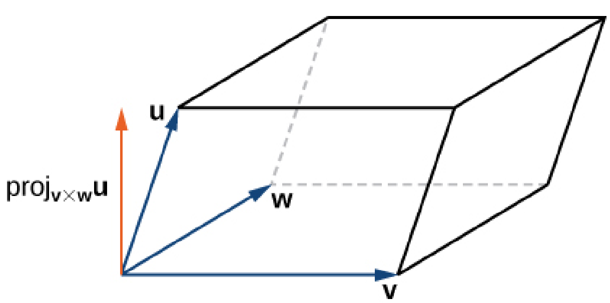
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in standard position. If $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in [link] that the area of this parallelogram is $\|\mathbf{u}\ \times\ \mathbf{v}\|$. Now suppose we add a third vector $\mathbf{w}$ that does not lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in [link]. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Note:
Volume of a Parallelepiped
The volume of a parallelepiped with adjacent edges given by the vectors $\mathbf{u},\mathbf{v},$ and $\mathbf{w}$ is the absolute value of the triple scalar product:
Equation:

$$V=|\mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w})|.$$

See [link].

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.



The height of the parallelepiped is given by $\|\text{proj}_{\text{v}\times\text{w}}\mathbf{u}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\mathbf{v}\ \times\ \mathbf{w}\|$. The height of the figure is given by $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

Equation:

$$\begin{aligned} V &= \|\text{proj}_{\mathbf{v}\ \times\ \mathbf{w}}\mathbf{u}\|\ \|\mathbf{v}\ \times\ \mathbf{w}\| \\ &= \left|\frac{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{v}\times\mathbf{w}\|}\right|\ \|\mathbf{v}\ \times\ \mathbf{w}\| \\ &= |\mathbf{u}\cdot(\mathbf{v}\ \times\ \mathbf{w})|. \end{aligned}$$

□

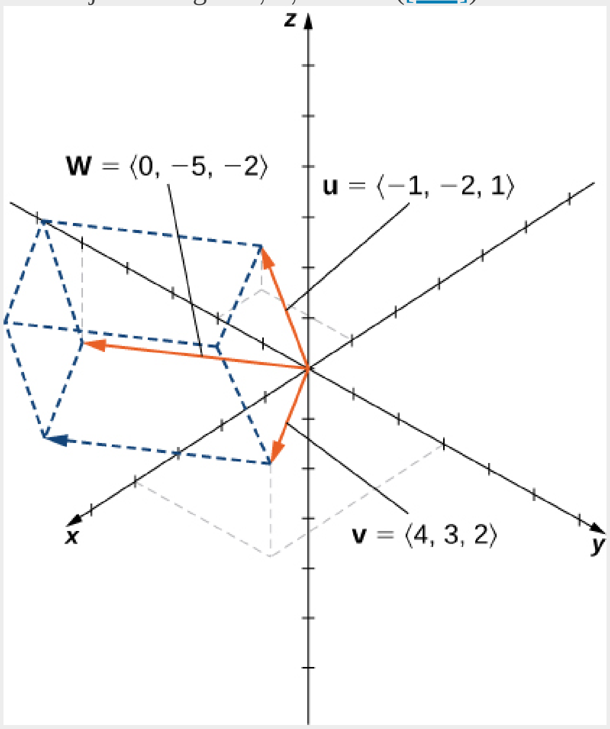
Example:

Exercise:

Problem:

Calculating the Volume of a Parallelepiped

Let $\mathbf{u}=\langle -1,-2,1\rangle, \mathbf{v}=\langle 4,3,2\rangle,$ and $\mathbf{w}=\langle 0,-5,-2\rangle$. Find the volume of the parallelepiped with adjacent edges $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ ([link]).



Solution:

We have
Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}-1 & -2 & 1\\ 4 & 3 & 2\\ 0 & -5 & -2\end{vmatrix} = (-1)\begin{vmatrix}3 & 2\\ -5 & -2\end{vmatrix} + 2\begin{vmatrix}4 & 2\\ 0 & -2\end{vmatrix} + \begin{vmatrix}4 & 3\\ 0 & -5\end{vmatrix}\\ &= (-1)(-6+10) + 2(-8-0) + (-20-0)\\ &= -4-16-20\\ &= -40.\end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3.

Note:
Exercise:

Problem:

Find the volume of the parallelepiped formed by the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{c} = 3\mathbf{j} + \mathbf{k}$.

Solution:

8 units^3

Hint

Calculate the triple scalar product by finding a determinant.

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let’s examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

Example:
Exercise:

Problem:
Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\mathbf{u} = \langle 2, 0, 5\rangle$, $\mathbf{v} = \langle 2, 2, 4\rangle$, and $\mathbf{w} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution:

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$:
Equation:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}2 & 0 & 5\\ 2 & 2 & 4\\ 1 & -1 & 3\end{vmatrix}\\ &= [2(2)(3)+(0)(4)(1)+5(2)(-1)]-[5(2)(1)+(2)(4)(-1)+(0)(2)(3)]\\ &= 2-2\\ &= 0.\end{aligned}$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

Note:
Exercise:

Problem: Are the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$, and $\mathbf{c}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ coplanar?

Solution:

No, the triple scalar product is $-4\neq 0$, so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

Hint

Calculate the triple scalar product.

Example:
Exercise:

Problem:
Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P=(9,-3,-2)$, $Q=(1,3,0)$, and $R=(-2,5,0)$.

Solution:

The plane must contain vectors $\vec{PQ}$ and $\vec{QR}$:
Equation:

$$\begin{aligned}\vec{PQ} &= \langle 1-9, 3-(-3), 0-(-2)\rangle = \langle -8, 6, 2\rangle\\ \vec{QR} &= \langle -2-1, 5-3, 0-0\rangle = \langle -3, 2, 0\rangle.\end{aligned}$$

The cross product $\vec{PQ} \times \vec{QR}$ produces a vector orthogonal to both $\vec{PQ}$ and $\vec{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:
Equation:

$$\begin{aligned}\vec{PQ} \times \vec{QR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix} \\ &= 0\mathbf{i} - 6\mathbf{j} - 16\mathbf{k} - (-18\mathbf{k} + 4\mathbf{i} + 0\mathbf{j}) \\ &= -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

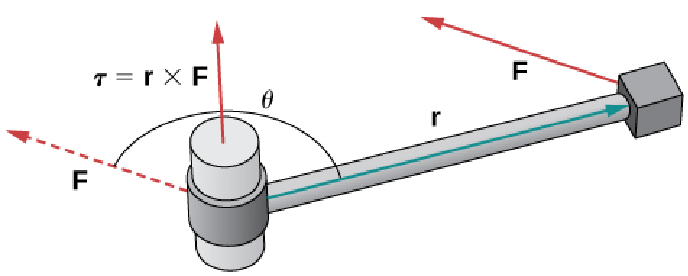
We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Note:
Definition
Torque, τ (the Greek letter *tau*), measures the tendency of a force to produce rotation about an axis of rotation. Let $\mathbf{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\mathbf{F}$ represent the force. Then torque is equal to the cross product of $\mathbf{r}$ and $\mathbf{F}$:
Equation:

$$\tau = \mathbf{r} \times \mathbf{F}.$$

See [link].



Torque measures how a force causes an object to rotate.

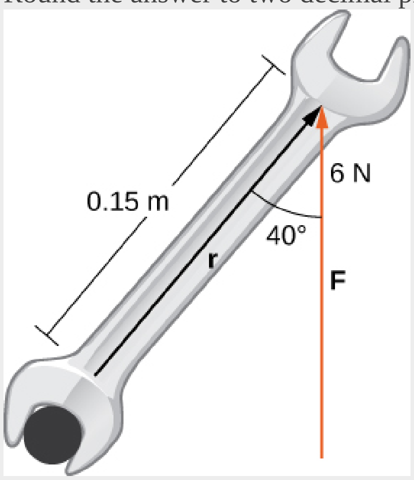
Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the

newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example:
Exercise:

Problem:
Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench ([link]). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.



Torque describes the twisting action of the wrench.

Solution:

Substitute the given information into the equation defining torque:
Equation:

$$\|\tau\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\| \, \|\mathbf{F}\| \sin\theta = (0.15\ \text{m})(6\ \text{N}) \sin 40^\circ \approx 0.58\ \text{N} \cdot \text{m}.$$

Note:
Exercise:

Problem:

Calculate the force required to produce $15\ \text{N} \cdot \text{m}$ torque at an angle of 30^o from a 150-cm rod.

Solution:

20 N

Hint

$\|\boldsymbol{\tau}\| = 15\text{N} \cdot \text{m}$ and $\|\mathbf{r}\| = 1.5\text{m}$

Key Concepts

- The cross product $\mathbf{u} \times \mathbf{v}$ of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Its length is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, is $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.
- The cross product satisfies the following properties for vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, and scalar c:
  - $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
  - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  - $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
  - $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
  - $\mathbf{v} \times \mathbf{v} = 0$
  - $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- The cross product of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.
- If vectors $\mathbf{u}$ and $\mathbf{v}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$.
- The triple scalar product of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
- The volume of a parallelepiped with adjacent edges given by vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.
- If the triple scalar product of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque τ measures the tendency of a force to produce rotation about an axis of rotation. If force $\mathbf{F}$ is acting at a distance $\mathbf{r}$ from the axis, then torque is equal to the cross product of $\mathbf{r}$ and $\mathbf{F}$: $\tau = \mathbf{r} \times \mathbf{F}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**
  $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

For the following exercises, the vectors $\mathbf{u}$ and $\mathbf{v}$ are given.

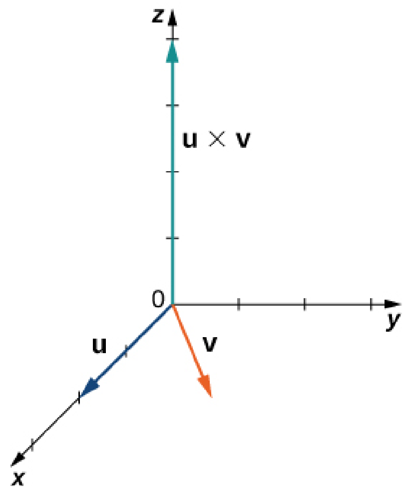
a. Find the cross product $\mathbf{u} \times \mathbf{v}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$. Express the answer in component form.
b. Sketch the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$.

Exercise:

Problem: $\mathbf{u} = \langle 2, 0, 0 \rangle$, $\mathbf{v} = \langle 2, 2, 0 \rangle$

Solution:

a. $\mathbf{u} \times \mathbf{v} = \langle 0, 0, 4 \rangle$;
b.



Exercise:

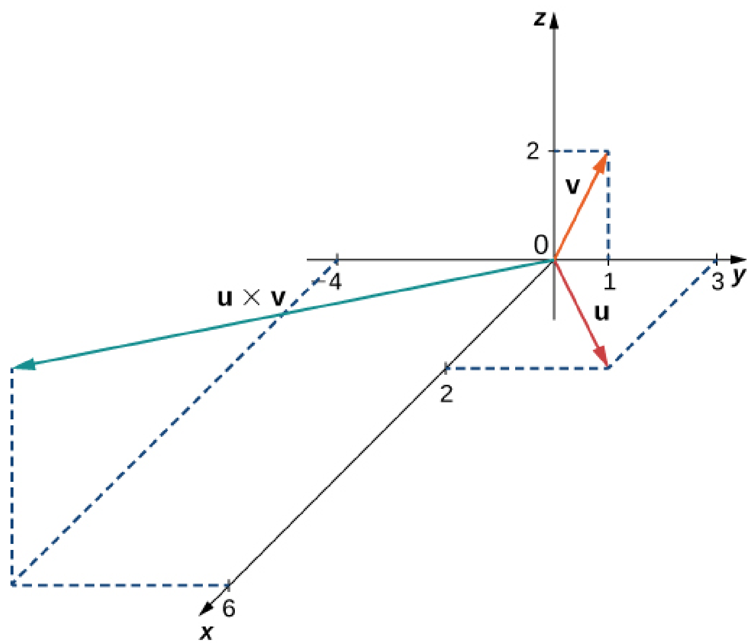
Problem: $\mathbf{u} = \langle 3, 2, -1 \rangle$, $\mathbf{v} = \langle 1, 1, 0 \rangle$

Exercise:

Problem: $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

Solution:

a. $\mathbf{u} \times \mathbf{v} = \langle 6, -4, 2 \rangle$;
b.



Exercise:

Problem: $\mathbf{u} = 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + \mathbf{k}$

Exercise:

Problem: Simplify $(\mathbf{i} \times \mathbf{i} - 2\mathbf{i} \times \mathbf{j} - 4\mathbf{i} \times \mathbf{k} + 3\mathbf{j} \times \mathbf{k}) \times \mathbf{i}$.

Solution:

$-2\mathbf{j} - 4\mathbf{k}$

Exercise:

Problem: Simplify $\mathbf{j} \times (\mathbf{k} \times \mathbf{j} + 2\mathbf{j} \times \mathbf{i} - 3\mathbf{j} \times \mathbf{j} + 5\mathbf{i} \times \mathbf{k})$.

In the following exercises, vectors $\mathbf{u}$ and $\mathbf{v}$ are given. Find unit vector $\mathbf{w}$ in the direction of the cross product vector $\mathbf{u} \times \mathbf{v}$. Express your answer using standard unit vectors.

Exercise:

Problem: $\mathbf{u} = \langle 3, -1, 2 \rangle$, $\mathbf{v} = \langle -2, 0, 1 \rangle$

Solution:

$\mathbf{w} = -\frac{1}{3\sqrt{6}}\mathbf{i} - \frac{7}{3\sqrt{6}}\mathbf{j} - \frac{2}{3\sqrt{6}}\mathbf{k}$

Exercise:

Problem: $\mathbf{u} = \langle 2, 6, 1 \rangle$, $\mathbf{v} = \langle 3, 0, 1 \rangle$

Exercise:

Problem: $\mathbf{u} = \vec{AB}$, $\mathbf{v} = \vec{AC}$, where $A(1, 0, 1)$, $B(1, -1, 3)$, and $C(0, 0, 5)$

Solution:

$\mathbf{w} = -\frac{4}{\sqrt{21}}\mathbf{i} - \frac{2}{\sqrt{21}}\mathbf{j} - \frac{1}{\sqrt{21}}\mathbf{k}$

Exercise:

Problem: $\mathbf{u} = \vec{OP}$, $\mathbf{v} = \vec{PQ}$, where $P(-1, 1, 0)$ and $Q(0, 2, 1)$

Exercise:

Problem:

Determine the real number α such that $\mathbf{u} \times \mathbf{v}$ and $\mathbf{i}$ are orthogonal, where $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + \alpha\mathbf{k}$.

Solution:

$\alpha = 10$

Exercise:

Problem:

Show that $\mathbf{u} \times \mathbf{v}$ and $2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}$ cannot be orthogonal for any α real number, where $\mathbf{u} = \mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \alpha\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

Exercise:

Problem: Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

Exercise:

Problem:

Show that $\mathbf{v} \times \mathbf{u}$ is orthogonal to $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} + \mathbf{v}) + \mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

Exercise:

Problem: Calculate the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

Solution:

$-3\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$

Exercise:

Problem: Calculate the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -4 \\ 1 & 6 & -1 \end{vmatrix}$.

For the following exercises, the vectors $\mathbf{u}$ and $\mathbf{v}$ are given. Use determinant notation to find vector $\mathbf{w}$ orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$.

Exercise:

Problem: $\mathbf{u} = \langle -1, 0, e^t \rangle$, $\mathbf{v} = \langle 1, e^{-t}, 0 \rangle$, where t is a real number

---

Solution:

$\mathbf{w} = \langle -1, e^t, -e^{-t} \rangle$

Exercise:

Problem: $\mathbf{u} = \langle 1, 0, x \rangle$, $\mathbf{v} = \left\langle \frac{2}{x}, 1, 0 \right\rangle$, where x is a nonzero real number

Exercise:

Problem:

Find vector $(\mathbf{a} - 2\mathbf{b}) \times \mathbf{c}$, where $\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$, $\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution:

$-26\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}$

Exercise:

Problem: Find vector $\mathbf{c} \times (\mathbf{a} + 3\mathbf{b})$, where $\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 9 \\ 0 & 1 & 0 \end{vmatrix}$, $\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ 7 & 1 & -1 \end{vmatrix}$, and $\mathbf{c} = \mathbf{i} - \mathbf{k}$.

Exercise:

Problem:

[T] Use the cross product $\mathbf{u} \times \mathbf{v}$ to find the acute angle between vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{k}$. Express the answer in degrees rounded to the nearest integer.

Solution:

72°

Exercise:

Problem:

[T] Use the cross product $\mathbf{u} \times \mathbf{v}$ to find the obtuse angle between vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$. Express the answer in degrees rounded to the nearest integer.

Exercise:

Problem:

Use the sine and cosine of the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ to prove Lagrange's identity: $\|\mathbf{u}\ \times\ \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.

Exercise:

Problem:

Verify Lagrange's identity $\|\mathbf{u}\ \times\ \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$ for vectors $\mathbf{u} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.

Exercise:

Problem:

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called *collinear* if there exists a nonzero scalar α such that $\mathbf{v} = \alpha\mathbf{u}$. Show that $\mathbf{u}$ and $\mathbf{v}$ are collinear if and only if $\mathbf{u}\ \times\ \mathbf{v} = \mathbf{0}$.

Exercise:

Problem:

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called *collinear* if there exists a nonzero scalar α such that $\mathbf{v} = \alpha\mathbf{u}$. Show that vectors $\vec{AB}$ and $\vec{AC}$ are collinear, where $A\,(4, 1, 0)$, $B\,(6, 5, -2)$, and $C\,(5, 3, -1)$.

Exercise:

Problem: Find the area of the parallelogram with adjacent sides $\mathbf{u} = \langle 3, 2, 0\rangle$ and $\mathbf{v} = \langle 0, 2, 1\rangle$.

Solution:

7

Exercise:

Problem: Find the area of the parallelogram with adjacent sides $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{k}$.

Exercise:

Problem: Consider points $A\,(3, -1, 2)$, $B\,(2, 1, 5)$, and $C\,(1, -2, -2)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\vec{AB}$ and $\vec{AC}$.
b. Find the area of triangle ABC.
c. Find the distance from point A to line BC.

Solution:

a. $5\sqrt{6}$; b. $\frac{5\sqrt{6}}{2}$; c. $\frac{5\sqrt{6}}{\sqrt{59}}$

Exercise:

Problem: Consider points $A\,(2, -3, 4)$, $B\,(0, 1, 2)$, and $C(-1, 2, 0)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\vec{AB}$ and $\vec{AC}$.
b. Find the area of triangle ABC.
c. Find the distance from point B to line AC.

In the following exercises, vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are given.

a. Find the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
b. Find the volume of the parallelepiped with the adjacent edges $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$.

Exercise:

Problem: $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = \mathbf{i} + \mathbf{k}$

Solution:

a. 2; b. 2

Exercise:

Problem: $\mathbf{u} = \langle -3, 5, -1 \rangle$, $\mathbf{v} = \langle 0, 2, -2 \rangle$, and $\mathbf{w} = \langle 3, 1, 1 \rangle$

Exercise:

Problem:

Calculate the triple scalar products $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$, where $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle 7, 6, 9 \rangle$, and $\mathbf{w} = \langle 4, 2, 7 \rangle$.

Solution:

$\mathbf{v} \cdot (\mathbf{u} \times \mathrm{w}) = -1$, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 1$

Exercise:

Problem:

Calculate the triple scalar products $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$ and $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$, where $\mathbf{u} = \langle 4, 2, -1 \rangle$, $\mathbf{v} = \langle 2, 5, -3 \rangle$, and $\mathbf{w} = \langle 9, 5, -10 \rangle$.

Exercise:

Problem: Find vectors $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ with a triple scalar product given by the determinant

$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 8 & 9 & 2 \end{vmatrix}$. Determine their triple scalar product.

Solution:

$\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 0, 2, 5 \rangle$, $\mathbf{c} = \langle 8, 9, 2 \rangle$; $\mathbf{a} \cdot (\mathrm{b} \times \mathrm{c}) = -9$

Exercise:

Problem: The triple scalar product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by the determinant

$\begin{vmatrix} 0 & -2 & 1 \\ 0 & 1 & 4 \\ 1 & -3 & 7 \end{vmatrix}$. Find vector $\mathbf{a} - \mathbf{b} + \mathbf{c}$.

Exercise:

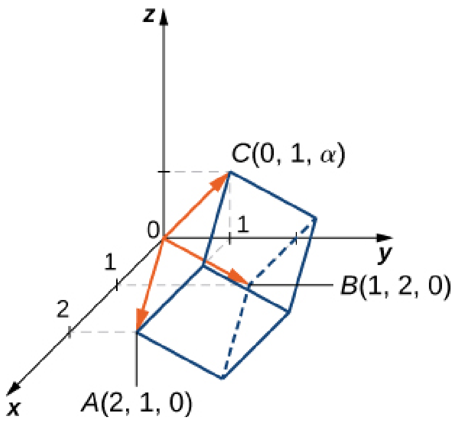
Problem:

Consider the parallelepiped with edges OA, OB, and OC, where $A\,(2,1,0)$, $B\,(1,2,0)$, and $C\,(0,1,\alpha)$.

a. Find the real number $\alpha > 0$ such that the volume of the parallelepiped is 3 units^3.
b. For $\alpha = 1$, find the height h from vertex C of the parallelepiped. Sketch the parallelepiped.

Solution:

a. $\alpha = 1$; b. $h = 1$,



Exercise:

Problem:

Consider points $A\,(\alpha,0,0)$, $B\,(0,\beta,0)$, and $C\,(0,0,\gamma)$, with α, β, and γ positive real numbers.

a. Determine the volume of the parallelepiped with adjacent sides $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$.
b. Find the volume of the tetrahedron with vertices O, A, B, and C. (*Hint*: The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.)
c. Find the distance from the origin to the plane determined by A, B, and C. Sketch the parallelepiped and tetrahedron.

Exercise:

Problem:

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be three-dimensional vectors and c be a real number. Prove the following properties of the cross product.

a. $\mathbf{u} \times \mathbf{u} = 0$
b. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
c. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
d. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

Exercise:

Problem:

Show that vectors $\mathbf{u} = \langle 1, 0, -8 \rangle$, $\mathbf{v} = \langle 0, 1, 6 \rangle$, and $\mathbf{w} = \langle -1, 9, 3 \rangle$ satisfy the following properties of the cross product.

a. $\mathbf{u} \times \mathbf{u} = 0$
b. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
c. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
d. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

Exercise:

Problem:

Nonzero vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are said to be *linearly dependent* if one of the vectors is a linear combination of the other two. For instance, there exist two nonzero real numbers α and β such that $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$. Otherwise, the vectors are called *linearly independent*. Show that $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are coplanar if and only if they are linear dependent.

Exercise:

Problem:

Consider vectors $\mathbf{u} = \langle 1, 4, -7 \rangle$, $\mathbf{v} = \langle 2, -1, 4 \rangle$, $\mathbf{w} = \langle 0, -9, 18 \rangle$, and $\mathbf{p} = \langle 0, -9, 17 \rangle$.

a. Show that $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are coplanar by using their triple scalar product
b. Show that $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are coplanar, using the definition that there exist two nonzero real numbers α and β such that $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$.
c. Show that $\mathbf{u}, \mathbf{v},$ and $\mathbf{p}$ are linearly independent—that is, none of the vectors is a linear combination of the other two.

Exercise:

Problem:

Consider points $A(0, 0, 2)$, $B(1, 0, 2)$, $C(1, 1, 2)$, and $D(0, 1, 2)$. Are vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ linearly dependent (that is, one of the vectors is a linear combination of the other two)?

Solution:

Yes, $\vec{AD} = \alpha\vec{AB} + \beta\vec{AC}$, where $\alpha = -1$ and $\beta = 1$.

Exercise:

Problem:

Show that vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}-\mathbf{j}$, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ are linearly independent—that is, there exist two nonzero real numbers α and β such that $\mathbf{i}+\mathbf{j}+\mathbf{k}=\alpha\left(\mathbf{i}+\mathbf{j}\right)+\beta\left(\mathbf{i}-\mathbf{j}\right)$.

Exercise:

Problem:

Let $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ be two-dimensional vectors. The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is not defined. However, if the vectors are regarded as the three-dimensional vectors $\tilde{\mathbf{u}}=\langle u_1,u_2,0\rangle$ and $\tilde{\mathbf{v}}=\langle v_1,v_2,0\rangle$, respectively, then, in this case, we can define the cross product of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$. In particular, in determinant notation, the cross product of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ is given by
Equation:

$$\tilde{\mathbf{u}}\times\tilde{\mathbf{v}}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & 0\\ v_1 & v_2 & 0\end{vmatrix}.$$

Use this result to compute $\left(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta\right)\times\left(\mathbf{i}sin\theta-\mathbf{j}cos\theta\right)$, where θ is a real number.

Solution:

$-\mathbf{k}$

Exercise:

Problem: Consider points $P\left(2,1\right)$, $Q\left(4,2\right)$, and $R\left(1,2\right)$.

a. Find the area of triangle $P,Q,\text{ and }R.$
b. Determine the distance from point R to the line passing through $P\text{ and }Q.$

Exercise:

Problem:

Determine a vector of magnitude 10 perpendicular to the plane passing through the *x*-axis and point $P\left(1,2,4\right)$.

Solution:

$\left\langle 0,\pm4\sqrt{5},\mp2\sqrt{5}\right\rangle$

Exercise:

Problem:

Determine a unit vector perpendicular to the plane passing through the *z*-axis and point $A\left(3,1,-2\right)$.

Exercise:

Problem:

Consider $\mathbf{u}$ and $\mathbf{v}$ two three-dimensional vectors. If the magnitude of the cross product vector $\mathbf{u} \times \mathbf{v}$ is k times larger than the magnitude of vector $\mathbf{u}$, show that the magnitude of $\mathbf{v}$ is greater than or equal to k, where k is a natural number.

Exercise:

Problem:

[T] Assume that the magnitudes of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are known. The function $f(\theta) = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$ defines the magnitude of the cross product vector $\mathbf{u} \times \mathbf{v}$, where $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

a. Graph the function f.
b. Find the absolute minimum and maximum of function f. Interpret the results.
c. If $\|\mathbf{u}\| = 5$ and $\|\mathbf{v}\| = 2$, find the angle between $\mathbf{u}$ and $\mathbf{v}$ if the magnitude of their cross product vector is equal to 9.

Exercise:

Problem:

Find all vectors $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ that satisfy the equation $\langle 1, 1, 1 \rangle \times \mathbf{w} = \langle -1, -1, 2 \rangle$.

Solution:

$\mathbf{w} = \langle w_3 - 1, w_3 + 1, w_3 \rangle$, where w_3 is any real number

Exercise:

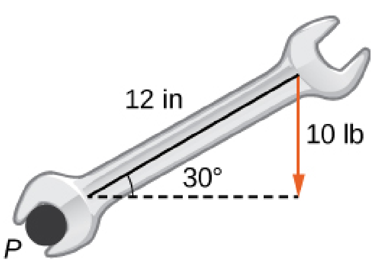
Problem:

Solve the equation $\mathbf{w} \times \langle 1, 0, -1 \rangle = \langle 3, 0, 3 \rangle$, where $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is a nonzero vector with a magnitude of 3.

Exercise:

Problem:

[T] A mechanic uses a 12-in. wrench to turn a bolt. The wrench makes a 30° angle with the horizontal. If the mechanic applies a vertical force of 10 lb on the wrench handle, what is the magnitude of the torque at point P (see the following figure)? Express the answer in foot-pounds rounded to two decimal places.



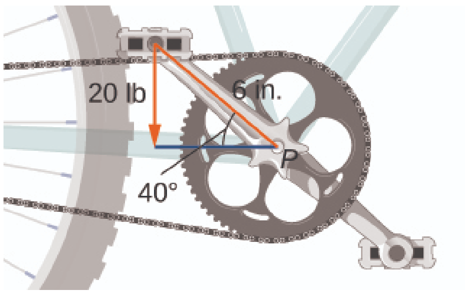
Solution:

8.66 ft-lb

Exercise:

Problem:

[T] A boy applies the brakes on a bicycle by applying a downward force of 20 lb on the pedal when the 6-in. crank makes a 40° angle with the horizontal (see the following figure). Find the torque at point P. Express your answer in foot-pounds rounded to two decimal places.



Exercise:

Problem:

[T] Find the magnitude of the force that needs to be applied to the end of a 20-cm wrench located on the positive direction of the *y*-axis if the force is applied in the direction $\langle 0, 1, -2\rangle$ and it produces a 100 N·m torque to the bolt located at the origin.

Solution:

250 N

Exercise:

Problem:

[T] What is the magnitude of the force required to be applied to the end of a 1-ft wrench at an angle of 35° to produce a torque of 20 N·m?

Exercise:

Problem:

[T] The force vector $\mathbf{F}$ acting on a proton with an electric charge of $1.6\ \times\ 10^{-19}\text{C}$ (in coulombs) moving in a magnetic field $\mathbf{B}$ where the velocity vector $\mathbf{v}$ is given by $\mathbf{F} = 1.6\ \times\ 10^{-19}\ (\mathbf{v}\ \times\ \mathbf{B})$ (here, $\mathbf{v}$ is expressed in meters per second, $\mathbf{B}$ is in tesla [T], and $\mathbf{F}$ is in newtons [N]). Find the force that acts on a proton that moves in the *xy*-plane at velocity $\mathbf{v} = 10^5\mathbf{i} + 10^5\mathbf{j}$ (in meters per second) in a magnetic field given by $\mathbf{B} = 0.3\mathbf{j}$.

Solution:

$\mathbf{F} = 4.8\ \times\ 10^{-15}\mathbf{k}\ \mathbf{N}$

Exercise:

Problem:

[T] The force vector $\mathbf{F}$ acting on a proton with an electric charge of 1.6×10^{-19}C moving in a magnetic field $\mathbf{B}$ where the velocity vector **v** is given by $\mathbf{F} = 1.6 \times 10^{-19} (\mathbf{v} \times \mathbf{B})$ (here, **v** is expressed in meters per second, $\mathbf{B}$ in T, and $\mathbf{F}$ in N). If the magnitude of force $\mathbf{F}$ acting on a proton is 5.9×10^{-17} N and the proton is moving at the speed of 300 m/sec in magnetic field $\mathbf{B}$ of magnitude 2.4 T, find the angle between velocity vector **v** of the proton and magnetic field $\mathbf{B}$. Express the answer in degrees rounded to the nearest integer.

Exercise:

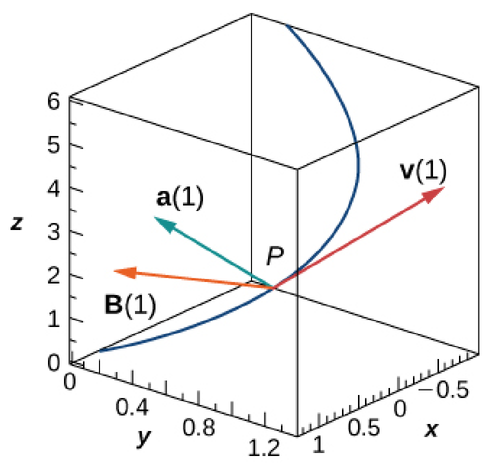
Problem:

[T] Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\mathbf{r}$ are expressed in centimeters and time in seconds. Let $\vec{OP}$ be the position vector of the particle after 1 sec.

a. Determine unit vector $\mathbf{B}(t)$ (called the *binormal unit vector*) that has the direction of cross product vector $\mathbf{v}(t) \times \mathbf{a}(t)$, where $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are the instantaneous velocity vector and, respectively, the acceleration vector of the particle after t seconds.
b. Use a CAS to visualize vectors $\mathbf{v}(1)$, $\mathbf{a}(1)$, and $\mathbf{B}(1)$ as vectors starting at point P along with the path of the particle.

Solution:

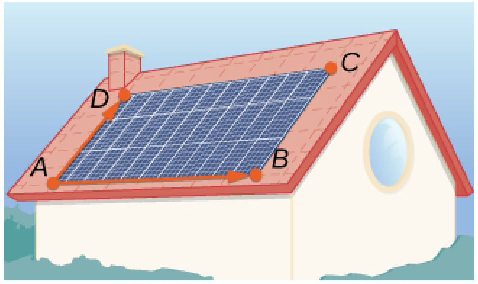
a. $\mathbf{B}(t) = \left\langle \frac{2\sin t}{\sqrt{5}}, -\frac{2\cos t}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$;
b.



Exercise:

Problem:

A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0)$, $B(8, 18, 0)$, $C(0, 18, 8)$, and $D(0, 0, 8)$ (see the following figure).



a. Find vector $\mathbf{n} = \vec{AB} \times \vec{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors.

b. Assume unit vector $\mathbf{s} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\mathbf{F} = 900\mathbf{s}$ (in watts per square meter $[\mathrm{W}/\mathrm{m}^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\mathbf{F}$ and $\mathbf{n}$ (expressed in watts).

c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (*Hint*: The angle between vectors $\mathbf{n}$ and $\mathbf{s}$ and the angle of elevation are complementary.)

Glossary

cross product
: $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$, where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

determinant
: a real number associated with a square matrix

parallelepiped
: a three-dimensional prism with six faces that are parallelograms

torque
: the effect of a force that causes an object to rotate

triple scalar product
: the dot product of a vector with the cross product of two other vectors: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

vector product
: the cross product of two vectors

Cylindrical and Spherical Coordinates

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

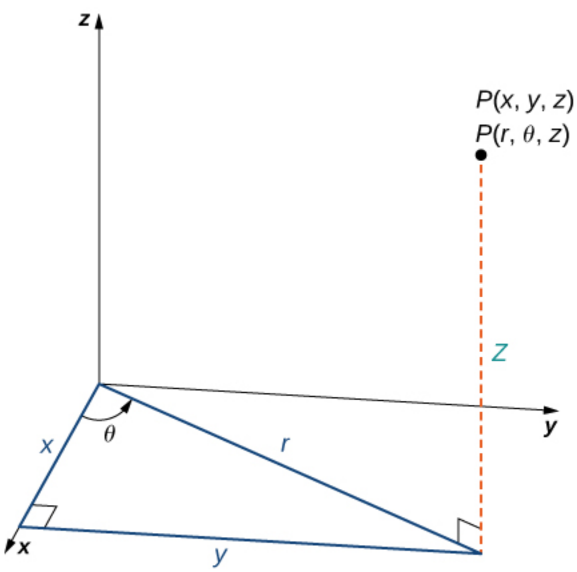
The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Note:
Definition
In the **cylindrical coordinate system**, a point in space ([link]) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the *xy*-plane
- z is the usual z-coordinate in the Cartesian coordinate system



The right triangle lies in the *xy*-plane. The

length of the hypotenuse is r and θ is the measure of the angle formed by the positive *x*-axis and the hypotenuse. The *z*-coordinate describes the location of the point above or below the *xy*-plane.

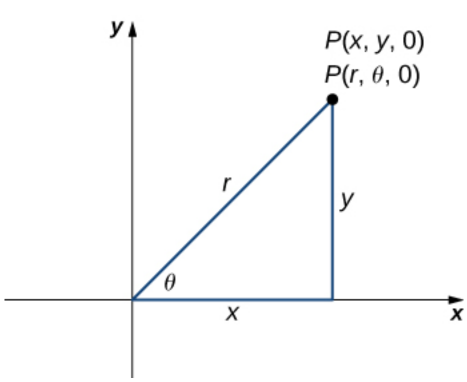
In the *xy*-plane, the right triangle shown in [link] provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Note:
Conversion between Cylindrical and Cartesian Coordinates
The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:
Equation:

$$\begin{array}{ccccl} x & = & r\cos\theta & & \text{These equations are used to convert from} \\ y & = & r\sin\theta & & \text{cylindrical coordinates to rectangular} \\ z & = & z & & \text{coordinates.} \\ & \text{and} & & & \\ r^2 & = & x^2+y^2 & & \text{These equations are used to convert from} \\ \tan\theta & = & \frac{y}{x} & & \text{rectangular coordinates to cylindrical} \\ z & = & z & & \text{coordinates.} \end{array}$$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \frac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the *xy*-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\frac{\pi}{2}$, $\frac{3\pi}{2}$, or 0, depending on the value of y.

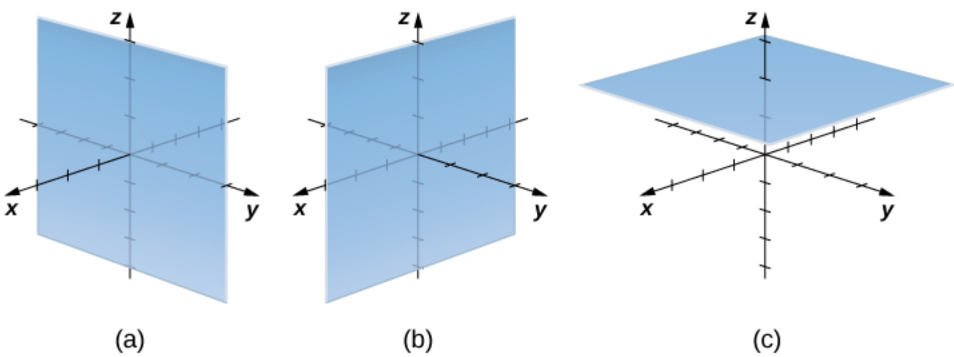
Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the *xy*-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in the following figure.



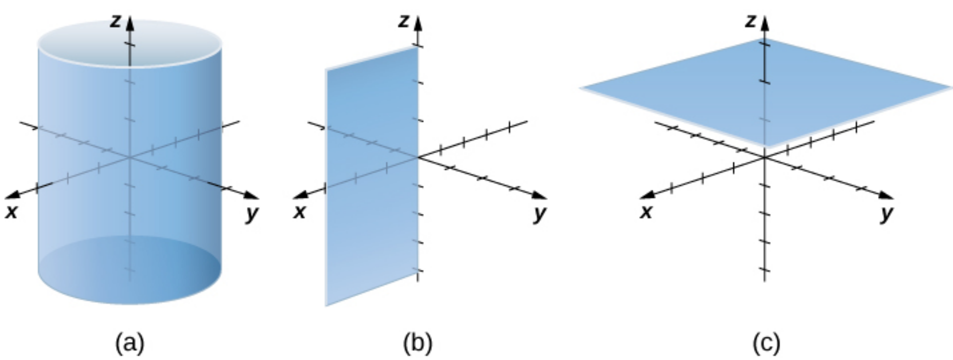
The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-

triangle relationships tell us that $x = r\cos\theta$, $y = r\sin\theta$, and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c$, $y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the *yz*-plane, the *xz*-plane, and the *xy*-plane, respectively. When we convert to cylindrical coordinates, the *z*-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the *xy*-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the *z*-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the *x*-axis, which gives us a half-plane that starts at the *z*-axis ([link] and [link]).



In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the *yz*-plane, (b) surfaces of the form $y = c$ are planes parallel to the *xz*-plane, and (c) surfaces of the form $z = c$ are planes parallel to the *xy*-plane.



In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the *x*-axis, and (c) surfaces of the form $z = c$ are planes parallel to the *xy*-plane.

Example:
Exercise:

Problem:
Converting from Cylindrical to Rectangular Coordinates

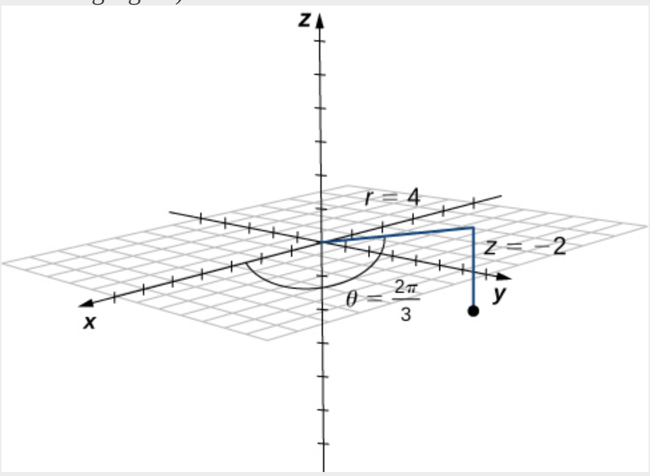
Plot the point with cylindrical coordinates $\left(4, \frac{2\pi}{3}, -2\right)$ and express its location in rectangular coordinates.

Solution:

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in [link]:
Equation:

$$\begin{aligned} x &= r\cos\theta = 4\cos\frac{2\pi}{3} = -2 \\ y &= r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3} \\ z &= -2. \end{aligned}$$

The point with cylindrical coordinates $\left(4, \frac{2\pi}{3}, -2\right)$ has rectangular coordinates $\left(-2, 2\sqrt{3}, -2\right)$ (see the following figure).



The projection of the point in the *xy*-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive *x*-axis. The point lies 2 units below the *xy*-plane.

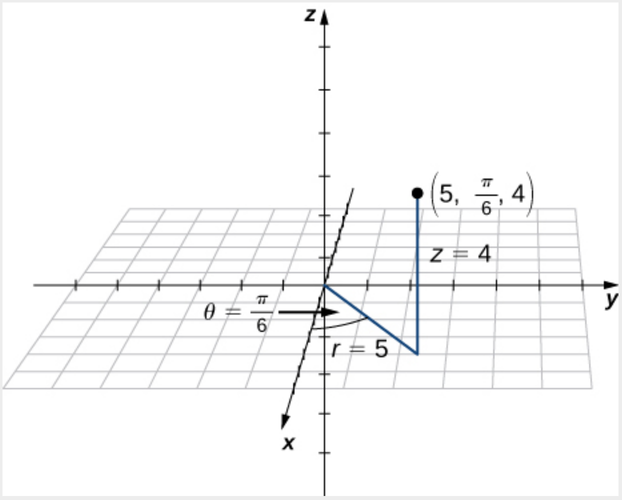
Note:
Exercise:

Problem:

Point R has cylindrical coordinates $\left(5, \frac{\pi}{6}, 4\right)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

Solution:

The rectangular coordinates of the point are $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4\right)$.



Hint

The first two components match the polar coordinates of the point in the *xy*-plane.

If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example:
Exercise:

Problem:
Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution:

Use the second set of equations from [link] to translate from rectangular to cylindrical coordinates:
Equation:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} = \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$. Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:
Equation:

$$\begin{aligned} \tan\theta &= \frac{y}{x} = \frac{-3}{1} \\ \theta &= \arctan(-3) \approx 5.03\,\text{rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:
Equation:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $\left(\sqrt{10}, 5.03, 5\right)$.

Note:
Exercise:

Problem: Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

Solution:

$\left(8\sqrt{2}, \frac{3\pi}{4}, -7\right)$

Hint

$r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

Example:
Exercise:

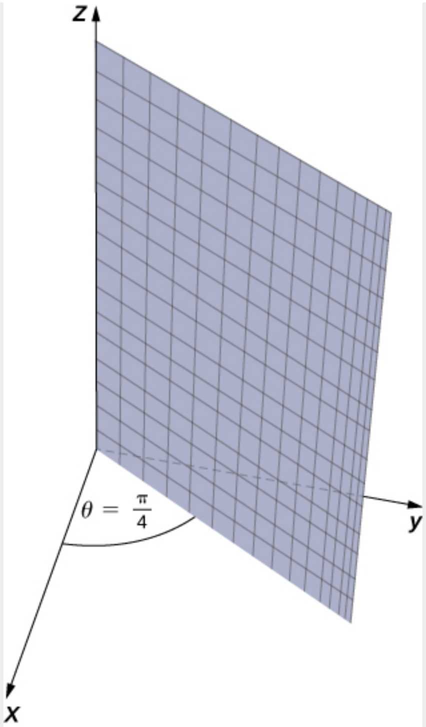
Problem:
Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta = \frac{\pi}{4}$
b. $r^2 + z^2 = 9$
c. $z = r$

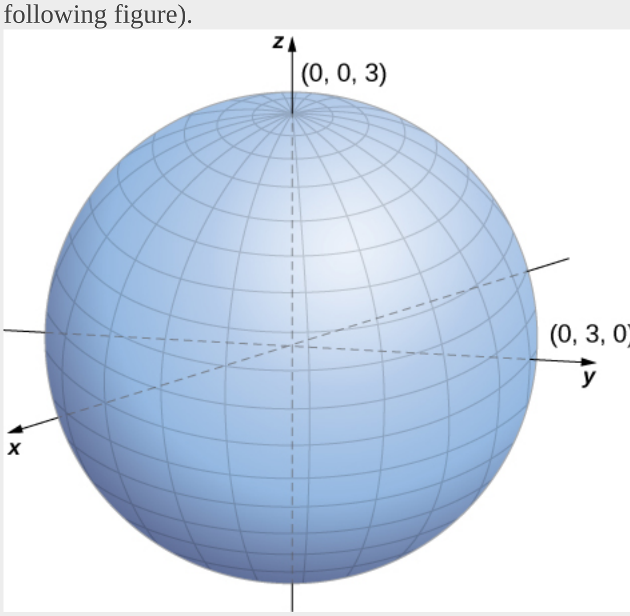
Solution:

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (see the following figure).



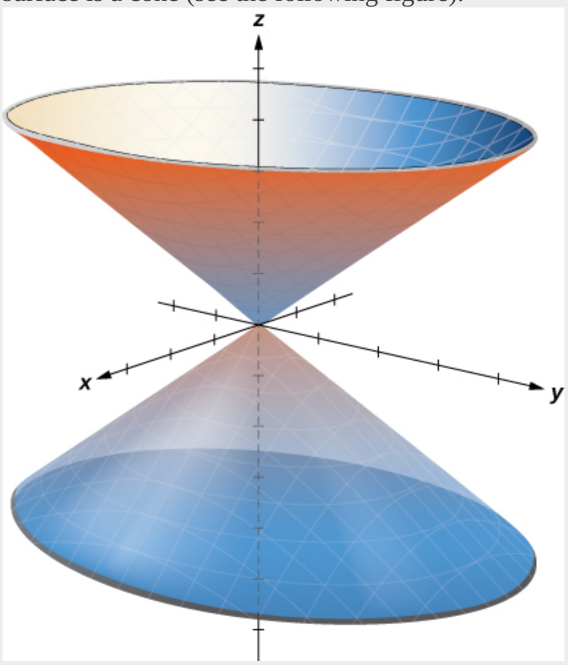
In polar coordinates, the equation $\theta = \pi/4$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2 = x^2 + y^2$ into equation $r^2 + z^2 = 9$ to express the rectangular form of the equation: $x^2 + y^2 + z^2 = 9$. This equation describes a sphere centered at the origin with radius 3 (see the following figure).



The sphere centered at the origin with radius 3 can be described by the cylindrical equation $r^2 + z^2 = 9$.

c. To describe the surface defined by equation $z = r$, is it useful to examine traces parallel to the *xy*-plane. For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (see the following figure).



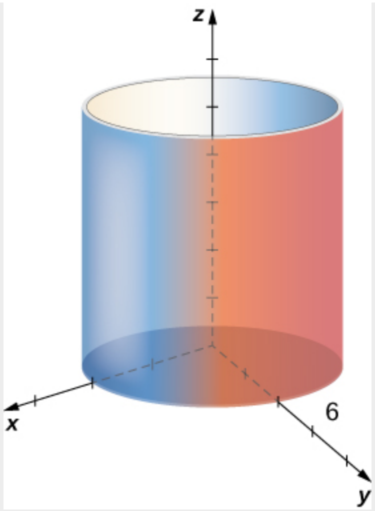
The traces in planes parallel to the *xy*-plane are circles. The radius of the circles increases as z increases.

Note:
Exercise:

Problem: Describe the surface with cylindrical equation $r = 6$.

Solution:

This surface is a cylinder with radius 6.



Hint

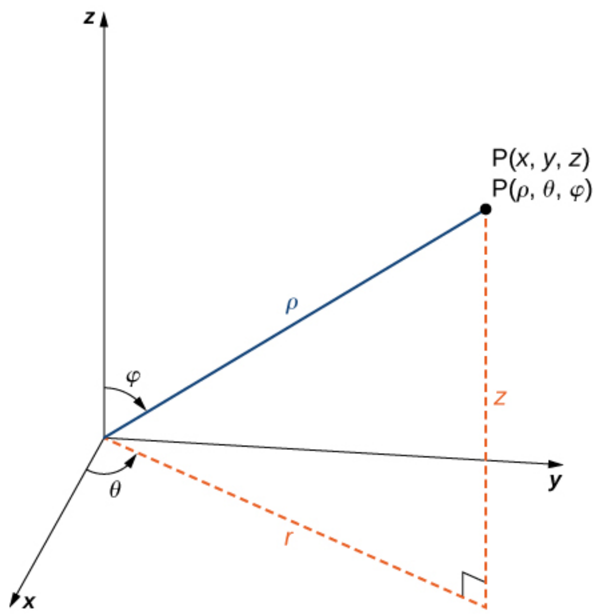
The θ and z components of points on the surface can take any value.

Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, location of a point in space is described using two distances $(r \text{ and } z)$ and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Note:
Definition
In the **spherical coordinate system**, a point P in space ([link]) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment OP, where O is the origin and $0 \leq \varphi \leq \pi$.



The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0,0,0)$ in spherical coordinates.

Note:
Converting among Spherical, Cylindrical, and Rectangular Coordinates
Rectangular coordinates (x,y,z) and spherical coordinates (ρ,θ,φ) of a point are related as follows:
Equation:

$$\begin{array}{rcll} x & = & \rho\sin\varphi\cos\theta & \text{These equations are used to convert from} \\ y & = & \rho\sin\varphi\sin\theta & \text{spherical coordinates to rectangular} \\ z & = & \rho\cos\varphi & \text{coordinates.} \\ & \text{and} & & \\ \rho^2 & = & x^2+y^2+z^2 & \text{These equations are used to convert from} \\ \tan\theta & = & \frac{y}{x} & \text{rectangular coordinates to spherical} \\ \varphi & = & \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). & \text{coordinates.} \end{array}$$

If a point has cylindrical coordinates (r,θ,z), then these equations define the relationship between cylindrical and spherical coordinates.
Equation:

$$\begin{aligned} r &= \rho \sin \varphi \\ \theta &= \theta \\ z &= \rho \cos \varphi \end{aligned}$$

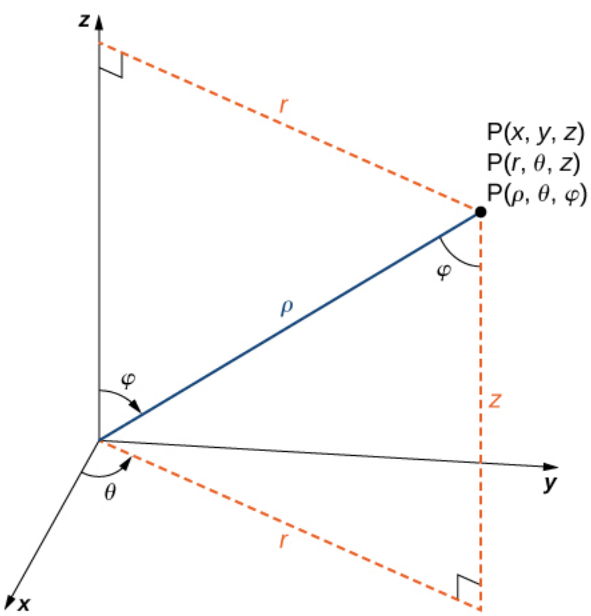
These equations are used to convert from spherical coordinates to cylindrical coordinates.

and

$$\begin{aligned} \rho &= \sqrt{r^2 + z^2} \\ \theta &= \theta \\ \varphi &= \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right) \end{aligned}$$

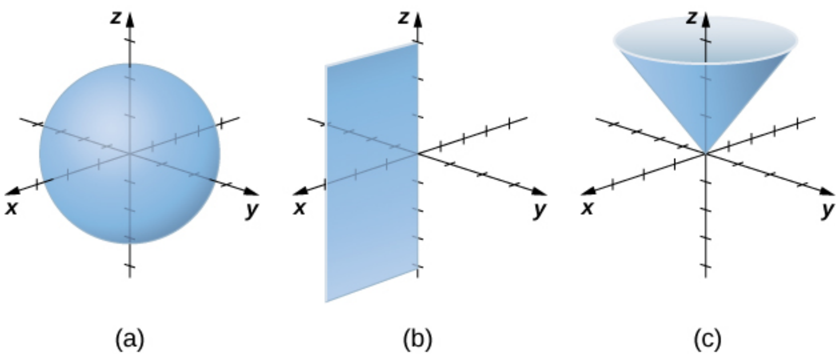
These equations are used to convert from cylindrical coordinates to spherical coordinates.

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at [link], it is easy to see that $r = \rho \sin \varphi$. Then, looking at the triangle in the *xy*-plane with r as its hypotenuse, we have $x = r \cos \theta = \rho \sin \varphi \cos \theta$. The derivation of the formula for y is similar. [link] also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan \theta = \frac{y}{x}$ to choose the correct value of θ.



The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let’s consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = c$. The points on these surfaces are at a fixed angle from the *z*-axis and form a half-cone ([link]).



In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example:
Exercise:

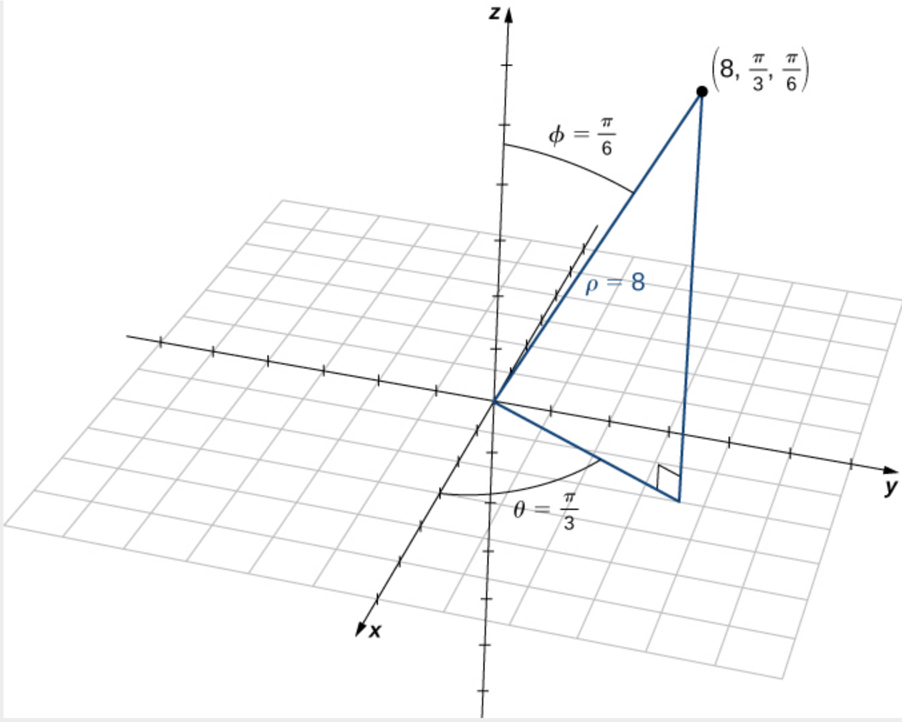
Problem:
Converting from Spherical Coordinates

Plot the point with spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ and express its location in both rectangular and cylindrical coordinates.

Solution:

Use the equations in [link] to translate between spherical and cylindrical coordinates ([link]):
Equation:

$$\begin{aligned} x &= \rho \sin \varphi \cos \theta = 8 \sin \left(\frac{\pi}{6}\right) \cos \left(\frac{\pi}{3}\right) = 8 \left(\frac{1}{2}\right) \frac{1}{2} = 2 \\ y &= \rho \sin \varphi \sin \theta = 8 \sin \left(\frac{\pi}{6}\right) \sin \left(\frac{\pi}{3}\right) = 8 \left(\frac{1}{2}\right) \frac{\sqrt{3}}{2} = 2\sqrt{3} \\ z &= \rho \cos \varphi = 8 \cos \left(\frac{\pi}{6}\right) = 8 \left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}. \end{aligned}$$



The projection of the point in the *xy*-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive *x*-axis. The point lies $4\sqrt{3}$ units above the *xy*-plane.

The point with spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ has rectangular coordinates $\left(2, 2\sqrt{3}, 4\sqrt{3}\right)$.

Finding the values in cylindrical coordinates is equally straightforward:
Equation:

$$\begin{aligned} r &= \rho \sin \varphi = 8 \sin \tfrac{\pi}{6} = 4 \\ \theta &= \theta \\ z &= \rho \cos \varphi = 8 \cos \tfrac{\pi}{6} = 4\sqrt{3}. \end{aligned}$$

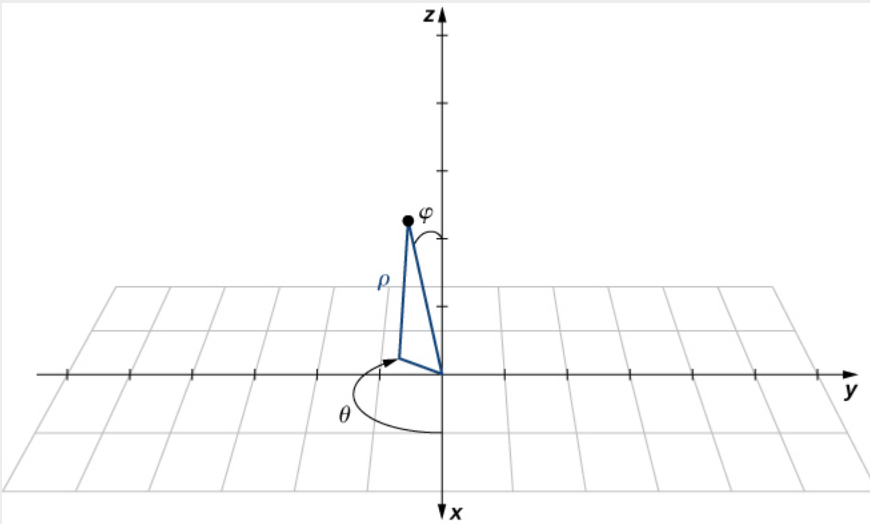
Thus, cylindrical coordinates for the point are $\left(4, \frac{\pi}{3}, 4\sqrt{3}\right)$.

Note:
Exercise:

Problem:

Plot the point with spherical coordinates $\left(2, -\frac{5\pi}{6}, \frac{\pi}{6}\right)$ and describe its location in both rectangular and cylindrical coordinates.

Solution:



Cartesian: $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right)$, cylindrical: $\left(1, -\frac{5\pi}{6}, \sqrt{3}\right)$

Hint

Converting the coordinates first may help to find the location of the point in space more easily.

Example:
Exercise:

Problem:
Converting from Rectangular Coordinates

Convert the rectangular coordinates $\left(-1, 1, \sqrt{6}\right)$ to both spherical and cylindrical coordinates.

Solution:

Start by converting from rectangular to spherical coordinates:
Equation:

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 = (-1)^2 + 1^2 + \left(\sqrt{6}\right)^2 = 8 \\ \rho &= 2\sqrt{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= \tfrac{1}{-1} \\ \theta &= \arctan(-1) = \tfrac{3\pi}{4}. \end{aligned}$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so
Equation:

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $\left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)$.

To find the cylindrical coordinates for the point, we need only find r:
Equation:

$$r = \rho \sin \varphi = 2\sqrt{2} \sin \left(\frac{\pi}{6}\right) = \sqrt{2}.$$

The cylindrical coordinates for the point are $\left(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6}\right)$.

Example:
Exercise:

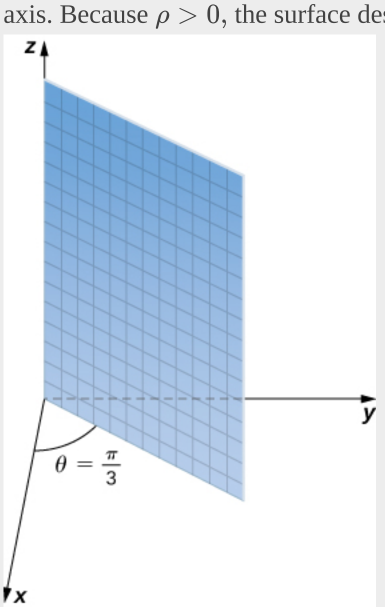
Problem:
Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$
b. $\varphi = \frac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin \theta \sin \varphi$

Solution:

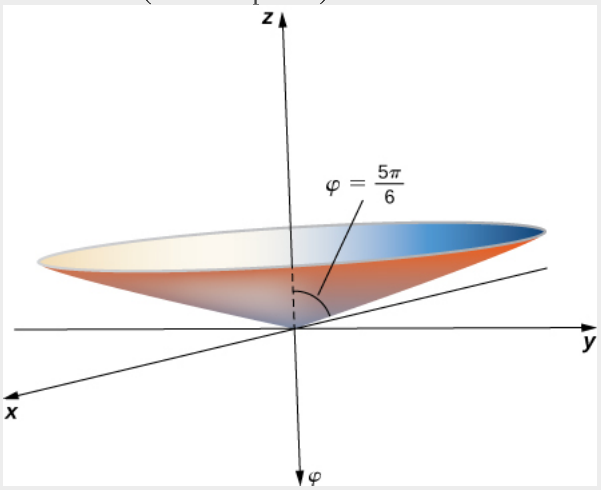
a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $\left(\rho, \frac{\pi}{3}, \varphi\right)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown in [link].



The surface described by equation $\theta = \frac{\pi}{3}$ is a half-plane.

plane.

b. Equation $\varphi = \frac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone ([link]). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).



The equation $\varphi = \frac{5\pi}{6}$ describes a cone.

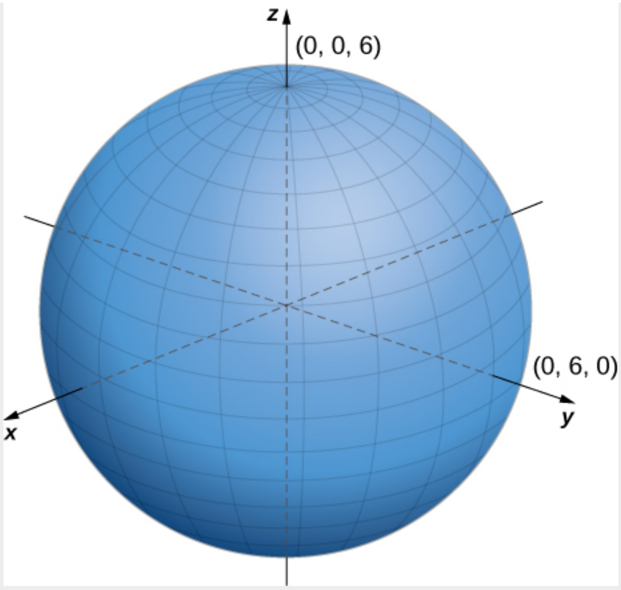
To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

Equation:

$$\begin{aligned} \frac{5\pi}{6} &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \cos\frac{5\pi}{6} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{\sqrt{3}}{2} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{3}{4} &= \frac{z^2}{x^2+y^2+z^2} \\ \frac{3x^2}{4} + \frac{3y^2}{4} + \frac{3z^2}{4} &= z^2 \\ \frac{3x^2}{4} + \frac{3y^2}{4} - \frac{z^2}{4} &= 0. \end{aligned}$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 ([link]).



Equation $\rho = 6$ describes a sphere with radius 6.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho \sin \varphi \sin \theta$ and $\rho^2 = x^2 + y^2 + z^2$:
Equation:

$$\begin{aligned}
\rho &= \sin \theta \sin \varphi & \\
\rho^2 &= \rho \sin \theta \sin \varphi & \text{Multiply both sides of the equation by } \rho. \\
x^2 + y^2 + z^2 &= y & \text{Substitute rectangular variables using the equations a} \\
x^2 + y^2 - y + z^2 &= 0 & \text{Subtract } y \text{ from both sides of the equation.} \\
x^2 + y^2 - y + \frac{1}{4} + z^2 &= \frac{1}{4} & \text{Complete the square.} \\
x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 &= \frac{1}{4}. & \text{Rewrite the middle terms as a perfect square.}
\end{aligned}$$

The equation describes a sphere centered at point $\left(0, \frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$.

Note:
Exercise:

Problem: Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Solution:

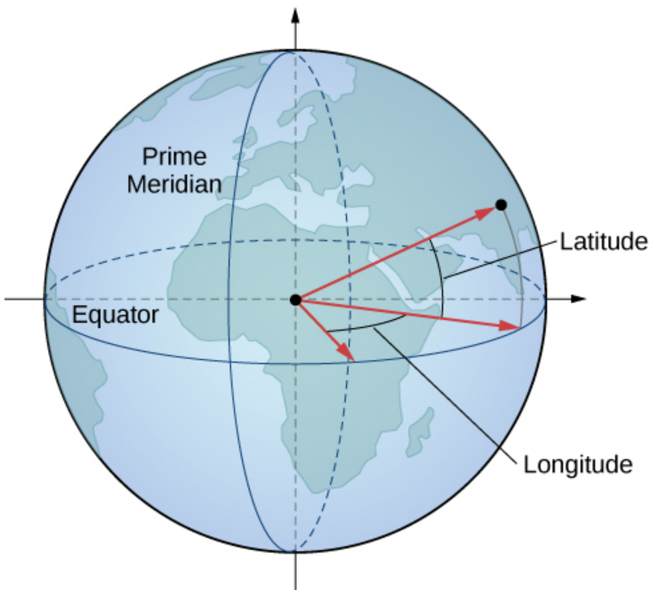
a. This is the set of all points 13 units from the origin. This set forms a sphere with radius 13. b. This set of points forms a half plane. The angle between the half plane and the positive *x*-axis is $\theta = \frac{2\pi}{3}$. c. Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \frac{\pi}{4}$ with the positive *z*-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

Hint

Think about what each component represents and what it means to hold that component constant.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in [link]. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.



In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive *z*-axis. The prime meridian represents the trace of the surface as it intersects the *xz*-plane. The equator is the trace of the sphere intersecting the *xy*-plane.

Example:
Exercise:

Problem:

Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is 40° N and the longitude is 83° W, which means that Columbus is 40° north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is 40°. In the same way, measuring from the prime meridian, Columbus lies 83° to the west. Express the location of Columbus in spherical coordinates.

Solution:

The radius of Earth is 4000 mi, so $\rho = 4000$. The intersection of the prime meridian and the equator lies on the positive *x*-axis. Movement to the west is then described with negative angle measures, which shows that $\theta = -83^\circ$, Because Columbus lies 40° north of the equator, it lies 50° south of the North Pole, so $\varphi = 50^\circ$. In spherical coordinates, Columbus lies at point $(4000, -83^\circ, 50^\circ)$.

Note:
Exercise:

Problem: Sydney, Australia is at 34°S and 151°E. Express Sydney's location in spherical coordinates.

Solution:

$(4000, 151^\circ, 124^\circ)$

Hint

Because Sydney lies south of the equator, we need to add 90° to find the angle measured from the positive *z*-axis.

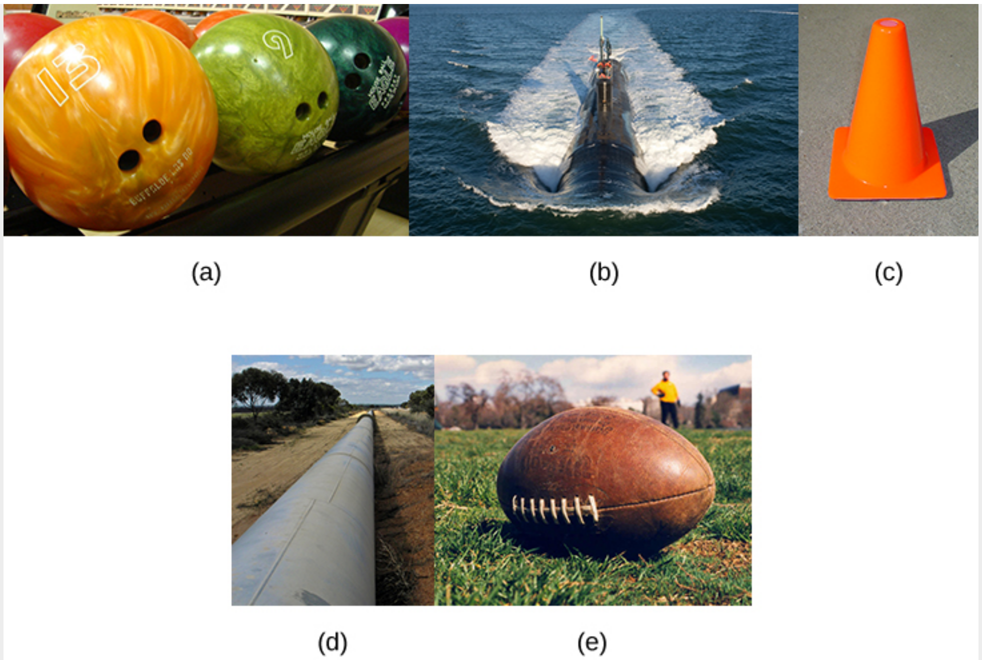
Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

Example:
Exercise:

Problem:
Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. *Note*: There is not enough information to set up or solve these problems; we simply select the coordinate system ([link]).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.



(credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

Solution:

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the *x*-, *y*- and *z*-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the *z*-axis with the axis of symmetry of the weight block.

b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The *z*-axis should probably point upward. The *x*- and *y*-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.

c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the *z*-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be best the best choice. In this case, however, we would likely choose to orient our *z*-axis with the center axis of the pipeline. The *x*-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the *z*-axis in a horizontal orientation, which is a little

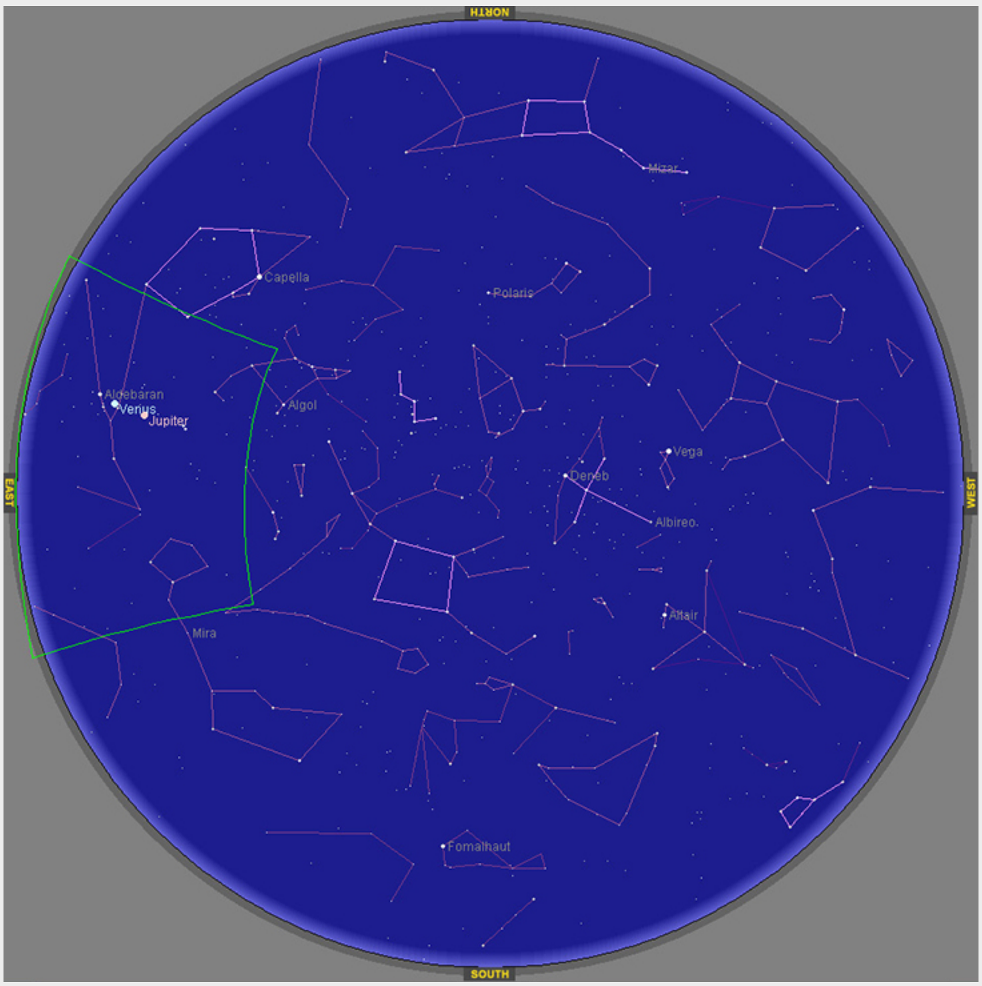
different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The *z*-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the *x*-axis is arbitrary.

Note:
Exercise:

Problem:

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Solution:

Spherical coordinates with the origin located at the center of the earth, the *z*-axis aligned with the North Pole, and the *x*-axis aligned with the prime meridian

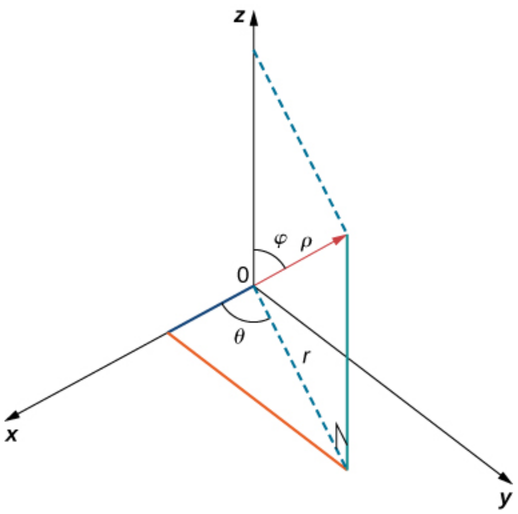
Hint

What kinds of symmetry are present in this situation?

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the *xy*-plane and z represents the point's projection onto the *z*-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$, and $z = z$.
- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive *z*-axis and line segment OP, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.
- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \frac{y}{x}$, and $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho\sin\varphi$, $\theta = \theta$, and $z = \rho\cos\varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2 + z^2}$, $\theta = \theta$, and $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$.

Use the following figure as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.



For the following exercises, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

Exercise:

Problem: $\left(4, \frac{\pi}{6}, 3\right)$

Solution:

$\left(2\sqrt{3}, 2, 3\right)$

Exercise:

Problem: $\left(3, \frac{\pi}{3}, 5\right)$

Exercise:

Problem: $\left(4, \frac{7\pi}{6}, 3\right)$

Solution:

$\left(-2\sqrt{3}, -2, 3\right)$

Exercise:

Problem: $(2, \pi, -4)$

For the following exercises, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

Exercise:

Problem: $\left(1, \sqrt{3}, 2\right)$

Solution:

$\left(2, \frac{\pi}{3}, 2\right)$

Exercise:

Problem: $(1, 1, 5)$

Exercise:

Problem: $(3, -3, 7)$

Solution:

$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right)$

Exercise:

Problem: $\left(-2\sqrt{2}, 2\sqrt{2}, 4\right)$

For the following exercises, the equation of a surface in cylindrical coordinates is given.

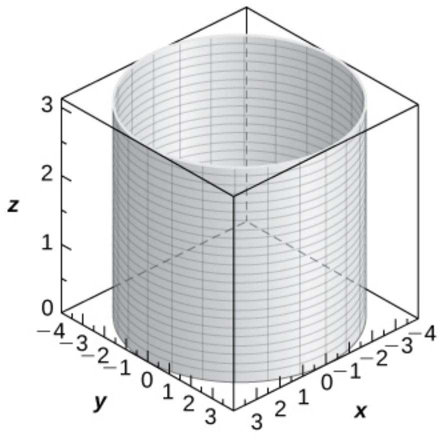
Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

Exercise:

Problem: **[T]** $r = 4$

Solution:

A cylinder of equation $x^2 + y^2 = 16$, with its center at the origin and rulings parallel to the *z*-axis,



Exercise:

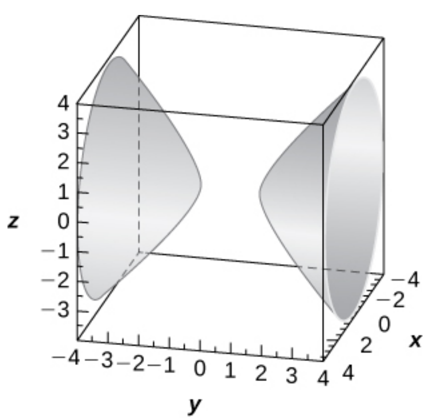
Problem: **[T]** $z = r^2 \cos^2 \theta$

Exercise:

Problem: **[T]** $r^2 \cos(2\theta) + z^2 + 1 = 0$

Solution:

Hyperboloid of two sheets of equation $-x^2 + y^2 - z^2 = 1$, with the *y*-axis as the axis of symmetry,



Exercise:

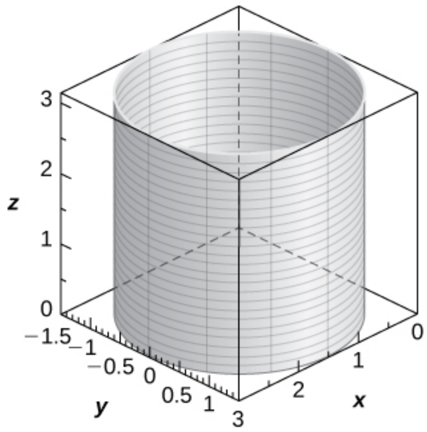
Problem: **[T]** $r = 3 \sin \theta$

Exercise:

Problem: **[T]** $r = 2 \cos \theta$

Solution:

Cylinder of equation $x^2 - 2x + y^2 = 0$, with a center at $(1, 0, 0)$ and radius 1, with rulings parallel to the *z*-axis,



Exercise:

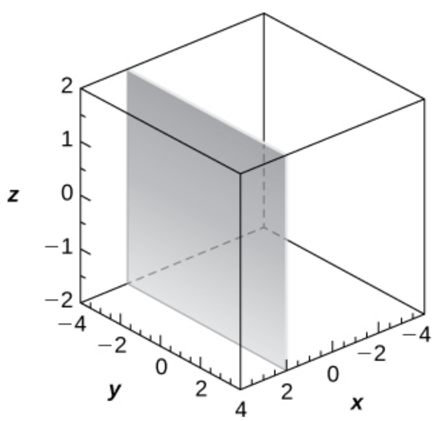
Problem: **[T]** $r^2 + z^2 = 5$

Exercise:

Problem: **[T]** $r = 2 \sec \theta$

Solution:

Plane of equation $x = 2$,



Exercise:

Problem: **[T]** $r = 3 \csc \theta$

For the following exercises, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

Exercise:

Problem: $z = 3$

Solution:

$z = 3$

Exercise:

Problem: $x = 6$

Exercise:

Problem: $x^2 + y^2 + z^2 = 9$

Solution:

$r^2 + z^2 = 9$

Exercise:

Problem: $y = 2x^2$

Exercise:

Problem: $x^2 + y^2 - 16x = 0$

Solution:

$r = 16 \cos \theta, r = 0$

Exercise:

Problem: $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

For the following exercises, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

Exercise:

Problem: $(3, 0, \pi)$

Solution:

$(0, 0, -3)$

Exercise:

Problem: $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$

Exercise:

Problem: $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution:

$\left(6, -6, \sqrt{2}\right)$

Exercise:

Problem: $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

For the following exercises, the rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

Exercise:

Problem: $(4, 0, 0)$

Solution:

$(4, 0, 90^\circ)$

Exercise:

Problem: $(-1, 2, 1)$

Exercise:

Problem: $(0, 3, 0)$

Solution:

$(3, 90^\circ, 90^\circ)$

Exercise:

Problem: $\left(-2, 2\sqrt{3}, 4\right)$

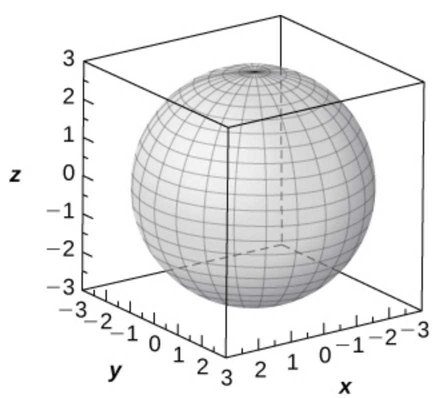
For the following exercises, the equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

Exercise:

Problem: [T] $\rho = 3$

Solution:

Sphere of equation $x^2 + y^2 + z^2 = 9$ centered at the origin with radius 3,



Exercise:

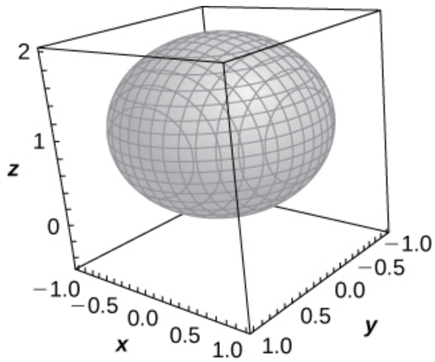
Problem: [T] $\varphi = \frac{\pi}{3}$

Exercise:

Problem: **[T]** $\rho = 2\cos\varphi$

Solution:

Sphere of equation $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0, 0, 1)$ with radius 1,



Exercise:

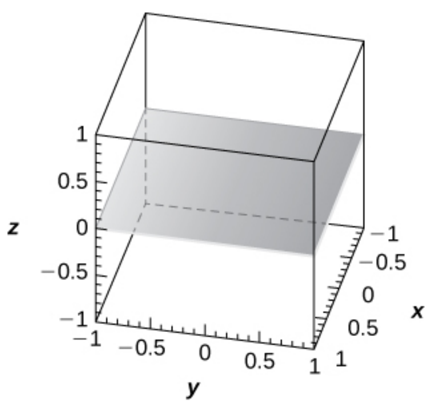
Problem: **[T]** $\rho = 4\csc\varphi$

Exercise:

Problem: **[T]** $\varphi = \frac{\pi}{2}$

Solution:

The *xy*-plane of equation $z = 0$,



Exercise:

Problem: **[T]** $\rho = 6\csc\varphi\sec\theta$

For the following exercises, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

Exercise:

Problem: $x^2 + y^2 - 3z^2 = 0,\ z \neq 0$

Solution:

$\varphi = \frac{\pi}{3}$ or $\varphi = \frac{2\pi}{3}$; Elliptic cone

Exercise:

Problem: $x^2 + y^2 + z^2 - 4z = 0$

Exercise:

Problem: $z = 6$

Solution:

$\rho \cos \varphi = 6$; Plane at $z = 6$

Exercise:

Problem: $x^2 + y^2 = 9$

For the following exercises, the cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

Exercise:

Problem: **[T]** $\left(1, \frac{\pi}{4}, 3\right)$

Solution:

$\left(\sqrt{10}, \frac{\pi}{4}, 0.3218\right)$

Exercise:

Problem: **[T]** $(5, \pi, 12)$

Exercise:

Problem: $\left(3, \frac{\pi}{2}, 3\right)$

Solution:

$\left(3\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$

Exercise:

Problem: $\left(3, -\frac{\pi}{6}, 3\right)$

For the following exercises, the spherical coordinates of a point are given. Find its associated cylindrical coordinates.

Exercise:

Problem: $\left(2, -\frac{\pi}{4}, \frac{\pi}{2}\right)$

Solution:

$\left(2, -\frac{\pi}{4}, 0\right)$

Exercise:

Problem: $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$

Exercise:

Problem: $\left(8, \frac{\pi}{3}, \frac{\pi}{2}\right)$

Solution:

$\left(8, \frac{\pi}{3}, 0\right)$

Exercise:

Problem: $\left(9, -\frac{\pi}{6}, \frac{\pi}{3}\right)$

For the following exercises, find the most suitable system of coordinates to describe the solids.

Exercise:

Problem:

The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

Solution:

Cartesian system, $\{(x, y, z) | 0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a\}$

Exercise:

Problem:

A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

Exercise:

Problem: A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

Solution:

Cylindrical system, $\left\{(r, \theta, z) \middle| r^2 + z^2 \leq 9, r \geq 3\cos\theta, 0 \leq \theta \leq 2\pi\right\}$

Exercise:

Problem:

A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

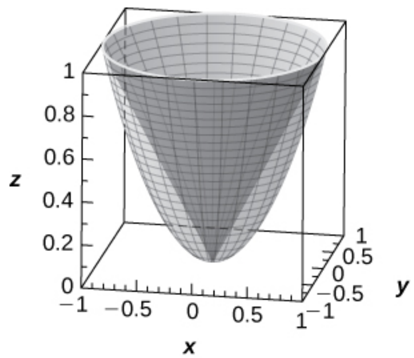
Exercise:

Problem:

[T] Use a CAS to graph in cylindrical coordinates the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$.

Solution:

The region is described by the set of points $\left\{(r,\theta,z) \middle| 0 \le r \le 1, 0 \le \theta \le 2\pi, r^2 \le z \le r\right\}$.



Exercise:

Problem:

[T] Use a CAS to graph in spherical coordinates the "ice cream-cone region" situated above the *xy*-plane between sphere $x^2+y^2+z^2=4$ and elliptical cone $x^2+y^2-z^2=0$.

Exercise:

Problem:

Washington, DC, is located at 39° N and 77° W (see the following figure). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.

Solution:

$(4000, -77^\circ, 51^\circ)$

Exercise:

Problem:

San Francisco is located at 37.78°N and 122.42°W. Assume the radius of Earth is 4000 mi. Express the location of San Francisco in spherical coordinates.

Exercise:

Problem:

Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17^\circ, 102.91^\circ)$.

Solution:

$43.17^{\circ}\mathrm{W}, 22.91^{\circ}\mathrm{S}$

Exercise:

Problem: Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38^{\circ}, 37.48^{\circ})$.

Exercise:

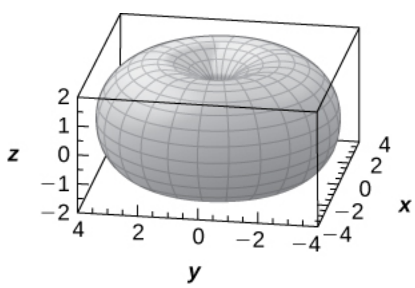
Problem:

[T] Consider the torus of equation $\left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 = 4R^2\left(x^2 + y^2\right)$, where $R \geq r > 0$.

a. Write the equation of the torus in spherical coordinates.
b. If $R = r$, the surface is called a *horn torus*. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R\sin\varphi$.
c. Use a CAS to graph the horn torus with $R = r = 2$ in spherical coordinates.

Solution:

a. $\rho = 0$, $\rho + R^2 - r^2 - 2R\sin\varphi = 0$;
c.



Exercise:

Problem:

[T] The “bumpy sphere” with an equation in spherical coordinates is $\rho = a + b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.

a. Show that the “bumpy sphere” is contained inside a sphere of equation $\rho = a + b$. Find the values of θ and φ at which the two surfaces intersect.
b. Use a CAS to graph the surface for $a = 14$, $b = 2$, $m = 4$, and $n = 6$ along with sphere $\rho = a + b$.
c. Find the equation of the intersection curve of the surface at b. with the cone $\varphi = \frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.

Chapter Review Exercises

For the following exercises, determine whether the statement is *true or false*. Justify the answer with a proof or a counterexample.

Exercise:

Problem: For vectors $\mathbf{a}$ and $\mathbf{b}$ and any given scalar c, $c\,(\mathbf{a}\cdot\mathbf{b}) = (c\mathbf{a})\cdot\mathbf{b}$.

Solution:

True

Exercise:

Problem: For vectors $\mathbf{a}$ and $\mathbf{b}$ and any given scalar c, $c\,(\mathbf{a}\ \times\ \mathbf{b}) = (c\mathbf{a})\ \times\ \mathbf{b}$.

Exercise:

Problem:

The symmetric equation for the line of intersection between two planes $x+y+z=2$ and $x+2y-4z=5$ is given by $-\frac{x-1}{6}=\frac{y-1}{5}=z$.

Solution:

False

Exercise:

Problem: If $\mathbf{a}\cdot\mathbf{b}=0$, then $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

For the following exercises, use the given vectors to find the quantities.

Exercise:

Problem: $\mathbf{a}=9\mathbf{i}-2\mathbf{j}, \mathbf{b}=-3\mathbf{i}+\mathbf{j}$

a. $3\mathbf{a}+\mathbf{b}$
b. $|\mathbf{a}|$
c. $\mathbf{a}\times|\mathbf{b}\times|\mathbf{a}$
d. $\mathbf{b}\times|\mathbf{a}$

Solution:

a. $\langle 24,-5\rangle$; b. $\sqrt{85}$; c. Can’t dot a vector with a scalar; d. -29

Exercise:

Problem: $\mathbf{a}=2\mathbf{i}+\mathbf{j}-9\mathbf{k}, \mathbf{b}=-\mathbf{i}+2\mathbf{k}, \mathbf{c}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$

a. $2\mathbf{a}-\mathbf{b}$
b. $|\mathbf{b}\ \times\ \mathbf{c}|$
c. $\mathbf{b}\ \times\ |\mathbf{b}\ \times\ \mathbf{c}|$
d. $\mathbf{c}\ \times\ |\mathbf{b}\ \times\ \mathbf{a}|$
e. $\text{proj}_{\mathbf{a}}\mathbf{b}$

Exercise:

Problem: Find the values of a such that vectors $\langle 2,4,a\rangle$ and $\langle 0,-1,a\rangle$ are orthogonal.

Solution:

$a = \pm 2$

For the following exercises, find the unit vectors.
Exercise:

Problem:

Find the unit vector that has the same direction as vector $\mathbf{v}$ that begins at $(0, -3)$ and ends at $(4, 10)$.

Exercise:

Problem:

Find the unit vector that has the same direction as vector $\mathbf{v}$ that begins at $(1, 4, 10)$ and ends at $(3, 0, 4)$.

Solution:

$\left\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right\rangle$

For the following exercises, find the area or volume of the given shapes.
Exercise:

Problem: The parallelogram spanned by vectors $\mathbf{a} = \langle 1, 13 \rangle$ and $\mathbf{b} = \langle 3, 21 \rangle$

Exercise:

Problem: The parallelepiped formed by $\mathbf{a} = \langle 1, 4, 1 \rangle$ and $\mathbf{b} = \langle 3, 6, 2 \rangle$, and $\mathbf{c} = \langle -2, 1, -5 \rangle$

Solution:

27

For the following exercises, find the vector and parametric equations of the line with the given properties.
Exercise:

Problem: The line that passes through point $(2, -3, 7)$ that is parallel to vector $\langle 1, 3, -2 \rangle$

Exercise:

Problem: The line that passes through points $(1, 3, 5)$ and $(-2, 6, -3)$

Solution:

$x = 1 - 3t, y = 3 + 3t, z = 5 - 8t, \mathbf{r}\,(t) = (1 - 3t)\mathbf{i} + 3\,(1 + t)\mathbf{j} + (5 - 8t)\mathbf{k}$

For the following exercises, find the equation of the plane with the given properties.
Exercise:

Problem: The plane that passes through point $(4, 7, -1)$ and has normal vector $\mathbf{n} = \langle 3, 4, 2 \rangle$

Exercise:

Problem: The plane that passes through points $(0, 1, 5), (2, -1, 6),\text{ and }(3, 2, 5)$.

Solution:

$-x + 3y + 8z = 43$

For the following exercises, find the traces for the surfaces in planes $x = k, y = k,$ and $z = k$. Then, describe and draw the surfaces.

Exercise:

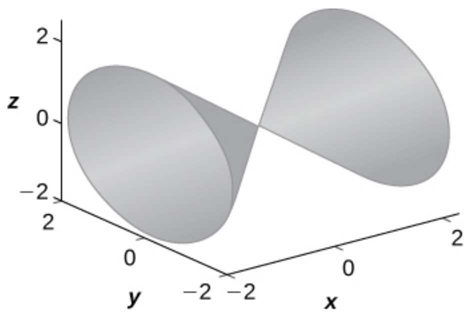
Problem: $9x^2 + 4y^2 - 16y + 36z^2 = 20$

Exercise:

Problem: $x^2 = y^2 + z^2$

Solution:

$x = k$ trace: $k^2 = y^2 + z^2$ is a circle, $y = k$ trace: $x^2 - z^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$), $z = k$ trace: $x^2 - y^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$). The surface is a cone.



For the following exercises, write the given equation in cylindrical coordinates and spherical coordinates.

Exercise:

Problem: $x^2 + y^2 + z^2 = 144$

Exercise:

Problem: $z = x^2 + y^2 - 1$

Solution:

Cylindrical: $z = r^2 - 1$, spherical: $\cos\varphi = \rho\sin^2\varphi - \frac{1}{\rho}$

For the following exercises, convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

Exercise:

Problem: $\rho^2\left(\sin^2(\varphi) - \cos^2(\varphi)\right) = 1$

Exercise:

Problem: $r^2 - 2r\cos(\theta) + z^2 = 1$

Solution:

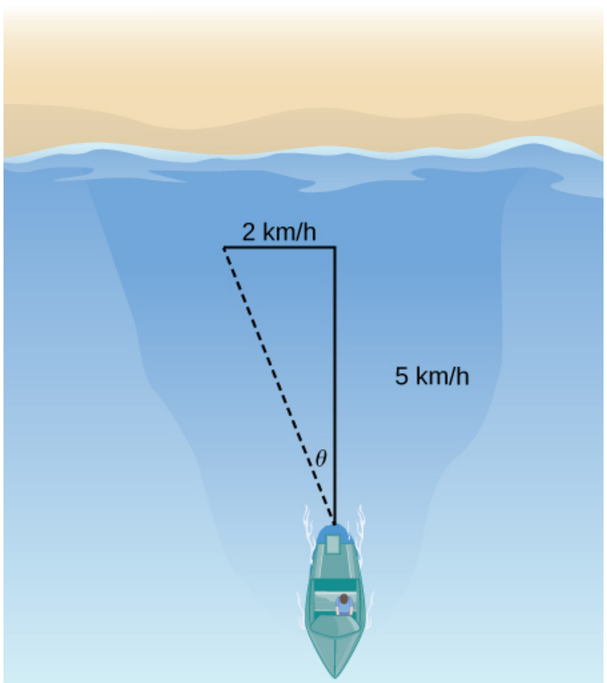
$x^2 - 2x + y^2 + z^2 = 1$, sphere

For the following exercises, consider a small boat crossing a river.

Exercise:

Problem:

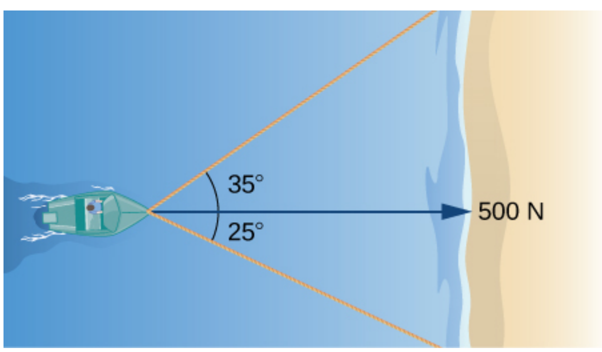
If the boat velocity is 5 km/h due north in still water and the water has a current of 2 km/h due west (see the following figure), what is the velocity of the boat relative to shore? What is the angle θ that the boat is actually traveling?



Exercise:

Problem:

When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of 25° and the other is at 35°. If the boat must be pulled straight and at a force of 500N, find the magnitude of force for each rope (see the following figure).



Solution:

331 N, and 244 N

Exercise:

Problem:

An airplane is flying in the direction of 52° east of north with a speed of 450 mph. A strong wind has a bearing 33° east of north with a speed of 50 mph. What is the resultant ground speed and bearing of the airplane?

Exercise:

Problem:

Calculate the work done by moving a particle from position $(1, 2, 0)$ to $(8, 4, 5)$ along a straight line with a force $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution:

15 J

The following problems consider your unsuccessful attempt to take the tire off your car using a wrench to loosen the bolts. Assume the wrench is 0.3 m long and you are able to apply a 200-N force.

Exercise:

Problem:

Because your tire is flat, you are only able to apply your force at a 60° angle. What is the torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

Exercise:

Problem:

Someone lends you a tire jack and you are now able to apply a 200-N force at an 80° angle. Is your resulting torque going to be more or less? What is the new resulting torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

Solution:

More, 59.09 J

Glossary

cylindrical coordinate system
: a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the *xy*-plane, and z represents the point's projection onto the *z*-axis

spherical coordinate system
: a way to describe a location in space with an ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive *z*-axis and line segment OP, where O is the origin and $0 \leq \varphi \leq \pi$

Vector-Valued Functions and Space Curves

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Note:
Definition
A **vector-valued function** is a function of the form
Equation:

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where the **component functions** *f*, *g*, and *h*, are real-valued functions of the parameter *t*. Vector-valued functions are also written in the form
Equation:

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle.$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

The parameter *t* can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of *t* might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of *t*. We often use *t* as a parameter because *t* can represent time.

Example:
Exercise:

Problem:
Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\mathbf{r}(0)$, $\mathbf{r}\left(\frac{\pi}{2}\right)$, and $\mathbf{r}\left(\frac{2\pi}{3}\right)$. Do any of these functions have domain restrictions?

a. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$
b. $\mathbf{r}(t) = 3\tan t\,\mathbf{i} + 4\sec t\,\mathbf{j} + 5t\,\mathbf{k}$

Solution:

a. To calculate each of the function values, substitute the appropriate value of t into the function:
Equation:

$$\begin{aligned}
\mathbf{r}(0) &= 4\cos(0)\mathbf{i} + 3\sin(0)\mathbf{j} \\
&= 4\mathbf{i} + 0\mathbf{j} = 4\mathbf{i} \\
\mathbf{r}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\mathbf{i} + 3\sin\left(\frac{\pi}{2}\right)\mathbf{j} \\
&= 0\mathbf{i} + 3\mathbf{j} = 3\mathbf{j} \\
\mathbf{r}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\mathbf{i} + 3\sin\left(\frac{2\pi}{3}\right)\mathbf{j} \\
&= 4\left(-\frac{1}{2}\right)\mathbf{i} + 3\left(\frac{\sqrt{3}}{2}\right)\mathbf{j} = -2\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}.
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ is all real numbers.

b. To calculate each of the function values, substitute the appropriate value of t into the function:
Equation:

$$\begin{aligned}
\mathbf{r}(0) &= 3\tan(0)\mathbf{i} + 4\sec(0)\mathbf{j} + 5(0)\mathbf{k} \\
&= 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} = 4\mathbf{j} \\
\mathbf{r}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\mathbf{i} + 4\sec\left(\frac{\pi}{2}\right)\mathbf{j} + 5\left(\frac{\pi}{2}\right)\mathbf{k}, \text{which does not exist} \\
\mathbf{r}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\mathbf{i} + 4\sec\left(\frac{2\pi}{3}\right)\mathbf{j} + 5\left(\frac{2\pi}{3}\right)\mathbf{k} \\
&= 3\left(-\sqrt{3}\right)\mathbf{i} + 4(-2)\mathbf{j} + \frac{10\pi}{3}\mathbf{k} \\
&= -3\sqrt{3}\mathbf{i} - 8\mathbf{j} + \frac{10\pi}{3}\mathbf{k}.
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\pi/2$, so the function is not defined for odd multiples of $\pi/2$. Therefore, $\text{dom}(\mathbf{r}(t)) = \left\{t \,\middle|\, t \neq \frac{(2n+1)\pi}{2}\right\}$, where n is any integer.

Note:
Exercise:

Problem:

For the vector-valued function $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (4t+1)\mathbf{j}$, evaluate $\mathbf{r}(0)$, $\mathbf{r}(1)$, and $\mathbf{r}(-4)$. Does this function have any domain restrictions?

Solution:

$\mathbf{r}(0) = \mathbf{j}$, $\mathbf{r}(1) = -2\mathbf{i} + 5\mathbf{j}$, $\mathbf{r}(-4) = 28\mathbf{i} - 15\mathbf{j}$

The domain of $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (4t+1)\mathbf{j}$ is all real numbers.

Hint

Substitute the appropriate values of t into the function.

[link] illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ consists of the set of all $(t, \mathbf{r}(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ consists of the set of all $(t, \mathbf{r}(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Example:
Exercise:

Problem:
Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

a. The plane curve represented by $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$
b. The plane curve represented by $\mathbf{r}(t) = 4\cos t^3\,\mathbf{i} + 3\sin t^3\,\mathbf{j}$, $0 \le t \le 2\pi$
c. The space curve represented by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4\pi$

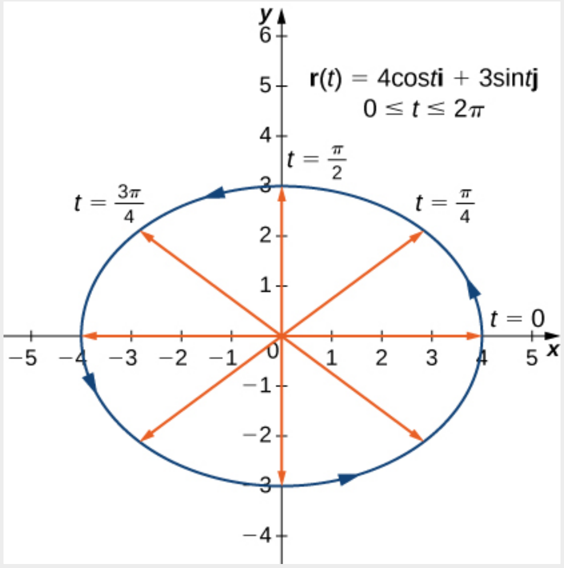
Solution:

a. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve ([link]). This curve turns out to be an ellipse centered at the origin.

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	π	$-4\mathbf{i}$

$\frac{\pi}{4}$	$2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
$\frac{\pi}{2}$	$3\mathbf{j}$	$\frac{3\pi}{2}$	$-3\mathbf{j}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{7\pi}{4}$	$2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
2π	$4\mathbf{i}$		

Table of Values for $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$



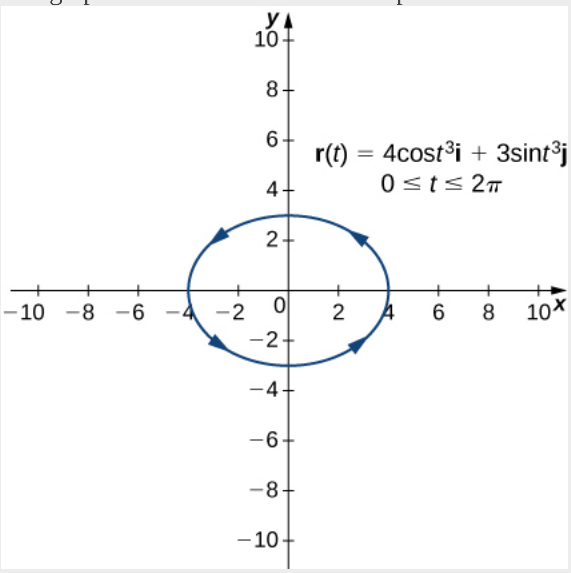
The graph of the first vector-valued function is an ellipse.

b. The table of values for $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$ is as follows:

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	π	$-4\mathbf{i}$
$\frac{\pi}{4}$	$2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
$\frac{\pi}{2}$	$3\mathbf{j}$	$\frac{3\pi}{2}$	$-3\mathbf{j}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{7\pi}{4}$	$2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
2π	$4\mathbf{i}$		

Table of Values for $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$

The graph of this curve is also an ellipse centered at the origin.



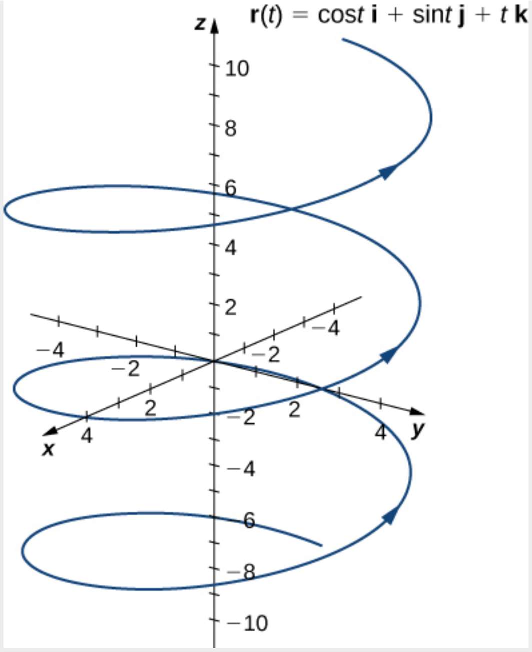
The graph of the second vector-valued function is also an ellipse.

c. We go through the same procedure for a three-dimensional vector function.

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	π	$-4\mathbf{j}+\pi\mathbf{k}$
$\frac{\pi}{4}$	$2\sqrt{2}\mathbf{i}+2\sqrt{2}\mathbf{j}+\frac{\pi}{4}\mathbf{k}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\mathbf{i}-2\sqrt{2}\mathbf{j}+\frac{5\pi}{4}\mathbf{k}$
$\frac{\pi}{2}$	$4\mathbf{j}+\frac{\pi}{2}\mathbf{k}$	$\frac{3\pi}{2}$	$-4\mathbf{j}+\frac{3\pi}{2}\mathbf{k}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\mathbf{i}+2\sqrt{2}\mathbf{j}+\frac{3\pi}{4}\mathbf{k}$	$\frac{7\pi}{4}$	$2\sqrt{2}\mathbf{i}-2\sqrt{2}\mathbf{j}+\frac{7\pi}{4}\mathbf{k}$
2π	$4\mathbf{i}+2\pi\mathbf{k}$		

Table of Values for $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k}$, $0\le t\le 4\pi$

The values then repeat themselves, except for the fact that the coefficient of k is always increasing ([link]). This curve is called a **helix**. Notice that if the **k** component is eliminated, then the function becomes $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, which is a unit circle centered at the origin.



The graph of the third vector-valued function is a helix.

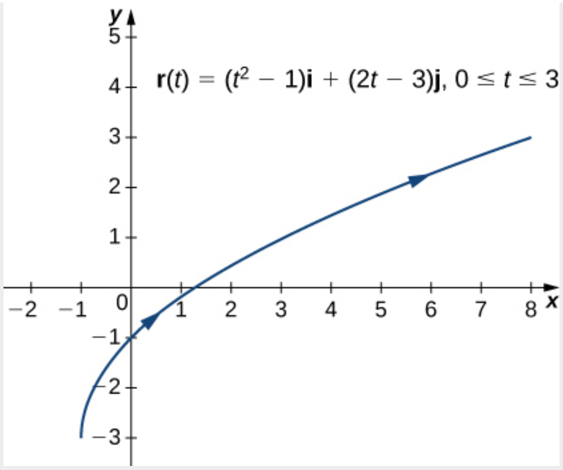
You may notice that the graphs in parts a. and b. are identical. This happens because the function describing curve b is a so-called **reparameterization** of the function describing curve a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace *t* with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which *t* is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization.

As mentioned, the name of the shape of the curve of the graph in [link]c. is a **helix** ([link]). The curve resembles a spring, with a circular cross-section looking down along the *z*-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\mathbf{r}\left(t\right)=4\cos t\mathbf{i}+3\sin t\mathbf{j}+t\mathbf{k}$ describes an elliptical helix. The projection of this helix into the $x,y\text{-plane}$ is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as *t* progresses from 0 to 4π.

Note:
Exercise:

Problem: Create a graph of the vector-valued function $\mathbf{r}\left(t\right)=\left(t^2-1\right)\mathbf{i}+\left(2t-3\right)\mathbf{j}$, $0\le t\le 3$.

Solution:



Hint

Start by making a table of values, then graph the vectors for each value of *t*.

At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\mathbf{r}\left(t\right)=f\left(t\right)\mathbf{i}+g\left(t\right)\mathbf{j}$, we can define $x=f\left(t\right)$ and $y=g\left(t\right)$. If a restriction exists on the values of *t* (for example, *t* is restricted to the interval $\left[a,b\right]$ for some constants $a<b$), then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y=f\left(x\right)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.

Limits and Continuity of a Vector-Valued Function

We now take a look at the **limit of a vector-valued function**. This is important to understand to study the calculus of vector-valued functions.

Note:
Definition
A vector-valued function **r** approaches the limit **L** as *t* approaches *a*, written
Equation:

$$\lim_{t\to a}\mathbf{r}\left(t\right)=\mathbf{L},$$

provided
Equation:

$$\lim_{t\to a}\left|\left|\mathbf{r}\left(t\right)-\mathbf{L}\right|\right|=0.$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Note:
Limit of a Vector-Valued Function
Let f, g, and h be functions of t. Then the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ as t approaches a is given by
Equation:

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j},$$

provided the limits $\lim_{t \to a} f(t)$ and $\lim_{t \to a} g(t)$ exist. Similarly, the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as t approaches a is given by
Equation:

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k},$$

provided the limits $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$ and $\lim_{t \to a} h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example:
Exercise:

Problem:
Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t \to 3} \mathbf{r}(t)$ for

a. $\mathbf{r}(t) = \left(t^2 - 3t + 4\right)\mathbf{i} + (4t + 3)\mathbf{j}$
b. $\mathbf{r}(t) = \frac{2t-4}{t+1}\mathbf{i} + \frac{t}{t^2+1}\mathbf{j} + (4t - 3)\mathbf{k}$

Solution:

a. Use [link] and substitute the value $t = 3$ into the two component expressions:
Equation:

$$\begin{aligned} \lim_{t \to 3} \mathbf{r}(t) &= \lim_{t \to 3} \left[\left(t^2 - 3t + 4\right)\mathbf{i} + (4t + 3)\mathbf{j}\right] \\ &= \left[\lim_{t \to 3} \left(t^2 - 3t + 4\right)\right]\mathbf{i} + \left[\lim_{t \to 3} (4t + 3)\right]\mathbf{j} \\ &= 4\mathbf{i} + 15\mathbf{j}. \end{aligned}$$

b. Use [link] and substitute the value $t = 3$ into the three component expressions:
Equation:

$$\begin{aligned} \lim_{t \to 3} \mathbf{r}(t) &= \lim_{t \to 3} \left(\frac{2t-4}{t+1}\mathbf{i} + \frac{t}{t^2+1}\mathbf{j} + (4t - 3)\mathbf{k}\right) \\ &= \left[\lim_{t \to 3} \left(\frac{2t-4}{t+1}\right)\right]\mathbf{i} + \left[\lim_{t \to 3} \left(\frac{t}{t^2+1}\right)\right]\mathbf{j} + \left[\lim_{t \to 3} (4t - 3)\right]\mathbf{k} \\ &= \frac{1}{2}\mathbf{i} + \frac{3}{10}\mathbf{j} + 9\mathbf{k}. \end{aligned}$$

Note:
Exercise:

Problem: Calculate $\lim_{t \to -2} \mathbf{r}(t)$ for the function $\mathbf{r}(t) = \sqrt{t^2 - 3t - 1}\,\mathbf{i} + (4t + 3)\mathbf{j} + \sin\frac{(t+1)\pi}{2}\,\mathbf{k}$.

Solution:

$\lim_{t \to -2} \mathbf{r}(t) = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

Hint

Use [link] from the preceding theorem.

Now that we know how to calculate the limit of a vector-valued function, we can define continuity at a point for such a function.

Note:
Definition
Let *f*, *g*, and *h* be functions of *t*. Then, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is continuous at point $t = a$ if the following three conditions hold:

1. $\mathbf{r}(a)$ exists
2. $\lim_{t \to a} \mathbf{r}(t)$ exists
3. $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

Similarly, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at point $t = a$ if the following three conditions hold:

1. $\mathbf{r}(a)$ exists
2. $\lim_{t \to a} \mathbf{r}(t)$ exists
3. $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

Key Concepts

- A vector-valued function is a function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where the component functions *f*, *g*, and *h* are real-valued functions of the parameter *t*.
- The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is called a *plane curve*. The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, or $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t),$
- **Limit of a vector-valued function**
  $\lim\limits_{t\to a}\mathbf{r}(t) = \left[\lim\limits_{t\to a} f(t)\right]\mathbf{i} + \left[\lim\limits_{t\to a} g(t)\right]\mathbf{j}$ or $\lim\limits_{t\to a}\mathbf{r}(t) = \left[\lim\limits_{t\to a} f(t)\right]\mathbf{i} + \left[\lim\limits_{t\to a} g(t)\right]\mathbf{j} + \left[\lim\limits_{t\to a} h(t)\right]\mathbf{k}$

Exercise:

Problem:

Give the component functions $x = f(t)$ and $y = g(t)$ for the vector-valued function $\mathbf{r}(t) = 3\sec t\mathbf{i} + 2\tan t\mathbf{j}$.

Solution:

$f(t) = 3\sec t, g(t) = 2\tan t$

Exercise:

Problem: Given $\mathbf{r}(t) = 3\sec t\mathbf{i} + 2\tan t\mathbf{j}$, find the following values (if possible).

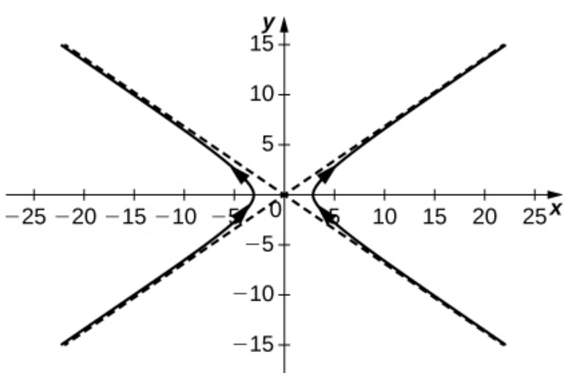
a. $\mathbf{r}\left(\frac{\pi}{4}\right)$
b. $\mathbf{r}(\pi)$
c. $\mathbf{r}\left(\frac{\pi}{2}\right)$

Exercise:

Problem:

Sketch the curve of the vector-valued function $\mathbf{r}(t) = 3\sec t\mathbf{i} + 2\tan t\mathbf{j}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.

Solution:



Exercise:

Problem: Evaluate $\lim\limits_{t\to 0}\left\langle e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right\rangle$.

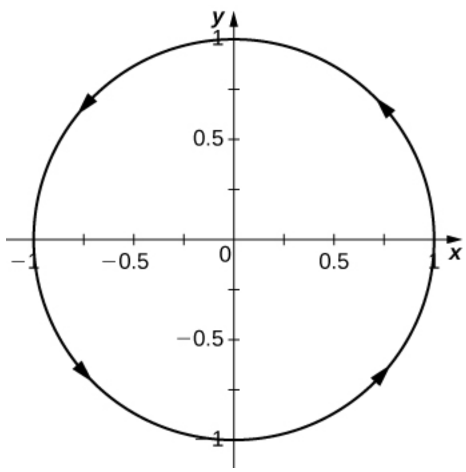
Exercise:

Problem: Given the vector-valued function $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, find the following values:

a. $\lim_{t \to \frac{\pi}{4}} \mathbf{r}(t)$
b. $\mathbf{r}\left(\frac{\pi}{3}\right)$
c. Is $\mathbf{r}(t)$ continuous at $t = \frac{\pi}{3}$?
d. Graph $\mathbf{r}(t)$.

Solution:

a. $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$, b. $\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, c. Yes, the limit as t approaches $\pi/3$ is equal to $\mathbf{r}\left(\pi/3\right)$, d.



Exercise:

Problem: Given the vector-valued function $\mathbf{r}(t) = \langle t, t^2 + 1 \rangle$, find the following values:

a. $\lim_{t \to -3} \mathbf{r}(t)$
b. $\mathbf{r}(-3)$
c. Is $\mathbf{r}(t)$ continuous at $x = -3$?
d. $\mathbf{r}(t+2) - \mathbf{r}(t)$

Exercise:

Problem: Let $\mathbf{r}(t) = e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln t\,\mathbf{k}$. Find the following values:

a. $\mathbf{r}\left(\frac{\pi}{4}\right)$
b. $\lim_{t \to \pi/4} \mathbf{r}(t)$
c. Is $\mathbf{r}(t)$ continuous at $t = t = \frac{\pi}{4}$?

Solution:

a. $\left\langle e^{\pi/4}, \frac{\sqrt{2}}{2}, \ln\left(\frac{\pi}{4}\right) \right\rangle$; b. $\left\langle e^{\pi/4}, \frac{\sqrt{2}}{2}, \ln\left(\frac{\pi}{4}\right) \right\rangle$; c. Yes

Find the limit of the following vector-valued functions at the indicated value of t.
Exercise:

Problem: $\lim_{t\to 4}\left\langle \sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan\left(\frac{\pi}{t}\right)\right\rangle$

Exercise:

Problem: $\lim_{t\to \pi/2} \mathbf{r}(t)$ for $\mathbf{r}(t) = e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln t\,\mathbf{k}$

Solution:

$\left\langle e^{\pi/2}, 1, \ln\left(\frac{\pi}{2}\right)\right\rangle$

Exercise:

Problem: $\lim_{t\to\infty}\left\langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t)\right\rangle$

Exercise:

Problem: $\lim_{t\to e^2}\left\langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln\left(t^2\right)}\right\rangle$

Solution:

$2e^2\,\mathbf{i} + \frac{2}{e^4}\,\mathbf{j} + 2\,\mathbf{k}$

Exercise:

Problem: $\lim_{t\to \pi/6}\left\langle \cos^2 t, \sin^2 t, 1\right\rangle$

Exercise:

Problem: $\lim_{t\to\infty} \mathbf{r}(t)$ for $\mathbf{r}(t) = 2e^{-t}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \ln(t-1)\,\mathbf{k}$

Solution:

The limit does not exist because the limit of $\ln(t-1)$ as t approaches infinity does not exist.

Exercise:

Problem:

Describe the curve defined by the vector-valued function $\mathbf{r}(t) = (1+t)\,\mathbf{i} + (2+5t)\,\mathbf{j} + (-1+6t)\,\mathbf{k}$.

Find the domain of the vector-valued functions.

Exercise:

Problem: Domain: $\mathbf{r}(t) = \left\langle t^2, \tan t, \ln t\right\rangle$

Solution:

$t > 0, t \neq (2k+1)\frac{\pi}{2}$, where k is an integer

Exercise:

Problem: Domain: $\mathbf{r}(t) = \left\langle t^2, \sqrt{t-3}, \frac{3}{2t+1}\right\rangle$

Exercise:

Problem: Domain: $\mathbf{r}(t) = \left\langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2) \right\rangle$

Solution:

$t > 3, t \neq n\pi$, where *n* is an integer

Let $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$ and use it to answer the following questions.

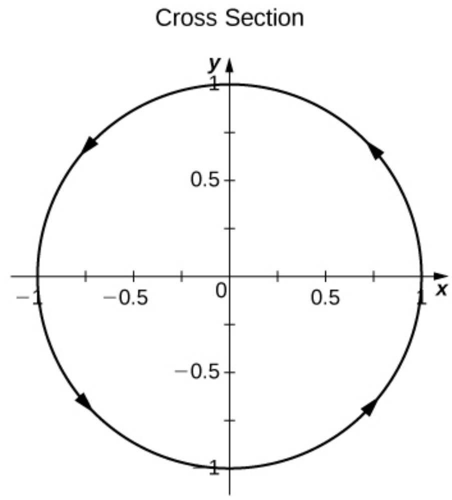
Exercise:

Problem: For what values of *t* is $\mathbf{r}(t)$ continuous?

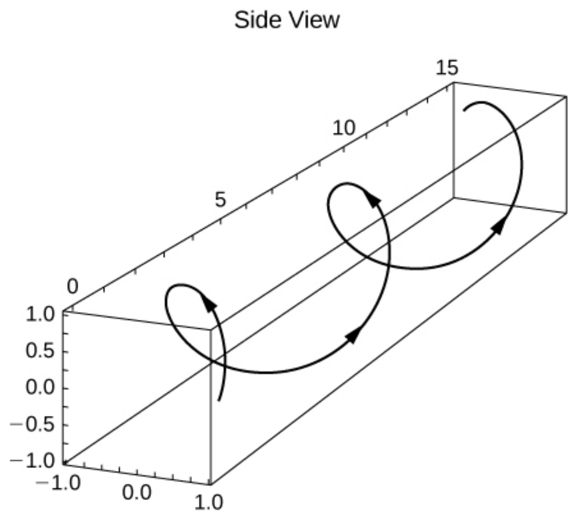
Exercise:

Problem: Sketch the graph of $\mathbf{r}(t)$.

Solution:



(a)



(b)

Exercise:

Problem: Find the domain of $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$.

Exercise:

Problem: For what values of *t* is $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$ continuous?

Solution:

All *t* such that $t \in (1, \infty)$

Eliminate the parameter *t*, write the equation in Cartesian coordinates, then sketch the graphs of the vector-valued functions.

Exercise:

Problem:

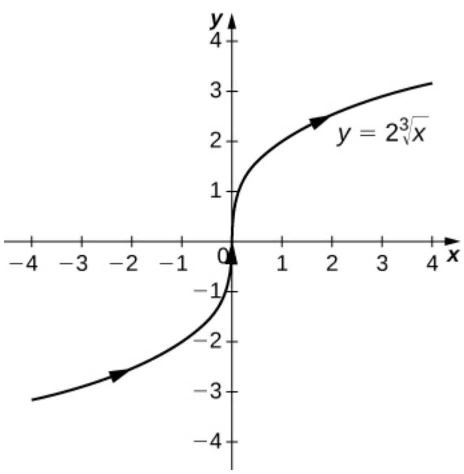
$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j}$ (*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for *x* in terms of *t* and substitute this result into the second equation.)

Exercise:

Problem: $\mathbf{r}(t) = t^3\,\mathbf{i} + 2t\,\mathbf{j}$

Solution:

$y = 2\sqrt[3]{x}$, a variation of the cube-root function



Exercise:

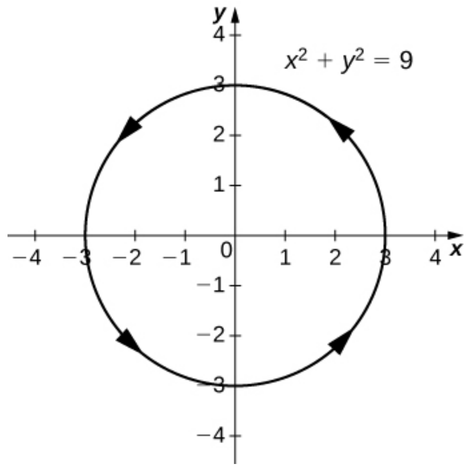
Problem: $\mathbf{r}(t) = 2\,(\sinh t)\,\mathbf{i} + 2\,(\cosh t)\,\mathbf{j}, t > 0$

Exercise:

Problem: $\mathbf{r}(t) = 3\,(\cos t)\,\mathbf{i} + 3\,(\sin t)\,\mathbf{j}$

Solution:

$x^2 + y^2 = 9$, a circle centered at $(0, 0)$ with radius 3, and a counterclockwise orientation



Exercise:

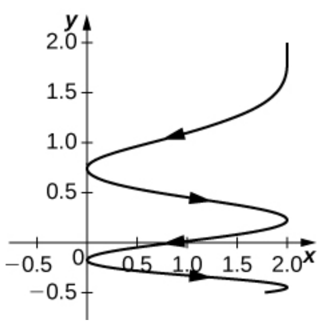
Problem: $\mathbf{r}(t) = \langle 3\sin t, 3\cos t \rangle$

Use a graphing utility to sketch each of the following vector-valued functions:

Exercise:

Problem: [T] $\mathbf{r}(t) = 2\cos t^2\,\mathbf{i} + (2 - \sqrt{t})\,\mathbf{j}$

Solution:



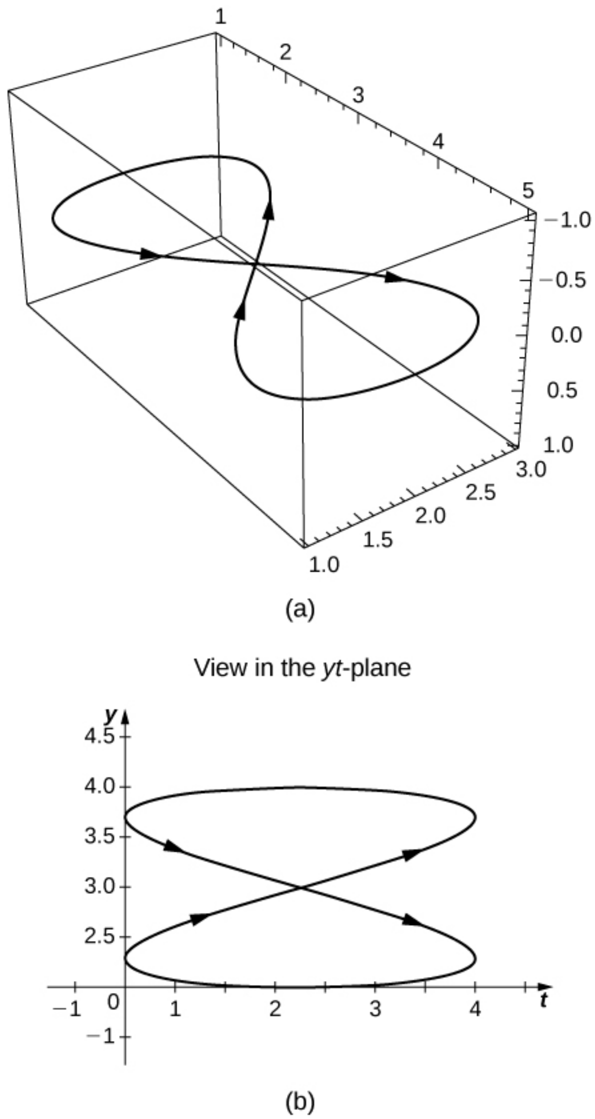
Exercise:

Problem: [T] $\mathbf{r}(t) = \left\langle e^{\cos(3t)}, e^{-\sin(t)} \right\rangle$

Exercise:

Problem: [T] $\mathbf{r}(t) = \langle 2 - \sin(2t), 3 + 2\cos t \rangle$

Solution:



Find a vector-valued function that traces out the given curve in the indicated direction.

Exercise:

Problem: $4x^2 + 9y^2 = 36$; clockwise and counterclockwise

Exercise:

Problem: $\mathbf{r}(t) = \langle t, t^2 \rangle$; from left to right

---

Solution:

For left to right, $y = x^2$, where t increases

Exercise:

Problem: The line through P and Q where P is $(1, 4, -2)$ and Q is $(3, 9, 6)$

Consider the curve described by the vector-valued function
$\mathbf{r}(t) = \left(50e^{-t}\cos t\right)\mathbf{i} + \left(50e^{-t}\sin t\right)\mathbf{j} + (5 - 5e^{-t})\mathbf{k}$.

Exercise:

Problem: What is the initial point of the path corresponding to $\mathbf{r}(0)$?

Solution:

$(50, 0, 0)$

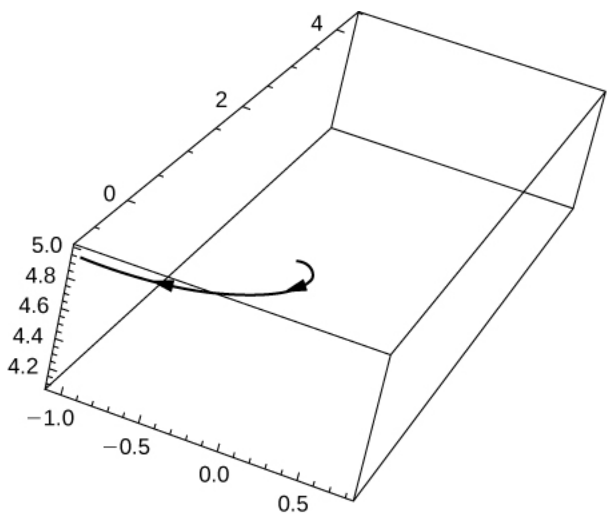
Exercise:

Problem: What is $\lim\limits_{t\to\infty} \mathbf{r}(t)$?

Exercise:

Problem: **[T]** Use technology to sketch the curve.

Solution:



Exercise:

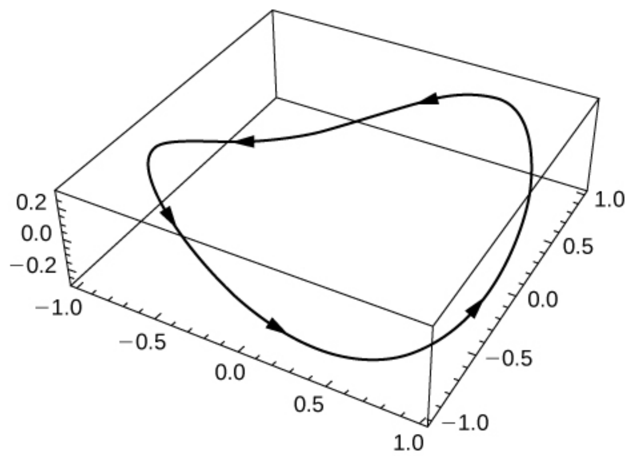
Problem: Eliminate the parameter *t* to show that $z = 5 - \frac{r}{10}$ where $r^2 = x^2 + y^2$.

Exercise:

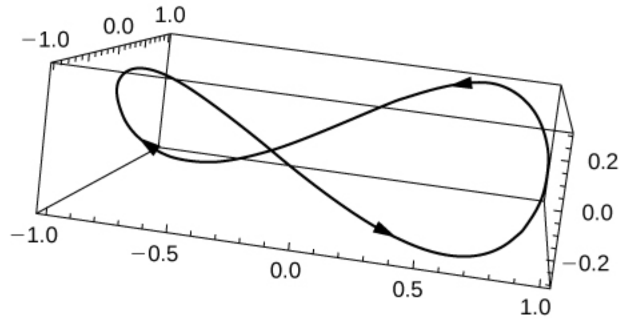
Problem:

[T] Let $r(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0.3\sin(2t)\,\mathbf{k}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

Solution:



(a)



(b)

Exercise:

Problem:

[T] Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the "valley." Then, use technology to graph the equation.

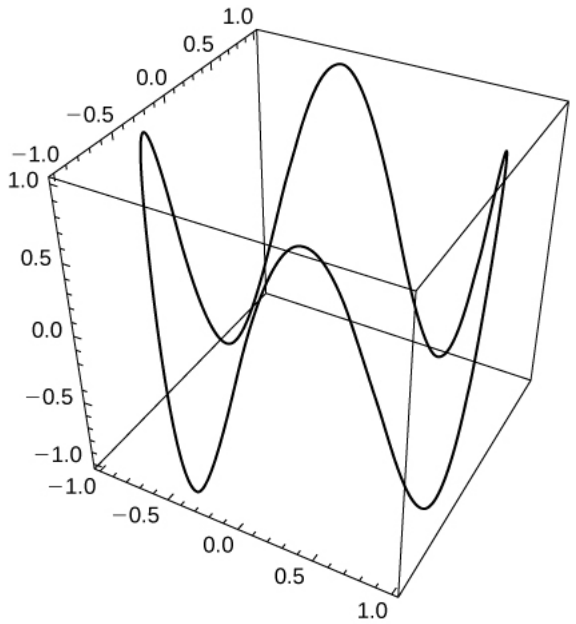
Exercise:

Problem:

Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

Solution:

One possibility is $r(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin(4t)\,\mathbf{k}$. By increasing the coefficient of *t* in the third component, the number of turning points will increase.



Exercise:

Problem:

a. Graph the curve $\mathbf{r}(t) = (4 + \cos(18t))\cos(t)\,\mathbf{i} + (4 + \cos(18t)\sin(t))\,\mathbf{j} + 0.3\sin(18t)\,\mathbf{k}$ using two viewing angles of your choice to see the overall shape of the curve.
b. Does the curve resemble a "slinky"?
c. What changes to the equation should be made to increase the number of coils of the slinky?

Glossary

component functions
: the component functions of the vector-valued function $\mathbf{r}\,(t) = f\,(t)\,\mathbf{i} + g\,(t)\,\mathbf{j}$ are $f\,(t)$ and $g\,(t)$, and the component functions of the vector-valued function $\mathbf{r}\,(t) = f\,(t)\,\mathbf{i} + g\,(t)\,\mathbf{j} + h\,(t)\,\mathbf{k}$ are $f\,(t)$, $g\,(t)$ and $h\,(t)$

helix
: a three-dimensional curve in the shape of a spiral

limit of a vector-valued function
: a vector-valued function $\mathbf{r}(t)$ has a limit $\mathbf{L}$ as t approaches a if $\lim\limits_{t\to a} \left|\mathbf{r}\,(t) - \mathbf{L}\right| = 0$

plane curve
: the set of ordered pairs $(f\,(t), g\,(t))$ together with their defining parametric equations $x = f\,(t)$ and $y = g\,(t)$

reparameterization
: an alternative parameterization of a given vector-valued function

space curve
: the set of ordered triples $(f\,(t), g\,(t), h\,(t))$ together with their defining parametric equations $x = f\,(t)$, $y = g\,(t)$ and $z = h\,(t)$

vector parameterization
: any representation of a plane or space curve using a vector-valued function

vector-valued function

a function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where the component functions *f*, *g*, and *h* are real-valued functions of the parameter *t*

Calculus of Vector-Valued Functions

- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Note:
Definition
The **derivative of a vector-valued function** $\mathbf{r}\left(t\right)$ is
Equation:

$$\mathbf{r}'\left(t\right)=\lim_{\Delta t\to 0}\frac{\mathbf{r}\left(t+\Delta t\right)-\mathbf{r}\left(t\right)}{\Delta t},$$

provided the limit exists. If $\mathbf{r}'\left(t\right)$ exists, then $\mathbf{r}$ is differentiable at t. If $\mathbf{r}'\left(t\right)$ exists for all t in an open interval $\left(a,b\right)$, then $\mathbf{r}$ is differentiable over the interval $\left(a,b\right)$. For the function to be differentiable over the closed interval $\left[a,b\right]$, the following two limits must exist as well:
Equation:

$$\mathbf{r}'\left(a\right)=\lim_{\Delta t\to 0^{+}}\frac{\mathbf{r}\left(a+\Delta t\right)-\mathbf{r}\left(a\right)}{\Delta t}\text{ and }\mathbf{r}'\left(b\right)=\lim_{\Delta t\to 0^{-}}\frac{\mathbf{r}\left(b+\Delta t\right)-\mathbf{r}\left(b\right)}{\Delta t}.$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example:
Exercise:

Problem:
Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function
Equation:

$$\mathbf{r}\left(t\right)=\left(3t+4\right)\mathbf{i}+\left({t}^{2}-4t+3\right)\mathbf{j}.$$

Solution:

Let’s use [link]:
Equation:

$$\begin{array}{cc}\hfill \mathbf{r}'\left(t\right)& =\underset{\Delta t\to 0}{\lim}\frac{\mathbf{r}\left(t+\Delta t\right)-\mathbf{r}\left(t\right)}{\Delta t}\hfill \\ & =\underset{\Delta t\to 0}{\lim}\frac{\left[\left(3\left(t+\Delta t\right)+4\right)\mathbf{i}+\left({\left(t+\Delta t\right)}^{2}-4\left(t+\Delta t\right)+3\right)\mathbf{j}\right]-\left[\left(3t+4\right)\mathbf{i}+\left({t}^{2}-4t+3\right)\mathbf{j}\right]}{\Delta t}\hfill \\ & =\underset{\Delta t\to 0}{\lim}\frac{\left(3t+3\Delta t+4\right)\mathbf{i}-\left(3t+4\right)\mathbf{i}+\left({t}^{2}+2t\Delta t+{\left(\Delta t\right)}^{2}-4t-4\Delta t+3\right)\mathbf{j}-\left({t}^{2}-4t+3\right)\mathbf{j}}{\Delta t}\hfill \\ & =\underset{\Delta t\to 0}{\lim}\frac{\left(3\Delta t\right)\mathbf{i}+\left(2t\Delta t+{\left(\Delta t\right)}^{2}-4\Delta t\right)\mathbf{j}}{\Delta t}\hfill \\ & =\underset{\Delta t\to 0}{\lim}\left(3\mathbf{i}+\left(2t+\Delta t-4\right)\mathbf{j}\right)\hfill \\ & =3\mathbf{i}+\left(2t-4\right)\mathbf{j}.\hfill \end{array}$$

Note:
Exercise:

Problem:

Use the definition to calculate the derivative of the function
$\mathbf{r}\left(t\right)=\left(2{t}^{2}+3\right)\mathbf{i}+\left(5t-6\right)\mathbf{j}.$

Solution:

$\mathbf{r}'\left(t\right)=4t\mathbf{i}+5\mathbf{j}$

Hint

Use [link].

Notice that in the calculations in [link], we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Note:
Differentiation of Vector-Valued Functions
Let *f*, *g*, and *h* be differentiable functions of *t*.

i. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$, then $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}$.
ii. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, then $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$.

Example:
Exercise:

Problem:
Calculating the Derivative of Vector-Valued Functions

Use [link] to calculate the derivative of each of the following functions.

a. $\mathbf{r}(t)=(6t+8)\mathbf{i}+\left(4t^2+2t-3\right)\mathbf{j}$
b. $\mathbf{r}(t)=3\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}$
c. $\mathbf{r}(t)=e^t\sin t\,\mathbf{i}+e^t\cos t\,\mathbf{j}-e^{2t}\mathbf{k}$

Solution:

We use [link] and what we know about differentiating functions of one variable.

a. The first component of $\mathbf{r}(t)=(6t+8)\mathbf{i}+\left(4t^2+2t-3\right)\mathbf{j}$ is $f(t)=6t+8$. The second component is $g(t)=4t^2+2t-3$. We have $f'(t)=6$ and $g'(t)=8t+2$, so the theorem gives $\mathbf{r}'(t)=6\mathbf{i}+(8t+2)\mathbf{j}$.
b. The first component is $f(t)=3\cos t$ and the second component is $g(t)=4\sin t$. We have $f'(t)=-3\sin t$ and $g'(t)=4\cos t$, so we obtain $\mathbf{r}'(t)=-3\sin t\,\mathbf{i}+4\cos t\,\mathbf{j}$.
c. The first component of $\mathbf{r}(t)=e^t\sin t\,\mathbf{i}+e^t\cos t\,\mathbf{j}-e^{2t}\mathbf{k}$ is $f(t)=e^t\sin t$, the second component is $g(t)=e^t\cos t$, and the third component is $h(t)=-e^{2t}$. We have $f'(t)=e^t(\sin t+\cos t)$, $g'(t)=e^t(\cos t-\sin t)$, and $h'(t)=-2e^{2t}$, so the theorem gives $\mathbf{r}'(t)=e^t(\sin t+\cos t)\mathbf{i}+e^t(\cos t-\sin t)\mathbf{j}-2e^{2t}\mathbf{k}$.

Note:
Exercise:

Problem: Calculate the derivative of the function
Equation:

$$\mathbf{r}(t) = (t \ln t)\,\mathbf{i} + \left(5e^{t}\right)\mathbf{j} + (\cos t - \sin t)\,\mathbf{k}.$$

Solution:

$\mathbf{r}'(t) = (1 + \ln t)\,\mathbf{i} + 5e^{t}\,\mathbf{j} - (\sin t + \cos t)\,\mathbf{k}$

Hint

Identify the component functions and use [link].

We can extend to vector-valued functions the properties of the derivative that we presented in the Introduction to Derivatives. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions: (1) for a real-valued function multiplied by a vector-valued function, (2) for the dot product of two vector-valued functions, and (3) for the cross product of two vector-valued functions.

Note:
Properties of the Derivative of Vector-Valued Functions
Let **r** and **u** be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.
Equation:

$$\begin{array}{llcll}
\text{i.} & \frac{d}{dt}[c\mathbf{r}(t)] & = & c\mathbf{r}'(t) & \text{Scalar multiple} \\
\text{ii.} & \frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] & = & \mathbf{r}'(t) \pm \mathbf{u}'(t) & \text{Sum and difference} \\
\text{iii.} & \frac{d}{dt}[f(t)\mathbf{u}(t)] & = & f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) & \text{Scalar product} \\
\text{iv.} & \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] & = & \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t) & \text{Dot product} \\
\text{v.} & \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] & = & \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t) & \text{Cross product} \\
\text{vi.} & \frac{d}{dt}[\mathbf{r}(f(t))] & = & \mathbf{r}'(f(t)) \cdot f'(t) & \text{Chain rule} \\
\text{vii.} & \text{If } \mathbf{r}(t) \cdot \mathbf{r}(t) & = & c, \text{ then } \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0. &
\end{array}$$

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule from the Introduction to Derivatives. Let $\mathbf{u}(t) = g(t)\mathbf{i} + h(t)\mathbf{j}$. Then

Equation:

$$\begin{aligned} \frac{d}{dt}[f(t)\mathbf{u}(t)] &= \frac{d}{dt}[f(t)(g(t)\mathbf{i} + h(t)\mathbf{j})] \\ &= \frac{d}{dt}[f(t)g(t)\mathbf{i} + f(t)h(t)\mathbf{j}] \\ &= \frac{d}{dt}[f(t)g(t)]\mathbf{i} + \frac{d}{dt}[f(t)h(t)]\mathbf{j} \\ &= (f'(t)g(t) + f(t)g'(t))\mathbf{i} + (f'(t)h(t) + f(t)h'(t))\mathbf{j} \\ &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t). \end{aligned}$$

To prove property iv. let $\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j}$ and $\mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$. Then

Equation:

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) \\ &= f_1{}'(t)f_2(t) + g_1{}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\ &= (f_1{}'\mathbf{i} + g_1{}'\mathbf{j})\cdot(f_2\mathbf{i} + g_2\mathbf{j}) + (f_1\mathbf{i} + g_1\mathbf{j})\cdot(f_2{}'\mathbf{i} + g_2{}'\mathbf{j}) \\ &= \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t). \end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

Equation:

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] &= \frac{d}{dt}[c] \\ \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\ 2\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\ \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0. \end{aligned}$$

□

Now for some examples using these properties.

Example:

Exercise:

Problem:

Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions
Equation:

$$\mathbf{r}(t) = (6t + 8)\mathbf{i} + \left(4t^2 + 2t - 3\right)\mathbf{j} + 5t\,\mathbf{k}$$

and
Equation:

$$\mathbf{u}(t) = \left(t^2 - 3\right)\mathbf{i} + (2t + 4)\mathbf{j} + \left(t^3 - 3t\right)\mathbf{k},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$
b. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)]$

Solution:

a. We have $\mathbf{r}'(t) = 6\mathbf{i} + (8t + 2)\mathbf{j} + 5\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \left(3t^2 - 3\right)\mathbf{k}$. Therefore, according to property iv.:
Equation:

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t) \\ &= (6\mathbf{i} + (8t + 2)\mathbf{j} + 5\mathbf{k}) \cdot \left(\left(t^2 - 3\right)\mathbf{i} + (2t + 4)\mathbf{j} + \left(t^3 - 3t\right)\mathbf{k}\right) \\ &\quad + \left((6t + 8)\mathbf{i} + \left(4t^2 + 2t - 3\right)\mathbf{j} + 5t\mathbf{k}\right) \cdot \left(2t\mathbf{i} + 2\mathbf{j} + \left(3t^2 - 3\right)\mathbf{k}\right) \\ &= 6\left(t^2 - 3\right) + (8t + 2)(2t + 4) + 5\left(t^3 - 3t\right) \\ &\quad + 2t(6t + 8) + 2\left(4t^2 + 2t - 3\right) + 5t\left(3t^2 - 3\right) \\ &= 20t^3 + 42t^2 + 26t - 16. \end{aligned}$$

b. First, we need to adapt property v. for this problem:
Equation:

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] = \mathbf{u}'(t) \times \mathbf{u}'(t) + \mathbf{u}(t) \times \mathbf{u}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\mathbf{u}''(t)$ represents the second derivative of $\mathbf{u}(t)$:

Equation:

$$\mathbf{u}''(t) = \frac{d}{dt}[\mathbf{u}'(t)] = \frac{d}{dt}[2t\,\mathbf{i} + 2\mathbf{j} + (3t^2 - 3)\,\mathbf{k}] = 2\,\mathbf{i} + 6t\,\mathbf{k}.$$

Therefore,

Equation:

$$\begin{aligned}
\tfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] &= \mathbf{0} + ((t^2-3)\,\mathbf{i} + (2t+4)\mathbf{j} + (t^3-3t)\,\mathbf{k}) \times (2\,\mathbf{i} + 6t\,\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2-3 & 2t+4 & t^3-3t \\ 2 & 0 & 6t \end{vmatrix} \\
&= 6t\,(2t+4)\,\mathbf{i} - (6t\,(t^2-3) - 2\,(t^3-3t))\,\mathbf{j} - 2\,(2t+4)\,\mathbf{k} \\
&= (12t^2+24t)\,\mathbf{i} + (12t-4t^3)\,\mathbf{j} - (4t+8)\,\mathbf{k}.
\end{aligned}$$

Note:
Exercise:

Problem:

Given the vector-valued functions $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - e^{2t}\,\mathbf{k}$ and $\mathbf{u}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, calculate $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}'(t)]$ and $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{r}(t)]$.

Solution:

$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 8e^{4t}$

$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{r}(t)]$
$= -(e^{2t}(\cos t + 2\sin t) + \cos 2t)\,\mathbf{i} + (e^{2t}(2t+1) - \sin 2t)\,\mathbf{j} + (t\cos t + \sin t - \cos 2t)\,\mathbf{k}$

Hint

Follow the same steps as in [link].

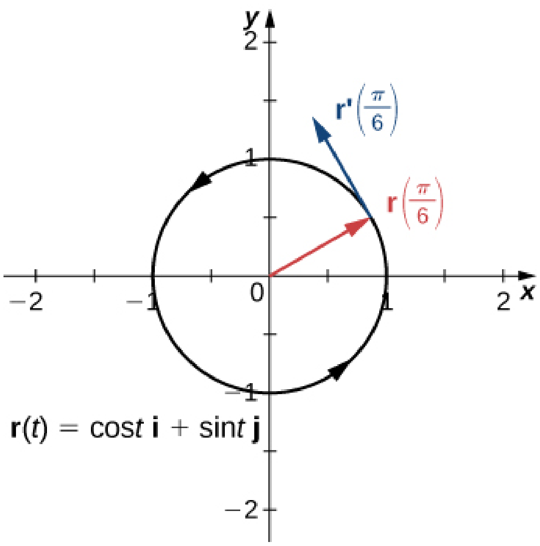
Tangent Vectors and Unit Tangent Vectors

Recall from the Introduction to Derivatives that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-

valued function $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. The derivative of this function is $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. If we substitute the value $t = \pi/6$ into both functions we get
Equation:

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \quad \text{and} \quad \mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}.$$

The graph of this function appears in [link], along with the vectors $\mathbf{r}\left(\frac{\pi}{6}\right)$ and $\mathbf{r}'\left(\frac{\pi}{6}\right)$.



The tangent line at a point is calculated from the derivative of the vector-valued function $\mathbf{r}(t)$.

Notice that the vector $\mathbf{r}'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \pi/6$. This is an example of a **tangent vector** to the plane curve defined by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$.

Note:
Definition
Let C be a curve defined by a vector-valued function **r,** and assume that $\mathbf{r}'(t)$ exists when $t = t_0$. A tangent vector **v** at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector **v** is tangent to curve *C*. Vector $\mathbf{r}'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\mathbf{r}'(t) \neq \mathbf{0}$. The **principal unit tangent vector** at *t* is defined to be
Equation:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

provided $\|\mathbf{r}'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\mathbf{r}'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example:
Exercise:

Problem:
Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$
b. $\mathbf{u}(t) = \left(3t^2 + 2t\right)\mathbf{i} + \left(2 - 4t^3\right)\mathbf{j} + (6t + 5)\,\mathbf{k}$

Solution:

a.

$$\begin{array}{lrcl} \text{First step:} & \mathbf{r}'(t) & = & -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} \\ \text{Second step:} & \|\mathbf{r}'(t)\| & = & \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\ \text{Third step:} & \mathbf{T}(t) & = & \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}}{1} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} \end{array}$$

b.

$$\begin{array}{lrcl} \text{First step:} & \mathbf{u}'(t) & = & (6t + 2)\,\mathbf{i} - 12t^2\,\mathbf{j} + 6\,\mathbf{k} \\ \text{Second step:} & \|\mathbf{u}'(t)\| & = & \sqrt{(6t + 2)^2 + \left(-12t^2\right)^2 + 6^2} \\ & & = & \sqrt{144t^4 + 36t^2 + 24t + 40} \\ & & = & 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\ \text{Third step:} & \mathbf{T}(t) & = & \frac{\mathbf{u}'(t)}{\|\mathbf{u}'(t)\|} = \frac{(6t+2)\,\mathbf{i} - 12t^2\,\mathbf{j} + 6\,\mathbf{k}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\ & & = & \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{i} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{j} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{k} \end{array}$$

Note:

Exercise:

Problem: Find the unit tangent vector for the vector-valued function
Equation:

$$\mathbf{r}\left(t\right) = \left(t^2 - 3\right)\mathbf{i} + \left(2t + 1\right)\mathbf{j} + \left(t - 2\right)\mathbf{k}.$$

Solution:

$\mathbf{T}\left(t\right) = \frac{2t}{\sqrt{4t^2+5}}\mathbf{i} + \frac{2}{\sqrt{4t^2+5}}\mathbf{j} + \frac{1}{\sqrt{4t^2+5}}\mathbf{k}$

Hint

Follow the same steps as in [link].

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time *t*, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Note:
Definition
Let *f*, *g*, and *h* be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral of a vector-valued function** $\mathbf{r}\left(t\right) = f\left(t\right)\mathbf{i} + g\left(t\right)\mathbf{j}$ is
   Equation:

   $$\int [f\left(t\right)\mathbf{i} + g\left(t\right)\mathbf{j}]dt = \left[\int f\left(t\right)dt\right]\mathbf{i} + \left[\int g\left(t\right)dt\right]\mathbf{j}.$$

   The **definite integral of a vector-valued function** is
   Equation:

$$\int_a^b [f(t)\mathbf{i} + g(t)\mathbf{j}]dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j}.$$

2. The indefinite integral of a vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is
   Equation:

$$\int [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}.$$

   The definite integral of the vector-valued function is
   Equation:

$$\int_a^b [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}.$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have
Equation:

$$\int f(t)dt = F(t) + C_1 \text{ and } \int g(t)dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then
Equation:

$$\begin{aligned}\int [f(t)\mathbf{i} + g(t)\mathbf{j}]dt &= \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} \\ &= (F(t) + C_1)\mathbf{i} + (G(t) + C_2)\mathbf{j} \\ &= F(t)\mathbf{i} + G(t)\mathbf{j} + C_1\mathbf{i} + C_2\mathbf{j} \\ &= F(t)\mathbf{i} + G(t)\mathbf{j} + \mathbf{C},\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$. Therefore, the integration constant becomes a constant vector.

Example:
Exercise:

Problem:
Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\int \left[\left(3t^2+2t\right)\mathbf{i}+(3t-6)\mathbf{j}+\left(6t^3+5t^2-4\right)\mathbf{k}\right]dt$

b. $\int \left[\left\langle t,t^2,t^3\right\rangle \times \left\langle t^3,t^2,t\right\rangle\right]dt$

c. $\int_0^{\pi/3} \left[\sin 2t\,\mathbf{i}+\tan t\,\mathbf{j}+e^{-2t}\mathbf{k}\right]dt$

Solution:

a. We use the first part of the definition of the integral of a space curve:
Equation:

$$\begin{aligned}&\int \left[\left(3t^2+2t\right)\mathbf{i}+(3t-6)\mathbf{j}+\left(6t^3+5t^2-4\right)\mathbf{k}\right]dt\\ &\qquad=\left[\int 3t^2+2t\,dt\right]\mathbf{i}+\left[\int 3t-6dt\right]\mathbf{j}+\left[\int 6t^3+5t^2-4dt\right]\mathbf{k}\\ &\qquad=\left(t^3+t^2\right)\mathbf{i}+\left(\tfrac{3}{2}t^2-6t\right)\mathbf{j}+\left(\tfrac{3}{2}t^4+\tfrac{5}{3}t^3-4t\right)\mathbf{k}+\mathbf{C}.\end{aligned}$$

b. First calculate $\left\langle t,t^2,t^3\right\rangle \times \left\langle t^3,t^2,t\right\rangle$:
Equation:

$$\begin{aligned}\left\langle t,t^2,t^3\right\rangle \times \left\langle t^3,t^2,t\right\rangle &=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ t&t^2&t^3\\ t^3&t^2&t\end{vmatrix}\\ &=\left(t^2\left(t\right)-t^3\left(t^2\right)\right)\mathbf{i}-\left(t^2-t^3\left(t^3\right)\right)\mathbf{j}+\left(t\left(t^2\right)-t^2\left(t^3\right)\right)\mathbf{k}\\ &=\left(t^3-t^5\right)\mathbf{i}+\left(t^6-t^2\right)\mathbf{j}+\left(t^3-t^5\right)\mathbf{k}.\end{aligned}$$

Next, substitute this back into the integral and integrate:
Equation:

$$\begin{aligned}\int \left[\left\langle t,t^2,t^3\right\rangle \times \left\langle t^3,t^2,t\right\rangle\right]dt &=\int \left(t^3-t^5\right)\mathbf{i}+\left(t^6-t^2\right)\mathbf{j}+\left(t^3-t^5\right)\mathbf{k}\,dt\\ &=\left(\frac{t^4}{4}-\frac{t^6}{6}\right)\mathbf{i}+\left(\frac{t^7}{7}-\frac{t^3}{3}\right)\mathbf{j}+\left(\frac{t^4}{4}-\frac{t^6}{6}\right)\mathbf{k}+\mathbf{C}.\end{aligned}$$

c. Use the second part of the definition of the integral of a space curve:
Equation:

$$\int_0^{\pi/3} \left[\sin 2t\,\mathbf{i} + \tan t\,\mathbf{j} + e^{-2t}\,\mathbf{k}\right] dt$$

$$= \left[\int_0^{\pi/3} \sin 2t dt\right]\mathbf{i} + \left[\int_0^{\pi/3} \tan t dt\right]\mathbf{j} + \left[\int_0^{\pi/3} e^{-2t} dt\right]\mathbf{k}$$

$$= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\mathbf{i} - \left(\ln\left(\cos t\right)\right)\Big|_0^{\pi/3}\mathbf{j} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\mathbf{k}$$

$$= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\mathbf{i} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln\left(\cos 0\right)\right)\mathbf{j} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\mathbf{k}$$

$$= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\mathbf{i} - \left(-\ln 2\right)\mathbf{j} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\mathbf{k}$$

$$= \tfrac{3}{4}\mathbf{i} + \left(\ln 2\right)\mathbf{j} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\mathbf{k}.$$

Note:
Exercise:

Problem: Calculate the following integral:
Equation:

$$\int_1^3 \left[(2t+4)\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j}\right] dt.$$

Solution:

$$\int_1^3 \left[(2t+4)\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j}\right] dt = 16\,\mathbf{i} + 10\,\mathbf{j}$$

Hint

Use the definition of the definite integral of a plane curve.

Key Concepts

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation from the Introduction to Derivatives also apply to vector-valued functions.
- The derivative of a vector-valued function $\mathbf{r}(t)$ is also a tangent vector to the curve. The unit tangent vector $\mathbf{T}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.

- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**
  $\mathbf{r}'(t) = \lim\limits_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$
- **Principal unit tangent vector**
  $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$
- **Indefinite integral of a vector-valued function**
  $\displaystyle\int [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}$
- **Definite integral of a vector-valued function**
  $\displaystyle\int_a^b [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}$

Compute the derivatives of the vector-valued functions.

Exercise:

Problem: $\mathbf{r}(t) = t^3\mathbf{i} + 3t^2\mathbf{j} + \frac{t^3}{6}\mathbf{k}$

Solution:

$\left\langle 3t^2, 6t, \frac{1}{2}t^2 \right\rangle$

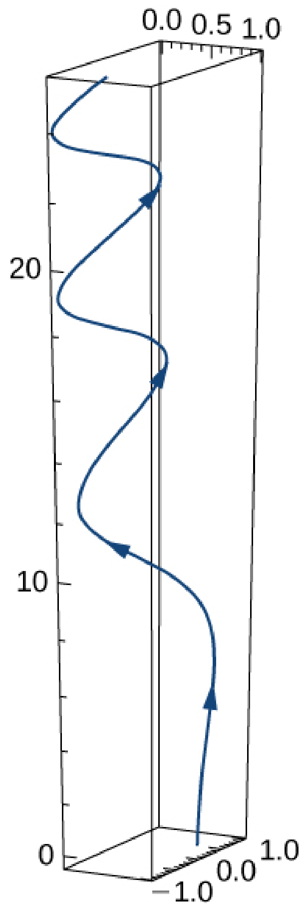
Exercise:

Problem: $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + e^t\mathbf{k}$

Exercise:

Problem:

$\mathbf{r}(t) = e^{-t}\mathbf{i} + \sin(3t)\mathbf{j} + 10\sqrt{t}\mathbf{k}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.



Solution:

$\left\langle -e^{-t}, 3\cos(3t), \frac{5}{\sqrt{t}} \right\rangle$

Exercise:

Problem:$\mathbf{r}(t) = e^t\,\mathbf{i} + 2e^t\,\mathbf{j} + \mathbf{k}$

Exercise:

Problem:$\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Solution:

$\langle 0, 0, 0 \rangle$

Exercise:

Problem:$\mathbf{r}(t) = te^t\,\mathbf{i} + t\ln(t)\,\mathbf{j} + \sin(3t)\,\mathbf{k}$

Exercise:

Problem:$\mathbf{r}(t) = \frac{1}{t+1}\,\mathbf{i} + \arctan(t)\,\mathbf{j} + \ln t^3\,\mathbf{k}$

Solution:

$$\left\langle \frac{-1}{(t+1)^2}, \frac{1}{1+t^2}, \frac{3}{t} \right\rangle$$

Exercise:

Problem:$\mathbf{r}(t) = \tan(2t)\,\mathbf{i} + \sec(2t)\,\mathbf{j} + \sin^2(t)\,\mathbf{k}$

Exercise:

Problem:$\mathbf{r}(t) = 3\,\mathbf{i} + 4\sin(3t)\,\mathbf{j} + t\cos(t)\,\mathbf{k}$

Solution:

$$\langle 0, 12\cos(3t), \cos t - t\sin t\rangle$$

Exercise:

Problem:$\mathbf{r}(t) = t^2\,\mathbf{i} + te^{-2t}\,\mathbf{j} - 5e^{-4t}\,\mathbf{k}$

For the following problems, find a tangent vector at the indicated value of *t*.

Exercise:

Problem:$\mathbf{r}(t) = t\,\mathbf{i} + \sin(2t)\,\mathbf{j} + \cos(3t)\,\mathbf{k}; t = \frac{\pi}{3}$

Solution:

$$\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$$

Exercise:

Problem:$\mathbf{r}(t) = 3t^3\,\mathbf{i} + 2t^2\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}; t = 1$

Exercise:

Problem:$\mathbf{r}(t) = 3e^t\,\mathbf{i} + 2e^{-3t}\,\mathbf{j} + 4e^{2t}\,\mathbf{k}; t = \ln(2)$

Solution:

$$\frac{1}{\sqrt{1060.5625}}\left\langle 6, -\frac{3}{4}, 32\right\rangle$$

Exercise:

Problem:$\mathbf{r}(t) = \cos(2t)\,\mathbf{i} + 2\sin t\,\mathbf{j} + t^2\,\mathbf{k}; t = \frac{\pi}{2}$

Find the unit tangent vector for the following parameterized curves.

Exercise:

Problem:$\mathbf{r}(t) = 6\,\mathbf{i} + \cos(3t)\,\mathbf{j} + 3\sin(4t)\,\mathbf{k}$, $0 \le t < 2\pi$

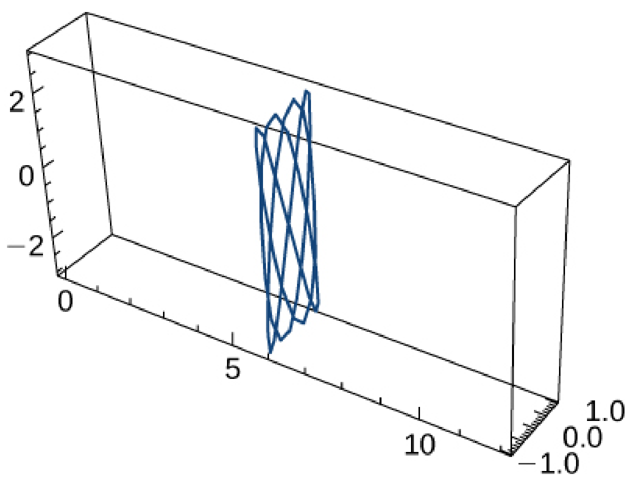
Solution:

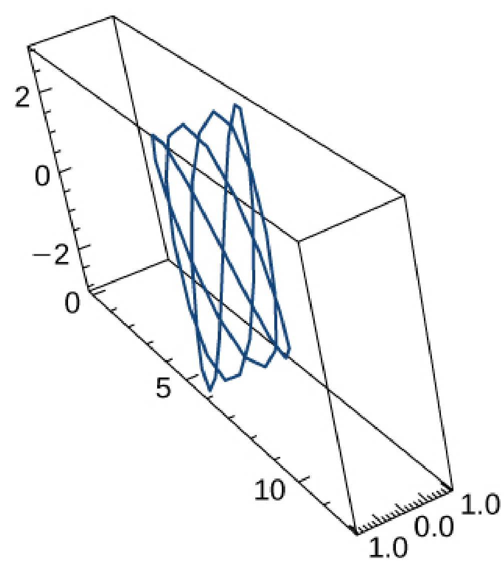
$\frac{1}{\sqrt{9\sin^2(3t)+144\cos^2(4t)}}\langle 0, -3\sin(3t), 12\cos(4t)\rangle$

Exercise:

Problem:

$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin t\,\mathbf{k}$, $0 \le t < 2\pi$. Two views of this curve are presented here:





Exercise:

Problem:$\mathbf{r}(t) = 3\cos(4t)\,\mathbf{i} + 3\sin(4t)\,\mathbf{j} + 5t\,\mathbf{k}, 1 \le t \le 2$

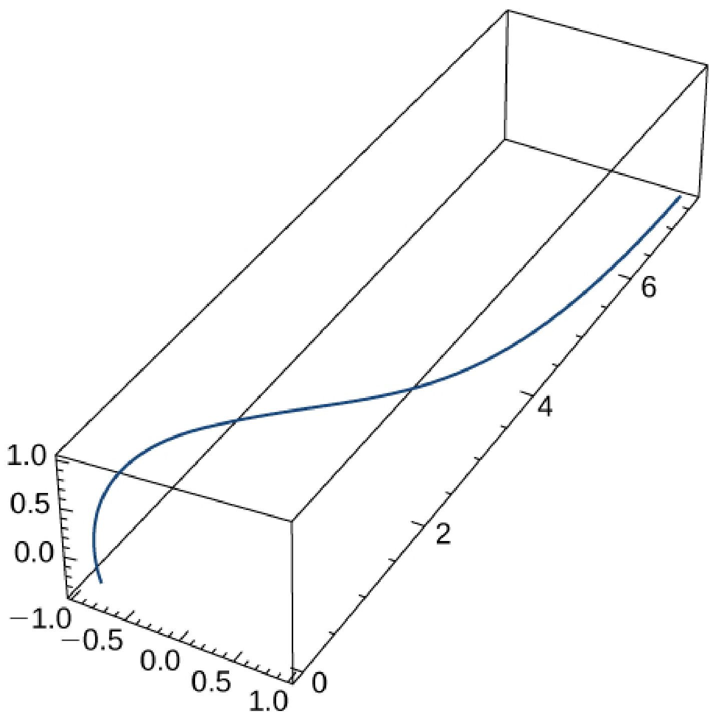
Solution:

$\mathbf{T}(t) = \frac{-12}{13}\sin(4t)\,\mathbf{i} + \frac{12}{13}\cos(4t)\,\mathbf{j} + \frac{5}{13}\,\mathbf{k}$

Exercise:

Problem:$\mathbf{r}(t) = t\,\mathbf{i} + 3t\,\mathbf{j} + t^2\,\mathbf{k}$

Let $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - t^4\,\mathbf{k}$ and $s(t) = \sin(t)\,\mathbf{i} + e^t\,\mathbf{j} + \cos(t)\,\mathbf{k}$. Here is the graph of the function:



Find the following.

Exercise:

Problem: $\frac{d}{dt}\left[r\left(t^2\right)\right]$

Solution:

$\left\langle 2t, 4t^3, -8t^7 \right\rangle$

Exercise:

Problem: $\frac{d}{dt}\left[t^2 \cdot s(t)\right]$

Exercise:

Problem: $\frac{d}{dt}[r(t) \cdot s(t)]$

Solution:

$\sin(t) + 2te^t - 4t^3\cos(t) + t\cos(t) + t^2e^t + t^4\sin(t)$

Exercise:

Problem:

Compute the first, second, and third derivatives of $\mathbf{r}(t) = 3t\,\mathbf{i} + 6\ln(t)\,\mathbf{j} + 5e^{-3t}\,\mathbf{k}$.

Exercise:

Problem: Find $\mathbf{r}\prime(t) \cdot \mathbf{r}\prime\prime(t)$ for $\mathbf{r}(t) = -3t^5\,\mathbf{i} + 5t\,\mathbf{j} + 2t^2\,\mathbf{k}$.

Solution:

$900t^7 + 16t$

Exercise:

The acceleration function, initial velocity, and initial position of a particle are $\mathbf{a}(t) = -5\cos t\,\mathbf{i} - 5\sin t\,\mathbf{j}$, $\mathbf{v}(0) = 9\,\mathbf{i} + 2\,\mathbf{j}$, and $\mathbf{r}(0) = 5\,\mathbf{i}$.

Problem: Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

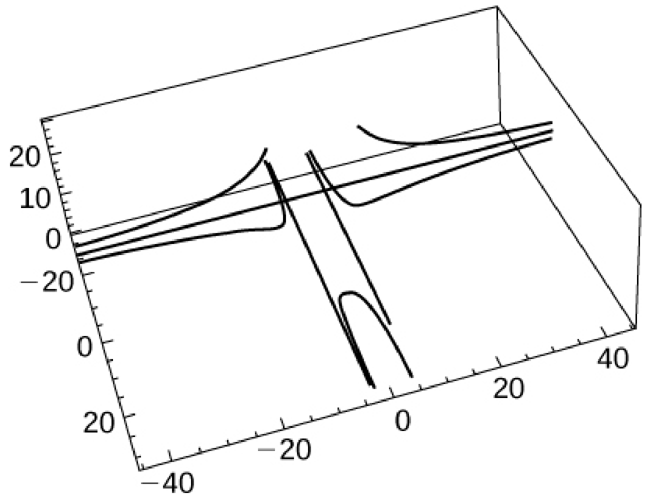
Exercise:

Problem: The position vector of a particle is $\mathbf{r}(t) = 5\sec(2t)\,\mathbf{i} - 4\tan(t)\,\mathbf{j} + 7t^2\,\mathbf{k}$.

a. Graph the position function and display a view of the graph that illustrates the asymptotic behavior of the function.
b. Find the velocity as t approaches but is not equal to $\pi/4$ (if it exists).

Solution:

a.



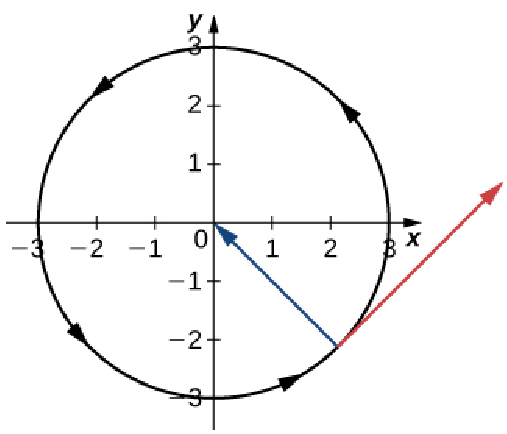
b. Undefined or infinite

Exercise:

Problem:

Find the velocity and the speed of a particle with the position function $\mathbf{r}(t) = \left(\frac{2t-1}{2t+1}\right)\mathbf{i} + \ln(1-4t^2)\mathbf{j}$. The speed of a particle is the magnitude of the velocity and is represented by $\|r'(t)\|$.

A particle moves on a circular path of radius b according to the function $\mathbf{r}(t) = b\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$, where ω is the angular velocity, $d\theta/dt$.



Exercise:

Problem: Find the velocity function and show that $\mathbf{v}(t)$ is always orthogonal to $\mathbf{r}(t)$.

Solution:

$\mathbf{r}\prime(t) = -b\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$. To show orthogonality, note that $\mathbf{r}\prime(t)\cdot\mathbf{r}(t) = 0$.

Exercise:

Problem: Show that the speed of the particle is proportional to the angular velocity.

Exercise:

Problem: Evaluate $\frac{d}{dt}[\mathbf{u}(t)\ \times\ \mathbf{u}'(t)]$ given $\mathbf{u}(t)=t^2\,\mathbf{i}-2t\mathbf{j}+\mathbf{k}$.

Solution:

$0\mathbf{i}+2\mathbf{j}+4t\mathbf{j}$

Exercise:

Problem:

Find the antiderivative of $\mathbf{r}\prime(t)=\cos(2t)\,\mathbf{i}-2\sin t\,\mathbf{j}+\frac{1}{1+t^2}\,\mathbf{k}$ that satisfies the initial condition $\mathbf{r}(0)=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

Exercise:

Problem: Evaluate $\displaystyle\int_0^3\|t\,\mathbf{i}+t^2\,\mathbf{j}\|dt$.

Solution:

$\frac{1}{3}\left(10^{3/2}-1\right)$

Exercise:

Problem:

An object starts from rest at point $P\,(1,2,0)$ and moves with an acceleration of $\mathbf{a}(t)=\mathbf{j}+2\mathbf{k}$, where $\|\,\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t=2$ sec.

Exercise:

Problem:

Show that if the speed of a particle traveling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

Solution:

$$\begin{aligned}
\|\mathbf{v}(t)\| &= k \\
\mathbf{v}(t)\cdot\mathbf{v}(t) &= k \\
\frac{d}{dt}(\mathbf{v}(t)\cdot\mathbf{v}(t)) &= \frac{d}{dt}k = 0 \\
\mathbf{v}(t)\cdot\mathbf{v}\prime(t) + \mathbf{v}\prime(t)\cdot\mathbf{v}(t) &= 0 \\
2\,\mathbf{v}(t)\cdot\mathbf{v}\prime(t) &= 0 \\
\mathbf{v}(t)\cdot\mathbf{v}\prime(t) &= 0.
\end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

Exercise:

Problem:

Given $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$ and $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, find $\frac{d}{dt}(\mathbf{r}(t)\ \times\ \mathbf{u}(t))$.

Exercise:

Problem: Given $\mathbf{r}\ (t) = \langle t + \cos t, t - \sin t\rangle$, find the velocity and the speed at any time.

Solution:

$\mathbf{v}(t) = \langle 1 - \sin t, 1 - \cos t\rangle$, speed $= -\mathbf{v}(t)\| = \sqrt{4 - 2(\sin t + \cos t)}$

Exercise:

Problem: Find the velocity vector for the function $\mathbf{r}(t) = \left\langle e^t, e^{-t}, 0\right\rangle$.

Exercise:

Problem: Find the equation of the tangent line to the curve $\mathbf{r}(t) = \left\langle e^t, e^{-t}, 0\right\rangle$ at $t = 0$.

Solution:

$x - 1 = t, y - 1 = -t, z - 0 = 0$

Exercise:

Problem:

Describe and sketch the curve represented by the vector-valued function
$\mathbf{r}(t) = \left\langle 6t, 6t - t^2\right\rangle$.

Exercise:

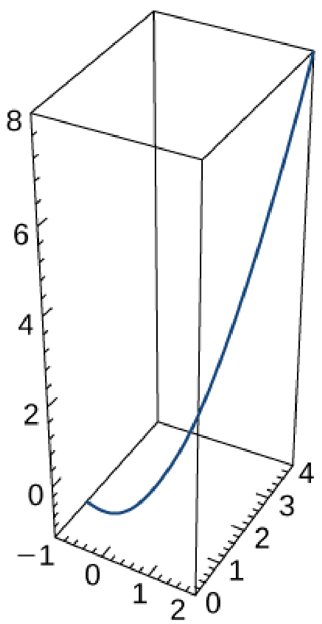
Problem:

Locate the highest point on the curve $\mathbf{r}(t) = \left\langle 6t, 6t - t^2\right\rangle$ and give the value of the function at this point.

Solution:

$\mathbf{r}(t) = \langle 18, 9 \rangle$ at $t = 3$

The position vector for a particle is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. The graph is shown here:



Exercise:

Problem: Find the velocity vector at any time.

Exercise:

Problem: Find the speed of the particle at time $t = 2$ sec.

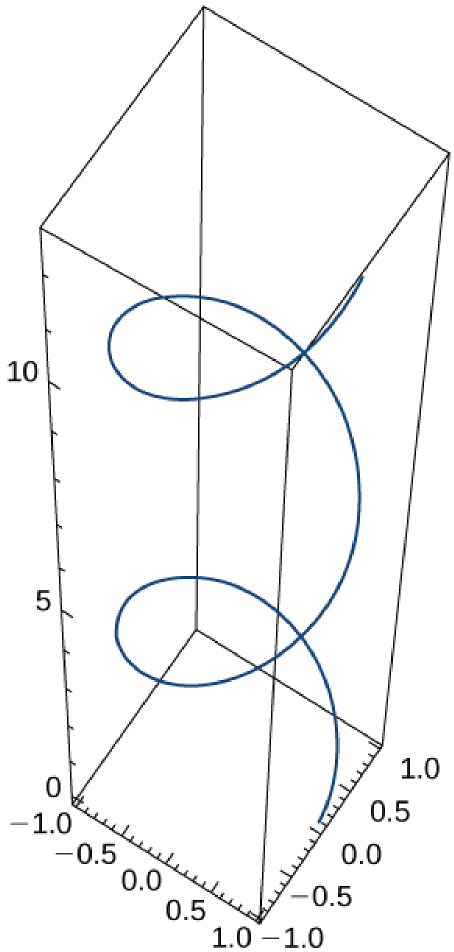
Solution:

$\sqrt{593}$

Exercise:

Problem: Find the acceleration at time $t = 2$ sec.

A particle travels along the path of a helix with the equation $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$. See the graph presented here:



Find the following:

Exercise:

Problem: Velocity of the particle at any time

Solution:

$\mathbf{v}(t) = \langle -\sin t, \cos t, 1 \rangle$

Exercise:

Problem: Speed of the particle at any time

Exercise:

Problem: Acceleration of the particle at any time

Solution:

$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{j}$

Exercise:

Problem: Find the unit tangent vector for the helix.

A particle travels along the path of an ellipse with the equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 0\,\mathbf{k}$. Find the following:

Exercise:

Problem: Velocity of the particle

Solution:

$\mathbf{v}(t) = \langle -\sin t, 2\cos t, 0\rangle$

Exercise:

Problem: Speed of the particle at $t = \frac{\pi}{4}$

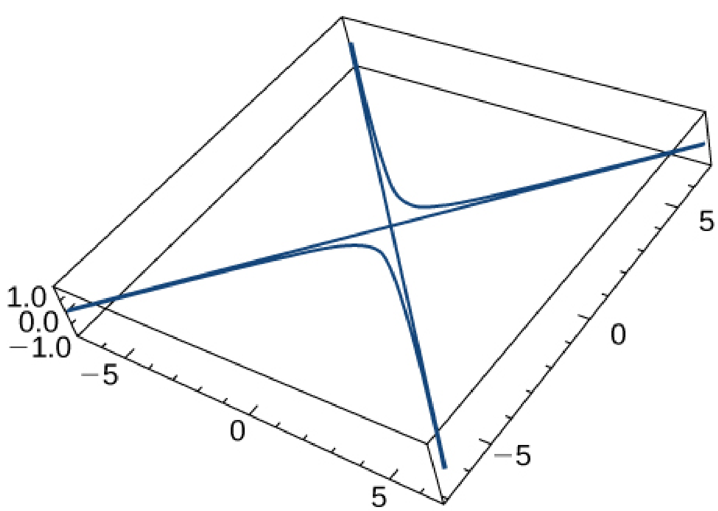
Exercise:

Problem: Acceleration of the particle at $t = \frac{\pi}{4}$

Solution:

$\mathbf{a}(t) = \left\langle -\frac{\sqrt{2}}{2}, -\sqrt{2}, 0\right\rangle$

Given the vector-valued function $\mathbf{r}(t) = \langle \tan t, \sec t, 0\rangle$ (graph is shown here), find the following:



Exercise:

Problem: Velocity

Exercise:

Problem: Speed

Solution:

$\|\mathbf{v}(t)\| = \sqrt{\sec^4 t + \sec^2 t \tan^2 t} = \sqrt{\sec^2 t(\sec^2 t + \tan^2 t)}$

Exercise:

Problem: Acceleration

Exercise:

Problem:

Find the minimum speed of a particle traveling along the curve $\mathbf{r}(t) = \langle t + \cos t, t - \sin t \rangle$ $t \in [0, 2\pi)$.

Solution:

2

Given $\mathbf{r}(t) = t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\cos t\,\mathbf{k}$ and $\mathbf{u}(t) = \frac{1}{t}\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\cos t\,\mathbf{k}$, find the following:

Exercise:

Problem: $\mathbf{r}(t) \times \mathbf{u}(t)$

Exercise:

Problem: $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{u}(t))$

Solution:

$\left\langle 0, 2\sin t\left(t - \frac{1}{t}\right) - 2\cos t\left(1 + \frac{1}{t^2}\right), 2\sin t\left(1 + \frac{1}{t^2}\right) + 2\cos t\left(t - \frac{2}{t}\right)\right\rangle$

Exercise:

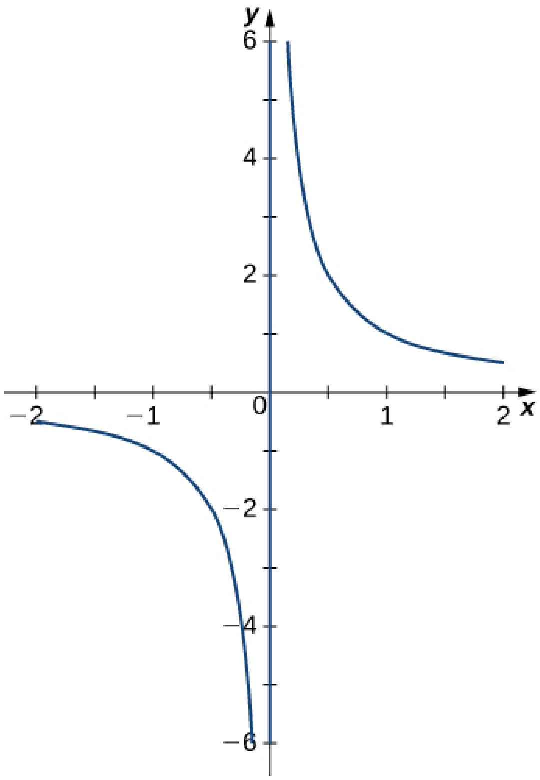
Problem:

Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

Find the unit tangent vector **T**(t) for the following vector-valued functions.

Exercise:

Problem: $\mathbf{r}(t) = \langle t, \frac{1}{t} \rangle$. The graph is shown here:



Solution:

$$\mathbf{T}(t) = \left\langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}} \right\rangle$$

Exercise:

Problem: $\mathbf{r}(t) = \langle t\cos t, t\sin t \rangle$

Exercise:

Problem: $\mathbf{r}(t) = \langle t+1, 2t+1, 2t+2 \rangle$

Solution:

$$\mathbf{T}(t) = \frac{1}{3}\langle 1, 2, 2 \rangle$$

Evaluate the following integrals:

Exercise:

Problem: $\int \left(e^t \mathbf{i} + \sin t \mathbf{j} + \frac{1}{2t-1} \mathbf{k} \right) dt$

Exercise:

Problem: $\int_0^1 \mathbf{r}(t) dt$, where $\mathbf{r}(t) = \left\langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t} \right\rangle$

Solution:

$\frac{3}{4} \mathbf{i} + \ln(2) \mathbf{j} + \left(1 - \frac{1}{e}\right) \mathbf{j}$

Glossary

definite integral of a vector-valued function
: the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

derivative of a vector-valued function
: the derivative of a vector-valued function $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \lim\limits_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$, provided the limit exists

indefinite integral of a vector-valued function
: a vector-valued function with a derivative that is equal to a given vector-valued function

principal unit tangent vector
: a unit vector tangent to a curve C

tangent vector
: to $\mathbf{r}(t)$ at $t = t_0$ any vector $\mathbf{v}$ such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector $\mathbf{v}$ is tangent to curve C

Motion in Space

- Describe the velocity and acceleration vectors of a particle moving in space.
- Explain the tangential and normal components of acceleration.
- State Kepler's laws of planetary motion.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Note:
Definition
Let $\mathbf{r}\left(t\right)$ be a twice-differentiable vector-valued function of the parameter *t* that represents the position of an object as a function of time. The **velocity vector** $\mathbf{v}\left(t\right)$ of the object is given by
Equation:

$$\text{Velocity}=\mathbf{v}\left(t\right)=\mathbf{r}'\left(t\right).$$

The **acceleration vector** $\mathbf{a}\left(t\right)$ is defined to be
Equation:

$$\text{Acceleration}=\mathbf{a}\left(t\right)=\mathbf{v}'\left(t\right)=\mathbf{r}''\left(t\right).$$

The *speed* is defined to be
Equation:

$$\text{Speed}=v\left(t\right)=\|\mathbf{v}\left(t\right)\|=\|\mathbf{r}'\left(t\right)\|=\frac{ds}{dt}.$$

Since $\mathbf{r}\left(t\right)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\mathbf{r}\left(t\right)=x\left(t\right)\mathbf{i}+y\left(t\right)\mathbf{j}$ and in three dimensions $\mathbf{r}\left(t\right)=x\left(t\right)\mathbf{i}+y\left(t\right)\mathbf{j}+z\left(t\right)\mathbf{k}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Quantity	Two Dimensions	Three Dimensions
Position	$\mathbf{r}\left(t\right)=x\left(t\right)\mathbf{i}+y\left(t\right)\mathbf{j}$	$\mathbf{r}\left(t\right)=x\left(t\right)\mathbf{i}+y\left(t\right)\mathbf{j}+z\left(t\right)\mathbf{k}$
Velocity	$\mathbf{v}\left(t\right)=x'\left(t\right)\mathbf{i}+y'\left(t\right)\mathbf{j}$	$\mathbf{v}\left(t\right)=x'\left(t\right)\mathbf{i}+y'\left(t\right)\mathbf{j}+z'\left(t\right)\mathbf{k}$
Acceleration	$\mathbf{a}\left(t\right)=x''\left(t\right)\mathbf{i}+y''\left(t\right)\mathbf{j}$	$\mathbf{a}\left(t\right)=x''\left(t\right)\mathbf{i}+y''\left(t\right)\mathbf{j}+z''\left(t\right)\mathbf{k}$
Speed	$v\left(t\right)=\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}$	$v\left(t\right)=\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2+\left(z'\left(t\right)\right)^2}$

Formulas for Position, Velocity, Acceleration, and Speed

Example:
Exercise:

Problem:
Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\mathbf{r}\left(t\right)=t^2\mathbf{i}+\sqrt{5-t^2}\mathbf{j}$, where t measures time in seconds.

a. Find the velocity, acceleration, and speed as functions of time.
b. Sketch the curve along with the velocity vector at time $t=1$.

Solution:

a. We use [link], [link], and [link]:
Equation:

$$\begin{aligned}\mathbf{v}\left(t\right)&=\mathbf{r}'\left(t\right)=2t\mathbf{i}-\tfrac{t}{\sqrt{5-t^2}}\mathbf{j}\\ \mathbf{a}\left(t\right)&=\mathbf{v}'\left(t\right)=2\mathbf{i}-5\left(5-t^2\right)^{-3/2}\mathbf{j}\\ v\left(t\right)&=\|\mathbf{r}'\left(t\right)\|\\ &=\sqrt{\left(2t\right)^2+\left(-\tfrac{t}{\sqrt{5-t^2}}\right)^2}\\ &=\sqrt{4t^2+\tfrac{t^2}{5-t^2}}\\ &=\sqrt{\tfrac{21t^2-4t^4}{5-t^2}}.\end{aligned}$$

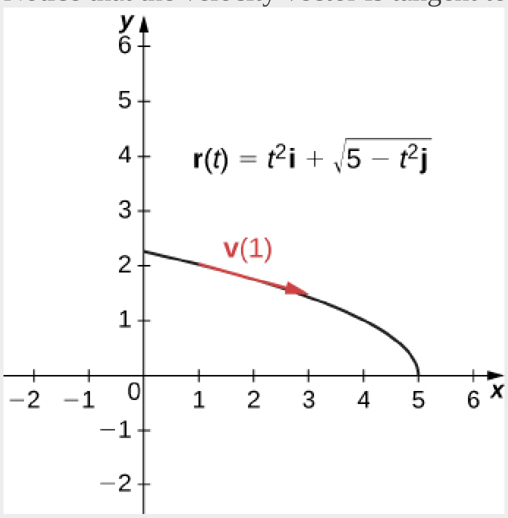
b. The graph of $\mathbf{r}\left(t\right)=t^2\mathbf{i}+\sqrt{5-t^2}\mathbf{j}$ is a portion of a parabola ([link]). The velocity vector at $t=1$ is
Equation:

$$\mathbf{v}\left(1\right)=\mathbf{r}'\left(1\right)=2\left(1\right)\mathbf{i}-\frac{1}{\sqrt{5-\left(1\right)^2}}\mathbf{j}=2\mathbf{i}-\frac{1}{2}\mathbf{j}$$

and the acceleration vector at $t = 1$ is
Equation:

$$\mathbf{a}(1) = \mathbf{v}'(1) = 2\mathbf{i} - 5\left(5 - (1)^2\right)^{-3/2}\mathbf{j} = 2\mathbf{i} - \frac{5}{8}\mathbf{j}.$$

Notice that the velocity vector is tangent to the path, as is always the case.



This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path.

Note:
Exercise:

Problem:

A particle moves in a path defined by the vector-valued function $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (2t - 4)\mathbf{j} + (t + 2)\mathbf{k}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Solution:

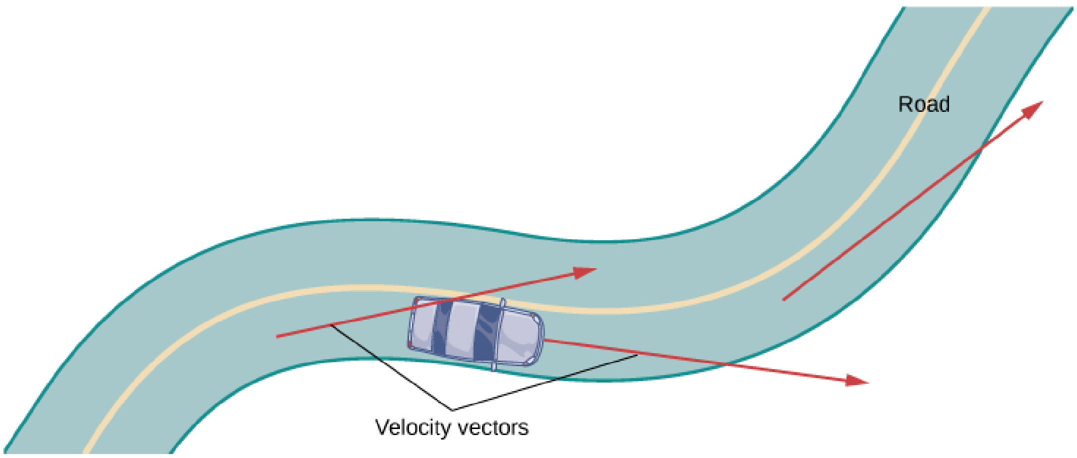
$\mathbf{v}(t) = \mathbf{r}'(t) = (2t - 3)\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{i}$
$v(t) = \|\mathbf{r}'(t)\| = \sqrt{(2t - 3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14}$

The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

Hint

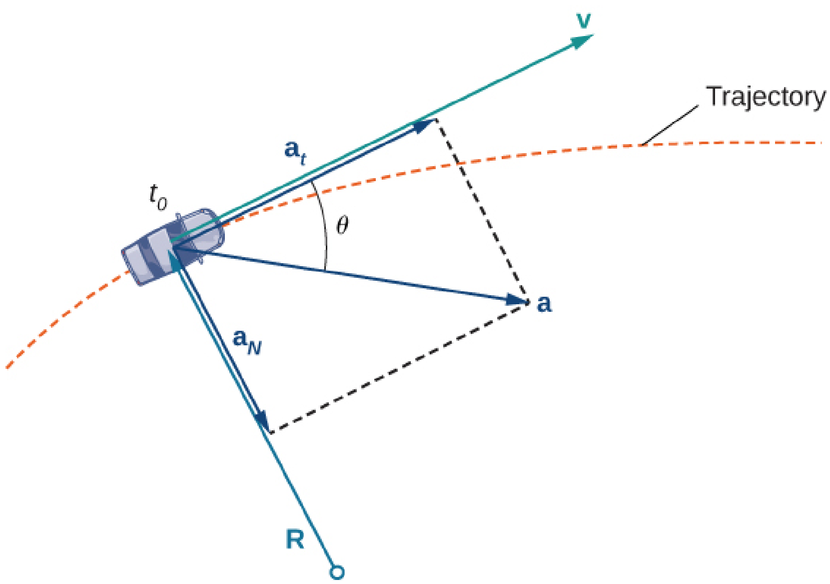
Use [link], [link], and [link].

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in the following figure.



At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies ([link]).



The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{N}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\mathbf{r}(t)$. The following theorem shows that the acceleration vector $\mathbf{a}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Note:
The Plane of the Acceleration Vector
The acceleration vector $\mathbf{a}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\mathbf{r}(t)$ lies in the plane formed by the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$ to C. Furthermore,
Equation:

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + [v(t)]^2\kappa\mathbf{N}(t).$$

Here, $v(t)$ is the speed of the object and κ is the curvature of C traced out by $\mathbf{r}(t)$.

Proof

Because $\mathbf{v}(t)=\mathbf{r}'(t)$ and $\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, we have $\mathbf{v}(t)=\|\mathbf{r}'(t)\|\mathbf{T}(t)=v(t)\mathbf{T}(t)$. Now we differentiate this equation:

Equation:

$$\mathbf{a}(t)=\mathbf{v}'(t)=\frac{d}{dt}(v(t)\mathbf{T}(t))=v'(t)\mathbf{T}(t)+v(t)\mathbf{T}'(t).$$

Since $\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, we know $\mathbf{T}'(t)=\|\mathbf{T}'(t)\|\mathbf{N}(t)$, so

Equation:

$$\mathbf{a}(t)=v'(t)\mathbf{T}(t)+v(t)\|\mathbf{T}'(t)\|\mathbf{N}(t).$$

A formula for curvature is $\kappa=\frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$, so $\|\mathbf{T}'(t)\|=\kappa\|\mathbf{r}'(t)\|=\kappa v(t)$. This gives $\mathbf{a}(t)=v'(t)\mathbf{T}(t)+\kappa(v(t))^2\mathbf{N}(t)$.

□

The coefficients of $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\mathbf{T}}$ to denote the tangential component and $a_{\mathbf{N}}$ to denote the normal component.

Note:
Tangential and Normal Components of Acceleration
Let $\mathbf{r}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\mathbf{a}(t)=\mathbf{r}''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are given by the formulas

Equation:

$$a_{\mathbf{T}}=\mathbf{a}\cdot\mathbf{T}=\frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

and

Equation:

$$a_{\mathbf{N}}=\mathbf{a}\cdot\mathbf{N}=\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}=\sqrt{\|\mathbf{a}\|^2-a_{\mathbf{T}}^2}.$$

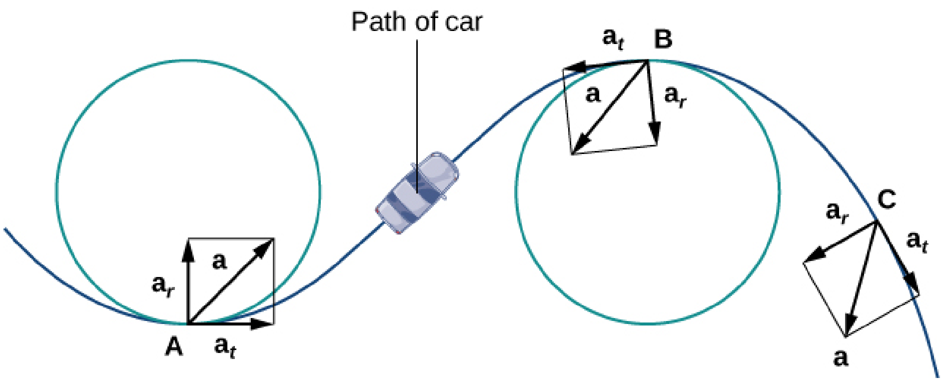
These components are related by the formula

Equation:

$$\mathbf{a}(t)=a_{\mathbf{T}}\mathbf{T}(t)+a_{\mathbf{N}}\mathbf{N}(t).$$

Here $\mathbf{T}(t)$ is the unit tangent vector to the curve defined by $\mathbf{r}(t)$, and $\mathbf{N}(t)$ is the unit normal vector to the curve defined by $\mathbf{r}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for noncircular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point *B* in [link] the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.



The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\mathbf{T}$ and $\mathbf{N}$, respectively.

Example:
Exercise:

Problem:
Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + (2t - 3)\mathbf{j} + \left(3t^2 - 3t\right)\mathbf{k}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as functions of t.
b. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at time $t = 2$.

Solution:

a. Let's start with [link]:

Equation:

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = 2t\,\mathbf{i} + 2\mathbf{j} + (6t-3)\,\mathbf{k} \\
\mathbf{a}(t) &= \mathbf{v}'(t) = 2\mathbf{i} + 6\mathbf{k} \\
a_{\mathbf{T}} &= \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} \\
&= \frac{(2t\,\mathbf{i}+2\mathbf{j}+(6t-3)\,\mathbf{k})\cdot(2\mathbf{i}+6\mathbf{k})}{\|2t\,\mathbf{i}+2\mathbf{j}+(6t-3)\,\mathbf{k}\|} \\
&= \frac{4t+6(6t-3)}{\sqrt{(2t)^2+2^2+(6t-3)^2}} \\
&= \frac{40t-18}{\sqrt{40t^2-36t+13}}.
\end{aligned}$$

Then we apply [link]:

Equation:

$$\begin{aligned}
a_{\mathbf{N}} &= \sqrt{\|\mathbf{a}\|^2 - \text{msup}} \\
&= \sqrt{\|2\mathbf{i}+6\mathbf{k}\|^2 - \left(\frac{40t-18}{\sqrt{40t^2-36t+13}}\right)^2} \\
&= \sqrt{4+36-\frac{(40t-18)^2}{40t^2-36t+13}} \\
&= \sqrt{\frac{40(40t^2-36t+13)-(1600t^2-1440t+324)}{40t^2-36t+13}} \\
&= \sqrt{\frac{196}{40t^2-36t+13}} \\
&= \frac{14}{\sqrt{40t^2-36t+13}}.
\end{aligned}$$

b. We must evaluate each of the answers from part a. at $t=2$:

Equation:

$$\begin{aligned}
a_{\mathbf{T}}(2) &= \frac{40(2)-18}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{80-18}{\sqrt{160-72+13}} = \frac{62}{\sqrt{101}} \\
a_{\mathbf{N}}(2) &= \frac{14}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{14}{\sqrt{160-72+13}} = \frac{14}{\sqrt{101}}.
\end{aligned}$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Note:
Exercise:

Problem:

An object moves in a path defined by the vector-valued function $\mathbf{r}(t) = 4t\,\mathbf{i} + t^2\mathbf{j}$, where t measures time in seconds.

a. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as functions of t.
b. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at time $t = -3$.

Solution:

a.
$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = 4\mathbf{i} + 2t\mathbf{j} \\ \mathbf{a}(t) &= \mathbf{v}'(t) = 2\mathbf{j} \\ a_{\mathbf{T}} &= \frac{2t}{\sqrt{t^2+4}}, a_{\mathbf{N}} = \frac{2}{\sqrt{t^2+4}} \end{aligned}$$

b. $a_{\mathbf{T}}(-3) = -\frac{6\sqrt{13}}{13}, a_{\mathbf{N}}(-3) = \frac{2\sqrt{13}}{13}$

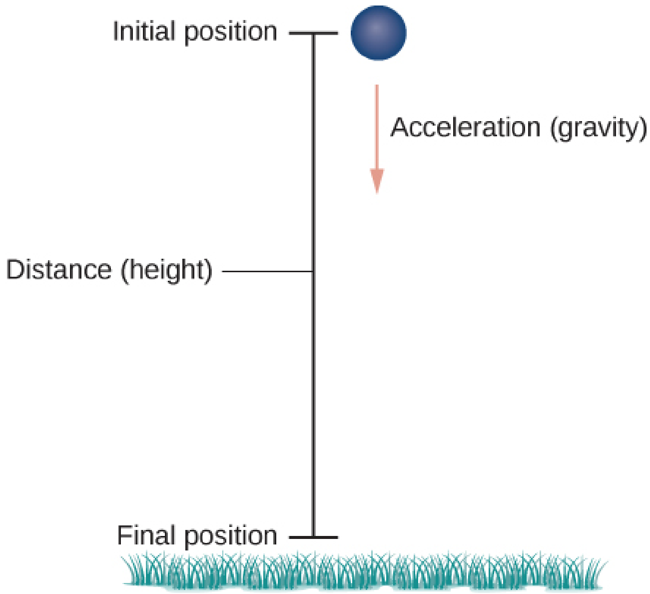
Hint

Use [link] and [link].

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as **projectile motion**. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive *y*-axis to be up, the negative *y*-axis to be down, and the positive *x*-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from to gravity, or $F_g = mg$, where F_g represents the force from gravity and g represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $F_g = -mg\mathbf{j}$, as shown in the following figure.



An object is falling under the influence of gravity.

Note:
Visit this website for a video showing projectile motion.

Newton's second law also tells us that $F = m\mathbf{a}$, where **a** represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

Equation:

$$\begin{aligned} F &= F_g \\ m\mathbf{a} &= -mg\mathbf{j} \\ \mathbf{a} &= -g\mathbf{j}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

Equation:

$$\mathbf{v}'(t) = -g\mathbf{j}.$$

By taking the antiderivative of each side of this equation we obtain

Equation:

$$\begin{aligned}\mathbf{v}(t) &= \int -g\mathbf{j}dt \\ &= -gt\mathbf{j} + \mathbf{C}_1\end{aligned}$$

for some constant vector $\mathbf{C}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\mathbf{v}(0) = \mathbf{v}_0$. Therefore, $\mathbf{v}(0) = -g(0)\mathbf{j} + \mathbf{C}_1 = \mathbf{v}_0$ and $\mathbf{C}_1 = \mathbf{v}_0$. This gives the velocity vector as $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$.

Next we use the fact that velocity $\mathbf{v}(t)$ is the derivative of position $\mathbf{s}(t)$. This gives the equation
Equation:

$$\mathbf{s}'(t) = -gt\mathbf{j} + \mathbf{v}_0.$$

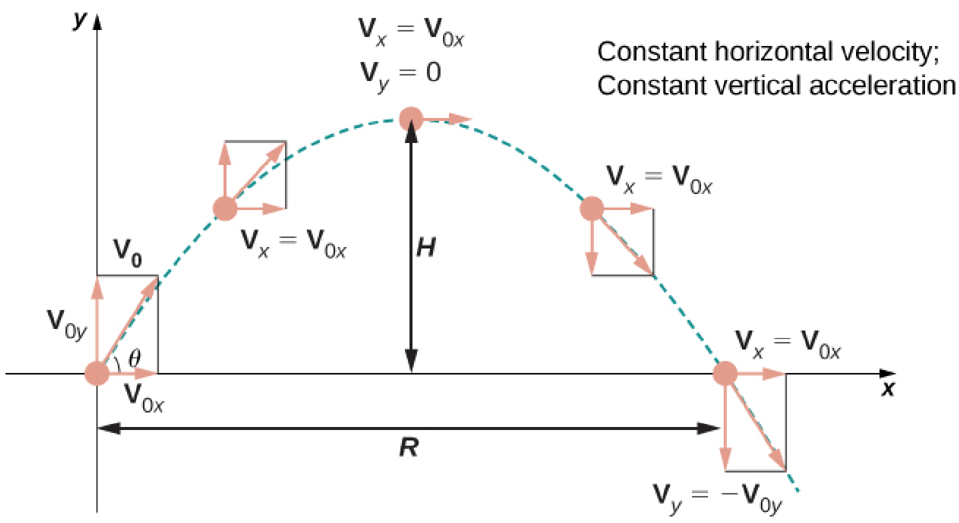
Taking the antiderivative of both sides of this equation leads to
Equation:

$$\begin{aligned}\mathbf{s}(t) &= \int -gt\mathbf{j} + \mathbf{v}_0 dt \\ &= -\tfrac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{C}_2,\end{aligned}$$

with another unknown constant vector $\mathbf{C}_2$. To determine the value of $\mathbf{C}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\mathbf{s}(0) = \mathbf{s}_0$. Therefore, $\mathbf{s}(0) = -(1/2)g(0)^2\mathbf{j} + \mathbf{v}_0(0) + \mathbf{C}_2 = \mathbf{s}_0$ and $\mathbf{C}_2 = \mathbf{s}_0$. This gives the position of the object at any time as
Equation:

$$\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0.$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed v_0. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\mathbf{s}_0 = \mathbf{0}$, as shown in the following figure.



Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\mathbf{v}_0 = v_0 \cos\theta \mathbf{i} + v_0 \sin\theta \mathbf{j}$. Then the equation for the position function $\mathbf{s}(t)$ becomes

Equation:

$$\begin{aligned} \mathbf{s}(t) &= -\tfrac{1}{2} g t^2 \mathbf{j} + v_0 t \cos\theta \mathbf{i} + v_0 t \sin\theta \mathbf{j} \\ &= v_0 t \cos\theta \mathbf{i} + v_0 t \sin\theta \mathbf{j} - \tfrac{1}{2} g t^2 \mathbf{j} \\ &= v_0 t \cos\theta \mathbf{i} + \left(v_0 t \sin\theta - \tfrac{1}{2} g t^2\right) \mathbf{j}. \end{aligned}$$

The coefficient of $\mathbf{i}$ represents the horizontal component of $\mathbf{s}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\mathbf{j}$ represents the vertical component of $\mathbf{s}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

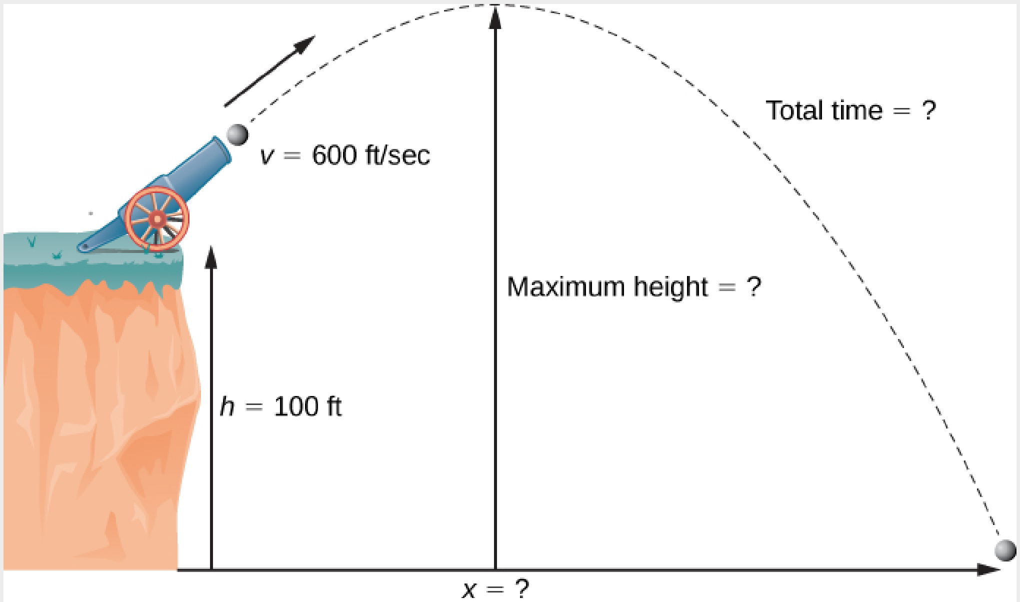
Example:
Exercise:

Problem:
Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water ([link]).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?

c. How far out to sea will the cannonball hit the water?



The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution:

We use the equation

Equation:

$$\mathbf{s}(t) = v_0 t \cos\theta \mathbf{i} + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) \mathbf{j}$$

with $\theta = 30^\circ$, $g = 32\ \text{ft/sec}^2$, and $v_0 = 600$ ft/sec. Then the position equation becomes

Equation:

$$\begin{aligned} \mathbf{s}(t) &= 600t(\cos 30)\mathbf{i} + \left(600t \sin 30 - \tfrac{1}{2}(32)t^2\right)\mathbf{j} \\ &= 300t\sqrt{3}\mathbf{i} + \left(300t - 16t^2\right)\mathbf{j}. \end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

Equation:

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{s}'(t) \\ &= 300\sqrt{3}\mathbf{i} + (300 - 32t)\mathbf{j}. \end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting

this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.
Equation:

$$\begin{aligned} \mathbf{s}\,(9.375) &= 300\,(9.375)\sqrt{3}\mathbf{i} + \left(300\,(9.375) - 16(9.375)^2\right)\mathbf{j} \\ &= 4871.39\mathbf{i} + 1406.25\mathbf{j} \end{aligned}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to -100. Setting the vertical component of $\mathbf{s}\,(t)$ equal to -100 and solving, we obtain
Equation:

$$\begin{aligned} 300t - 16t^2 &= -100 \\ 16t^2 - 300t - 100 &= 0 \\ 4t^2 - 75t - 25 &= 0 \\ t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)} \\ &= \frac{75 \pm \sqrt{6025}}{8} \\ &= \frac{75 \pm 5\sqrt{241}}{8}. \end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\mathbf{s}\,(t)$:
Equation:

$$\begin{aligned} \mathbf{s}\,(19.08) &= 300\,(19.08)\sqrt{3}\mathbf{i} + \left(300\,(19.08) - 16(19.08)^2\right)\mathbf{j} \\ &= 9914.26\,\mathbf{i} - 100.7424\mathbf{j}. \end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to -100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

Note:
Exercise:

Problem:

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Solution:

967.15 m

Hint

The equation for the position vector needs to account for the height of the archer in meters.

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

Equation:

$$\mathbf{s}\left(t\right)=v_0 t\cos\theta\,\mathbf{i}+\left(v_0 t\sin\theta-\frac{1}{2}gt^2\right)\mathbf{j}.$$

Setting the second component equal to zero and solving for *t* yields

Equation:

$$\begin{aligned} v_0 t\sin\theta-\tfrac{1}{2}gt^2 &= 0 \\ t\left(v_0\sin\theta-\tfrac{1}{2}gt\right) &= 0. \end{aligned}$$

Therefore, either $t=0$ or $t=\frac{2v_0\sin\theta}{g}$. We are interested in the second value of *t*, so we substitute this into $\mathbf{s}\left(t\right)$, which gives

Equation:

$$\begin{aligned} \mathbf{s}\left(\tfrac{2v_0\sin\theta}{g}\right) &= v_0\left(\tfrac{2v_0\sin\theta}{g}\right)\cos\theta\,\mathbf{i}+\left(v_0\left(\tfrac{2v_0\sin\theta}{g}\right)\sin\theta-\tfrac{1}{2}g\left(\tfrac{2v_0\sin\theta}{g}\right)^2\right)\mathbf{j} \\ &= \left(\tfrac{2v_0^2\sin\theta\cos\theta}{g}\right)\mathbf{i} \\ &= \tfrac{v_0^2\sin 2\theta}{g}\mathbf{i}. \end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

Equation:

$$R=\frac{v_0^2\sin 2\theta}{g}\mathbf{i}.$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of **i** with respect to θ and set it equal to zero:
Equation:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{v_0^2 \sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2 \cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ.\end{aligned}$$

This value of θ is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by
Equation:

$$\mathbf{s}\left(\frac{2v_0 \sin 45}{g}\right) = \frac{v_0^2 \sin 90}{g}\mathbf{i} = \frac{v_0^2}{g}\mathbf{j}.$$

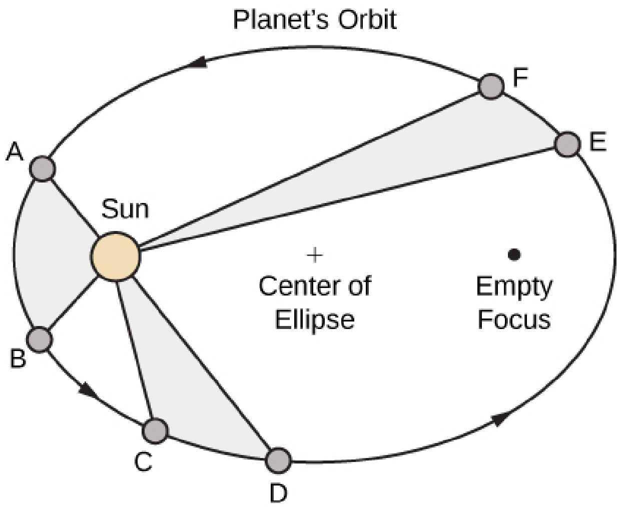
Therefore, the range for an angle of 45° is v_0^2/g.

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as **Kepler's laws of planetary motion**. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Note:
Kepler's Laws of Planetary Motion

i. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
ii. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) ([link]).
iii. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the law of harmonies).



Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

Equation:

$$T_p^2 = D_p^3$$

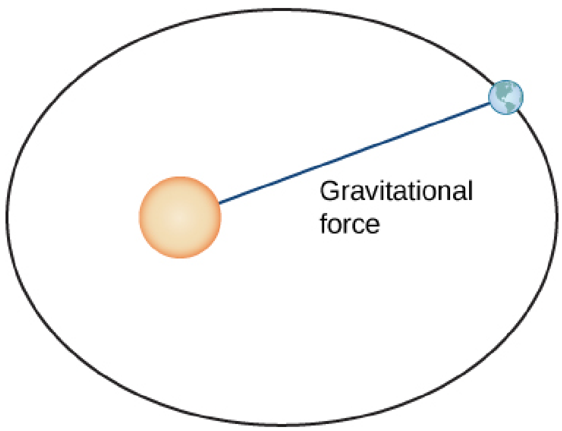
for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\mathbf{r}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using

his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\frac{GmM}{\|\mathbf{r}\|^2}$ ([link]).



The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\mathbf{a}(t) = \mathbf{v}'(t)$, we obtain
Equation:

$$m\mathbf{v}'(t) = -\frac{GmM}{\|\mathbf{r}\|^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|},$$

which can be rewritten as
Equation:

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{\|\mathbf{r}\|^3}\mathbf{r}.$$

This equation shows that the vectors $d\mathbf{v}/dt$ and $\mathbf{r}$ are parallel to each other, so $d\mathbf{v}/dt \times \mathbf{r} = \mathbf{0}$. Next, let's differentiate $\mathbf{r} \times \mathbf{v}$ with respect to time:
Equation:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{v} + \mathbf{0} = \mathbf{0}.$$

This proves that $\mathbf{r} \times \mathbf{v}$ is a constant vector, which we call $\mathbf{C}$. Since $\mathbf{r}$ and $\mathbf{v}$ are both perpendicular to $\mathbf{C}$ for all values of t, they must lie in a plane perpendicular to $\mathbf{C}$. Therefore, the motion of the planet lies in

a plane.

Next we calculate the expression $d\mathbf{v}/dt \times \mathbf{C}$:
Equation:

$$\frac{d\mathbf{v}}{dt} \times \mathbf{C} = -\frac{GM}{\|\mathbf{r}\|^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = -\frac{GM}{\|\mathbf{r}\|^3}[(\mathbf{r}\cdot\mathbf{v})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\mathbf{v}].$$

The last equality in [link] is from the triple cross product formula (Introduction to Vectors in Space). We need an expression for $\mathbf{r}\cdot\mathbf{v}$. To calculate this, we differentiate $\mathbf{r}\cdot\mathbf{r}$ with respect to time:
Equation:

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} + \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 2\mathbf{r}\cdot\mathbf{v}.$$

Since $\mathbf{r}\cdot\mathbf{r} = \|\mathbf{r}\|^2$, we also have
Equation:

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d}{dt}\|\mathbf{r}\|^2 = 2\,\|\mathbf{r}\|\frac{d}{dt}\|\mathbf{r}\|.$$

Combining [link] and [link], we get
Equation:

$$\begin{aligned} 2\mathbf{r}\cdot\mathbf{v} &= 2\,\|\mathbf{r}\|\tfrac{d}{dt}\|\mathbf{r}\| \\ \mathbf{r}\cdot\mathbf{v} &= \|\mathbf{r}\|\tfrac{d}{dt}\|\mathbf{r}\|. \end{aligned}$$

Substituting this into [link] gives us
Equation:

$$\begin{aligned} \tfrac{d\mathbf{v}}{dt} \times \mathbf{C} &= -\tfrac{GM}{\|\mathbf{r}\|^3}[(\mathbf{r}\cdot\mathbf{v})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\mathbf{v}] \\ &= -\tfrac{GM}{\|\mathbf{r}\|^3}\left[\|\mathbf{r}\|\left(\tfrac{d}{dt}\|\mathbf{r}\|\right)\mathbf{r} - \|\mathbf{r}\|^2\mathbf{v}\right] \\ &= -GM\left[\tfrac{1}{\|\mathbf{r}\|^2}\left(\tfrac{d}{dt}\|\mathbf{r}\|\right)\mathbf{r} - \tfrac{1}{\|\mathbf{r}\|}\mathbf{v}\right] \\ &= GM\left[\tfrac{\mathbf{v}}{\|\mathbf{r}\|} - \tfrac{\mathbf{r}}{\|\mathbf{r}\|^2}\left(\tfrac{d}{dt}\|\mathbf{r}\|\right)\right]. \end{aligned}$$

However,
Equation:

$$\begin{aligned} \tfrac{d}{dt}\tfrac{\mathbf{r}}{\|\mathbf{r}\|} &= \tfrac{\frac{d}{dt}(\mathbf{r})\|\mathbf{r}\| - \mathbf{r}\frac{d}{dt}\|\mathbf{r}\|}{\|\mathbf{r}\|^2} \\ &= \tfrac{\frac{d\mathbf{r}}{dt}}{\|\mathbf{r}\|} - \tfrac{\mathbf{r}}{\|\mathbf{r}\|^2}\tfrac{d}{dt}\|\mathbf{r}\| \\ &= \tfrac{\mathbf{v}}{\|\mathbf{r}\|} - \tfrac{\mathbf{r}}{\|\mathbf{r}\|^2}\tfrac{d}{dt}\|\mathbf{r}\|. \end{aligned}$$

Therefore, [link] becomes
Equation:

$$\frac{d\mathbf{v}}{dt} \times \mathbf{C} = GM\left(\frac{d}{dt}\frac{\mathbf{r}}{\|\mathbf{r}\|}\right).$$

Since $\mathbf{C}$ is a constant vector, we can integrate both sides and obtain
Equation:

$$\mathbf{v} \times \mathbf{C} = GM\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{D},$$

where $\mathbf{D}$ is a constant vector. Our goal is to solve for $\|\mathbf{r}\|$. Let’s start by calculating $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{C})$:
Equation:

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{C}) = \mathbf{r} \cdot \left(GM\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{D}\right) = GM\frac{\|\mathbf{r}\|^2}{\|\mathbf{r}\|} + \mathbf{r} \cdot \mathbf{D} = GM\,\|\mathbf{r}\| + \mathbf{r} \cdot \mathbf{D}.$$

However, $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{C}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{C}$, so
Equation:

$$(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{C} = GM\,\|\mathbf{r}\| + \mathbf{r} \cdot \mathbf{D}.$$

Since $\mathbf{r} \times \mathbf{v} = \mathbf{C}$, we have
Equation:

$$\|\mathbf{C}\|^2 = GM\,\|\mathbf{r}\| + \mathbf{r} \cdot \mathbf{D}.$$

Note that $\mathbf{r} \cdot \mathbf{D} = \|\mathbf{r}\|\,\|\mathbf{D}\|\cos\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{D}$. Therefore,
Equation:

$$\|\mathbf{C}\|^2 = GM\,\|\mathbf{r}\| + \|\mathbf{r}\|\,\|\mathbf{D}\|\cos\theta.$$

Solving for $\|\mathbf{r}\|$,
Equation:

$$\|\mathbf{r}\| = \frac{\|\mathbf{C}\|^2}{GM + \|\mathbf{D}\|\cos\theta} = \frac{\|\mathbf{C}\|^2}{GM}\left(\frac{1}{1 + e\cos\theta}\right),$$

where $e = \|\mathbf{D}\|/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds

too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Example:
Exercise:

Problem:
Using Kepler's Third Law for Nonheliocentric Orbits

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

Equation:

$$P^2 = \frac{4\pi^2 a^3}{G\,(m+M)},$$

where *m* is the mass of the Moon and *M* is the mass of Earth, *a* represents the length of the major axis of the elliptical orbit, and *P* represents the period.

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11}\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{sec}^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution:

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

Equation:

$$27.3\text{ days} \times \frac{24\text{ hr}}{1\text{ day}} \times \frac{3600\,\mathrm{sec}}{1\,\mathrm{hour}} = 2{,}358{,}720\,\mathrm{sec}.$$

Substitute all the data into [link] and solve for *a:*

Equation:

$$\begin{aligned}
(2{,}358{,}720\,\mathrm{sec})^2 &= \frac{4\pi^2 a^3}{\left(6.67\times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{sec}^2}\right)(7.35\,x\,10^{22}\mathrm{kg}+5.97\,x\,10^{24}\mathrm{kg})} \\
5.563 \times 10^{12} &= \frac{4\pi^2 a^3}{(6.67\times 10^{-11}\,\mathrm{m}^3)(6.04\,x\,10^{24})} \\
\left(5.563 \times 10^{12}\right)\left(6.67 \times 10^{-11}\mathrm{m}^3\right)\left(6.04 \times 10^{24}\right) &= 4\pi^2 a^3 \\
a^3 &= \frac{2.241\times 10^{27}}{4\pi^2}\mathrm{m}^3 \\
a &= 3.84 \times 10^8\mathrm{m} \\
&\approx 384{,}000\text{ km}.
\end{aligned}$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Note:
Exercise:

Problem:

Titan is the largest moon of Saturn. The mass of Titan is approximately 1.35×10^{23} kg. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67 \times 10^{-11} \text{m}^3/\text{kg} \cdot \text{sec}^2$ to estimate the distance from Titan to Saturn.

Solution:

$a = 1.224 \times 10^9 \text{m} \approx 1{,}224{,}000 \text{ km}$

Hint

Make sure your units agree, then use [link].

Example:
Exercise:

Problem:
Chapter Opener: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T = 76.1$ years. What is the average distance of Halley's comet from the Sun?

Solution:

Using the equation $T^2 = D^3$ with $T = 76.1$, we obtain $D^3 = 5791.21$, so $D \approx 17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary/factsheet/cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where *a* is the length of the semimajor axis and *c* is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from *a* gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary/factsheet/cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

Equation:

$$P = a + c = 17.96 + 17.37 = 35.33\text{ A.U.}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

Note:

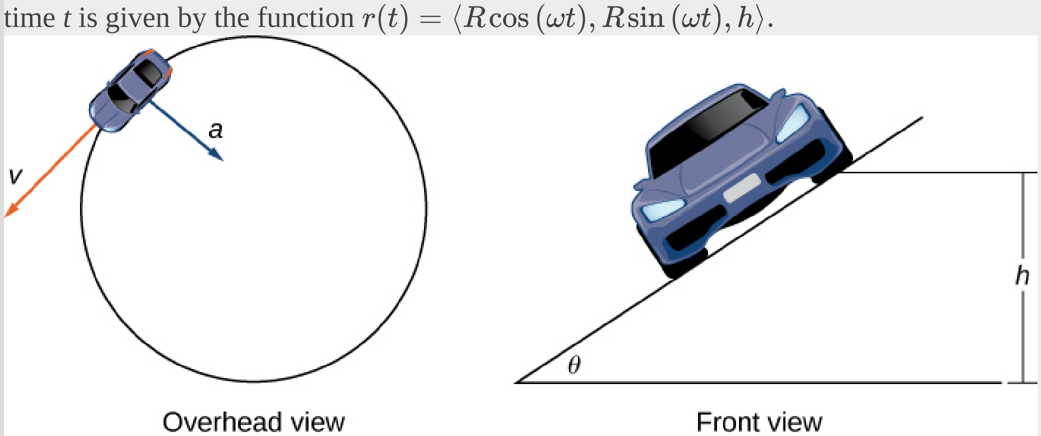
Navigating a Banked Turn

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

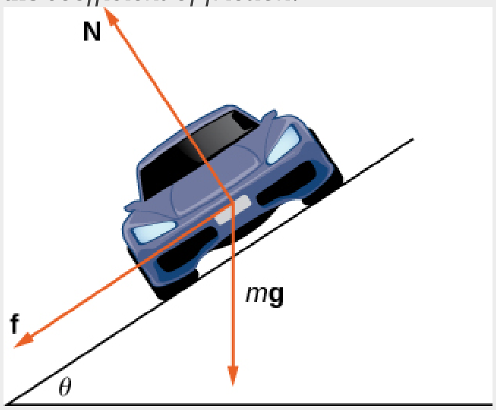
A car of mass *m* moves with constant angular speed *ω* around a circular curve of radius *R* ([link]). The curve is banked at an angle *θ*. If the height of the car off the ground is *h*, then the position of the car at time *t* is given by the function $r(t) = \langle R\cos(\omega t), R\sin(\omega t), h\rangle$.



Views of a race car moving around a track.

1. Find the velocity function $\mathbf{v}(t)$ of the car. Show that $\mathbf{v}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi r)/\,|\mathbf{v}| = (2\pi)/\omega$.
3. Find the acceleration $\mathbf{a}$. Show that this vector points toward the center of the circle and that $|\mathbf{a}| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\mathbf{F}_{\text{cent}}$. This force points toward the center of the circle (not toward the ground). Show that $|\mathbf{F}_{\text{cent}}| = \left(m|\mathbf{v}|^2\right)/R$.

   As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force ([link]). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $|\mathbf{f}| = \mu|\mathbf{N}|$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

   The car has three forces acting on it: gravity (denoted by $m\mathbf{g}$), the friction force $\mathbf{f}$, and the force exerted by the road $\mathbf{N}$.

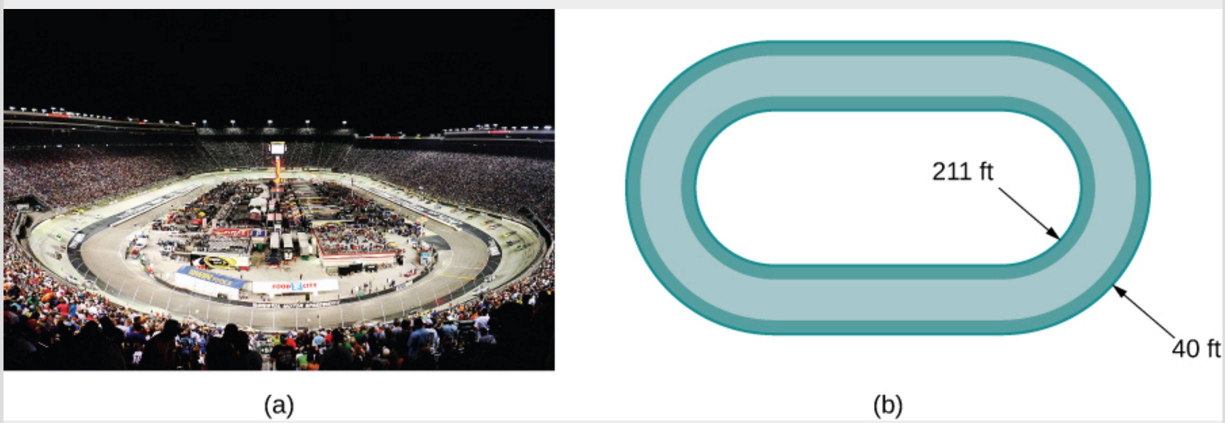
   Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

   Equation:

   $$|\mathbf{F}_{\text{cent}}| = \frac{mv_{\text{max}}^2}{R}.$$

   The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.
5. Show that $|\mathbf{N}|\cos\theta = mg + |\mathbf{f}|\sin\theta$. Conclude that $|\mathbf{N}| = (mg)/\,(\cos\theta - \mu\sin\theta)$.

6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that
   Equation:
   $$\mathbf{F}_{\text{cent}} = |\mathbf{N}|\sin\theta + |\mathbf{f}|\cos\theta.$$
   Conclude that
   Equation:
   $$|\mathbf{F}_{\text{cent}|} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}mg.$$
7. Show that $\text{v}_{\text{max}}^2 = ((\sin\theta + \mu\cos\theta)/\,(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.
   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.
   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in [link]. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)



At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.
Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?

11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

Key Concepts

- If $\mathbf{r}(t)$ represents the position of an object at time t, then $\mathbf{r}'(t)$ represents the velocity and $\mathbf{r}''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\mathbf{r}(t)$. The tangential and normal components of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

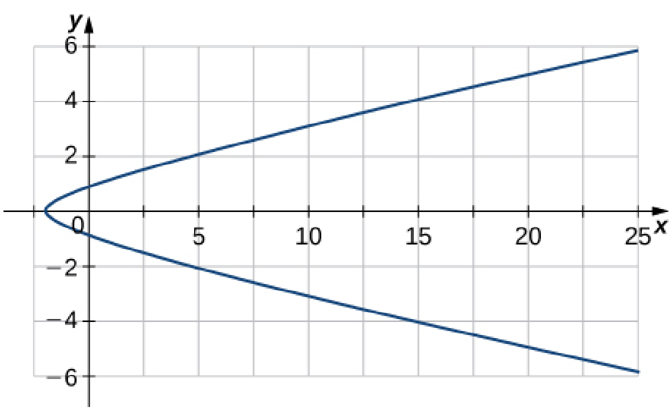
Key Equations

- **Velocity**
  $\mathbf{v}(t) = \mathbf{r}'(t)$
- **Acceleration**
  $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
- **Speed**
  $v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt}$
- **Tangential component of acceleration**
  $a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$
- **Normal component of acceleration**
  $a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$

Exercise:

Problem:

Given $\mathbf{r}(t) = (3t^2 - 2)\mathbf{i} + (2t - \sin(t))\mathbf{j}$, find the velocity of a particle moving along this curve.



Solution:

$\mathbf{v}(t) = (6t)\mathbf{i} + (2 - \cos(t))\mathbf{j}$

Exercise:

Problem:

Given $\mathbf{r}(t) = (3t^2 - 2)\mathbf{i} + (2t - \sin(t))\mathbf{j}$, find the acceleration vector of a particle moving along the curve in the preceding exercise.

Given the following position functions, find the velocity, acceleration, and speed in terms of the parameter t.

Exercise:

Problem: $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, t^2 \rangle$

Solution:

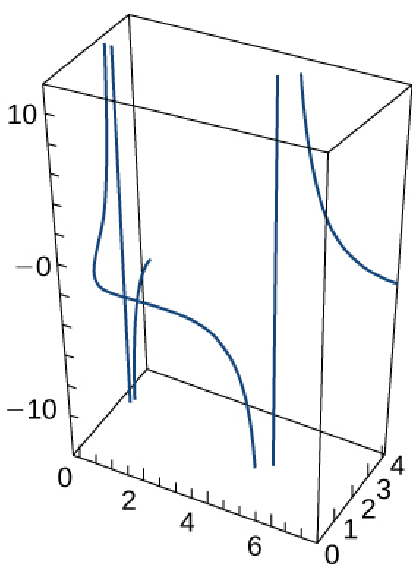
$\mathbf{v}(t) = \langle -3\sin t, 3\cos t, 2t \rangle$, $\mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2 \rangle$, speed $= \sqrt{9 + 4t^2}$

Exercise:

Problem: $\mathbf{r}(t) = e^{-t}\mathbf{i} + t^2\mathbf{j} + \tan t\,\mathbf{k}$

Exercise:

Problem: $\mathbf{r}(t) = 2\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$. The graph is shown here:



Solution:

$\mathbf{v}(t) = -2\sin\ t\mathbf{j} + 3\ \cos\ t\,\mathbf{k}$, $\mathbf{a}(t) = -2\cos t\mathbf{j} - 3\sin t\mathbf{k}$, speed $= \sqrt{4\sin^2(t) + 9_2^{\cos}(t)}$

Find the velocity, acceleration, and speed of a particle with the given position function.

Exercise:

Problem: $\mathbf{r}(t)=\langle t^2-1,t\rangle$

Exercise:

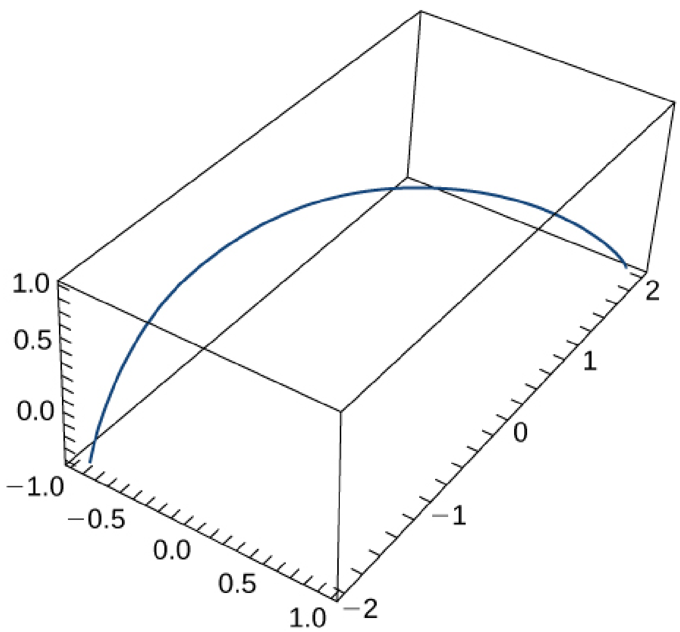
Problem: $\mathbf{r}(t)=\langle e^t,e^{-t}\rangle$

Solution:

$\mathbf{v}(t)=e^t\mathbf{i}-e^{-t}\mathbf{j}$, $\mathbf{a}(t)=e^t\mathbf{i}+e^{-t}\mathbf{j}$, $\|\mathbf{v}(t)\|\sqrt{e^{2t}+e^{-2t}}$

Exercise:

Problem: $\mathbf{r}(t)=\langle \sin t,t,\cos t\rangle$. The graph is shown here:



Exercise:

Problem:

The position function of an object is given by $\mathbf{r}(t)=\langle t^2,5t,t^2-16t\rangle$. At what time is the speed a minimum?

Solution:

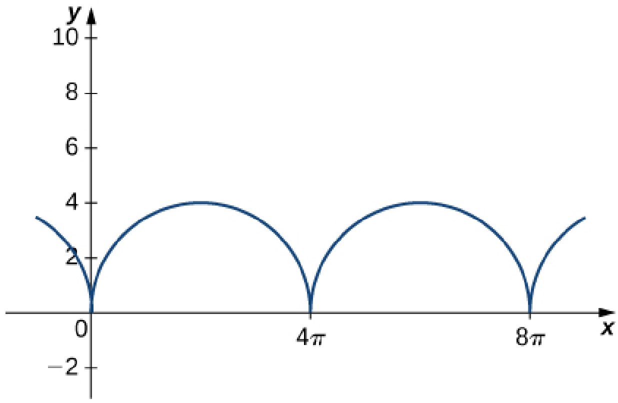
$t=4$

Exercise:

Problem:

Let $\mathbf{r}(t) = r\cosh(\omega t)\mathbf{i} + r\sinh(\omega t)\mathbf{j}$. Find the velocity and acceleration vectors and show that the acceleration is proportional to $\mathbf{r}(t)$.

Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\mathbf{r}(t) = (\omega t - \sin(\omega t))\,\mathbf{i} + (1 - \cos(\omega t))\,\mathbf{j}$, where ω is the angular velocity of the circle:



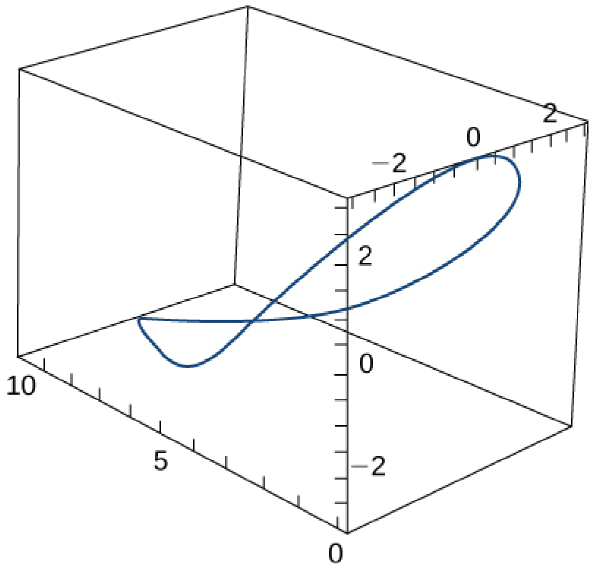
Exercise:

Problem: Find the equations for the velocity, acceleration, and speed of the particle at any time.

Solution:

$\mathbf{v}(t) = (\omega - \omega\cos(\omega t))\,\mathbf{i} + (\omega\sin(\omega t))\,\mathbf{j},$
$\mathbf{a}(t) = \left(\omega^2\sin(\omega t)\right)\mathbf{i} + \left(\omega^2\cos(\omega t)\right)\mathbf{j},$
$\text{speed} = \sqrt{\omega^2 - 2\omega^2\cos(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)} = \sqrt{2\omega^2(1 - \cos(\omega t))}$

A person on a hang glider is spiraling upward as a result of the rapidly rising air on a path having position vector $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:



Find the following quantities:

Exercise:

Problem: The velocity and acceleration vectors

Exercise:

Problem: The glider’s speed at any time

Solution:

$$\|\mathbf{v}(t)\| = \sqrt{9 + 4t^2}$$

Exercise:

Problem: The times, if any, at which the glider’s acceleration is orthogonal to its velocity

Given that $\mathbf{r}(t) = \langle e^{-5t}\sin t, e^{-5t}\cos t, 4e^{-5t}\rangle$ is the position vector of a moving particle, find the following quantities:

Exercise:

Problem: The velocity of the particle

Solution:

$$\mathbf{v}(t) = \langle e^{-5t}(\cos t - 5\sin t), -e^{-5t}(\sin t + 5\cos t), -20e^{-5t}\rangle$$

Exercise:

Problem: The speed of the particle

Exercise:

Problem: The acceleration of the particle

Solution:

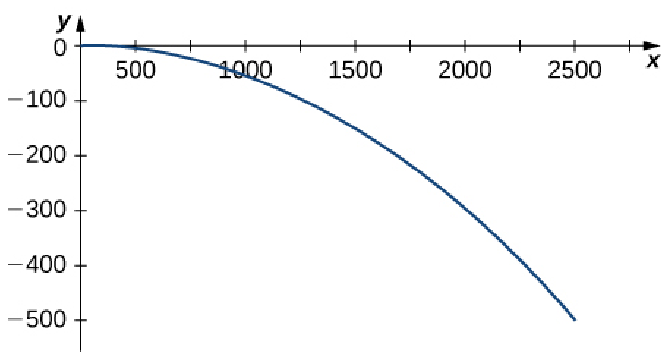
$$\mathbf{a}(t) = \langle e^{-5t}(-\sin t - 5\cos t) - 5e^{-5t}(\cos t - 5\sin t),$$
$$-e^{-5t}(\cos t - 5\sin t) + 5e^{-5t}(\sin t + 5\cos t), 100e^{-5t}\rangle$$

Exercise:

Problem:

Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is traveling at 55 mph.

A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal. The graph is shown here:



Exercise:

Problem: At what time does the projectile reach maximum height?

Solution:

44.185 sec

Exercise:

Problem: What is the approximate maximum height of the projectile?

Exercise:

Problem: At what time is the maximum range of the projectile attained?

Solution:

$t = 88.37$ sec

Exercise:

Problem: What is the maximum range?

Exercise:

Problem: What is the total flight time of the projectile?

Solution:

88.37 sec

A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:

Exercise:

Problem: Determine the maximum height of the projectile.

Exercise:

Problem: Determine the range of the projectile.

Solution:

The range is approximately 886.29 m.

Exercise:

Problem:

A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?

Exercise:

Problem:

A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity necessary to achieve this range.

Solution:

$\mathbf{v} = 42.16$ m/sec

Exercise:

Problem:

Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Exercise:

Problem:

The acceleration of an object is given by $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}$. The velocity at $t = 1$ sec is $\mathbf{v}(1) = 5\mathbf{j}$ and the position of the object at $t = 1$ sec is $\mathbf{r}(1) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. Find the object's position at any time.

Solution:

$\mathbf{r}(t) = 0\mathbf{i} + \left(\frac{1}{6}t^3 + 4.5t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$

Exercise:

Problem: Find $\mathbf{r}(t)$ given that $\mathbf{a}(t) = -32\mathbf{j}$, $\mathbf{v}(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$, and $\mathbf{r}(0) = \mathbf{0}$.

Exercise:

Problem:

Find the tangential and normal components of acceleration for $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$ at $t = 0$.

Solution:

$a_T = 0,\ a_N = a\omega^2$

Exercise:

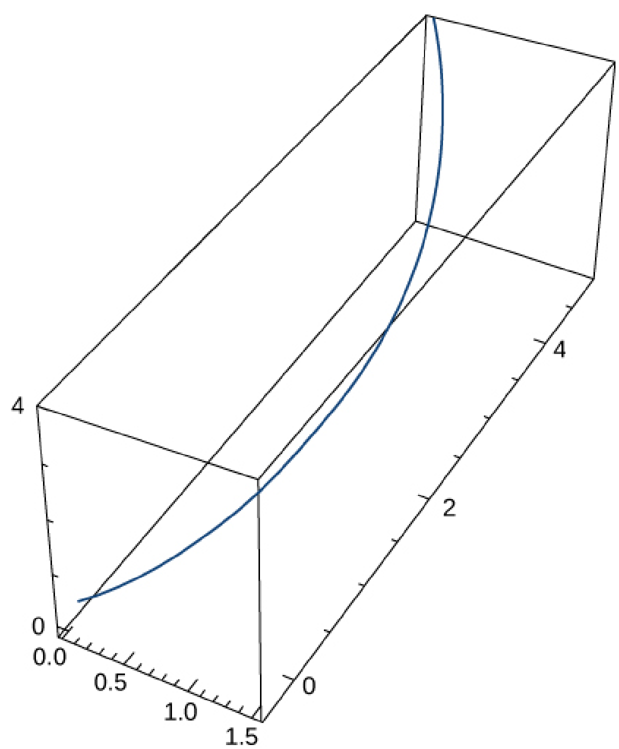
Problem:

Given $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$ and $t = 1$, find the tangential and normal components of acceleration.

For each of the following problems, find the tangential and normal components of acceleration.

Exercise:

Problem: $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$. The graph is shown here:



Solution:

$a_T = \sqrt{3}e^t$, $a_N = \sqrt{2}e^t$

Exercise:

Problem: $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 1 \rangle$

Exercise:

Problem: $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$

Solution:

$a_T = 2t$, $a_N = 4 + 2t^2$

Exercise:

Problem: $\mathbf{r}(t) = \left\langle \frac{2}{3}(1+t)^{3/2}, \frac{2}{3}(1-t)^{3/2}, \sqrt{2}t \right\rangle$

Exercise:

Problem: $\mathbf{r}(t) = \left\langle 6t, 3t^2, 2t^3 \right\rangle$

Solution:

$a_T \frac{6t+12t^3}{\sqrt{1+t^4+t^2}}$, $a_N = 6\sqrt{\frac{1+4t^2+t^4}{1+t^2+t^4}}$

Exercise:

Problem: $\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

Exercise:

Problem: $\mathbf{r}(t) = 3\cos(2\pi t)\,\mathbf{i} + 3\sin(2\pi t)\,\mathbf{j}$

Solution:

$a_T = 0$, $a_N = 2\sqrt{3}\pi$

Exercise:

Problem:

Find the position vector-valued function $\mathbf{r}(t)$, given that $\mathbf{a}(t) = \mathbf{i} + e^t\mathbf{j}$, $\mathbf{v}(0) = 2\mathbf{j}$, and $\mathbf{r}(0) = 2\mathbf{i}$.

Exercise:

Problem:

The force on a particle is given by $\mathbf{f}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\mathbf{v}(0) = v_0\mathbf{j}$. Find the path of the particle of mass $\mathbf{m}$. (Recall, $\mathbf{F} = m \cdot \mathbf{a}$.)

Solution:

$\mathbf{r}(t) = \left(\frac{-1}{m}\cos t + c + \frac{1}{m}\right)\mathbf{i} + \left(\frac{-\sin t}{m} + \left(v_0 + \frac{1}{m}\right)t\right)\mathbf{j}$

Exercise:

Problem:

An automobile that weighs 2700 lb makes a turn on a flat road while traveling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

Exercise:

Problem:

Using Kepler's laws, it can be shown that $v_0 = \sqrt{\frac{2GM}{r_0}}$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

Solution:

10.94 km/sec

Exercise:

Problem:

Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5$ A.U.

Suppose that the position function for an object in three dimensions is given by the equation $\mathbf{r}(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + 3t\mathbf{k}$.

Exercise:

Problem: Show that the particle moves on a circular cone.

Exercise:

Problem: Find the angle between the velocity and acceleration vectors when $t = 1.5$.

Exercise:

Problem: Find the tangential and normal components of acceleration when $t = 1.5$.

Solution:

$a_T = 0.43\ \mathrm{m/sec}^2$,
$a_N = 2.46\ \mathrm{m/sec}^2$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.
Exercise:

Problem:

A parametric equation that passes through points P and Q can be given by $\mathbf{r}(t) = \langle t^2, 3t+1, t-2 \rangle$, where $P(1,4,-1)$ and $Q(16,11,2)$.

Exercise:

Problem: $\frac{d}{dt}[\mathbf{u}(t)\ \times\ \mathbf{u}(t)] = 2\mathbf{u}'(t)\ \times\ \mathbf{u}(t)$

Solution:

False, $\frac{d}{dt}[\mathbf{u}(t)\ \times\ \mathbf{u}(t)] = 0$

Exercise:

Problem:

The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $1/r$.

Exercise:

Problem: The speed of a particle with a position function $\mathbf{r}(t)$ is $(\mathbf{r}'(t))/(|\mathbf{r}'(t)|)$.

Solution:

False, it is $|\mathbf{r}'(t)|$

Find the domains of the vector-valued functions.
Exercise:

Problem: $\mathbf{r}(t) = \left\langle \sin(t), \ln(t), \sqrt{t} \right\rangle$

Exercise:

Problem: $\mathbf{r}(t) = \left\langle e^t, \frac{1}{\sqrt{4-t}}, \sec(t) \right\rangle$

Solution:

$t < 4, t \neq \frac{n\pi}{2}$

Sketch the curves for the following vector equations. Use a calculator if needed.

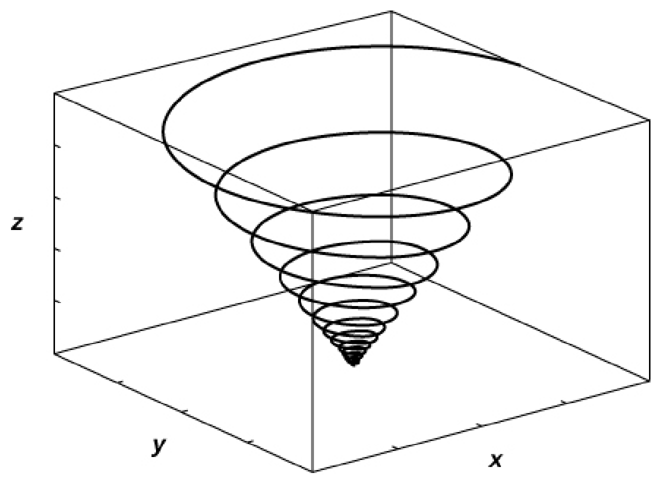
Exercise:

Problem: [T] $\mathbf{r}(t) = \langle t^2, t^3 \rangle$

Exercise:

Problem: [T] $\mathbf{r}(t) = \langle \sin(20t)e^{-t}, \cos(20t)e^{-t}, e^{-t} \rangle$

Solution:



Find a vector function that describes the following curves.

Exercise:

Problem: Intersection of the cylinder $x^2 + y^2 = 4$ with the plane $x + z = 6$

Exercise:

Problem: Intersection of the cone $z = \sqrt{x^2 + y^2}$ and plane $z = y - 4$

Solution:

$\mathbf{r}(t) = \left\langle t, 2 - \frac{t^2}{8}, -2 - \frac{t^2}{8} \right\rangle$

Find the derivatives of $\mathbf{u}(t)$, $\mathbf{u}'(t)$, $\mathbf{u}'(t) \times \mathbf{u}(t)$, $\mathbf{u}(t) \times \mathbf{u}'(t)$, and $\mathbf{u}(t) \cdot \mathbf{u}'(t)$. Find the unit tangent vector.

Exercise:

Problem: $\mathbf{u}(t) = \langle e^t, e^{-t} \rangle$

Exercise:

Problem: $\mathbf{u}(t) = \langle t^2, 2t+6, 4t^5-12 \rangle$

Solution:

$\mathbf{u}'(t) = \langle 2t, 2, 20t^4 \rangle$, $\mathbf{u}\prime\prime(t) = \langle 2, 0, 80t^3 \rangle$,
$\frac{d}{dt}[\mathbf{u}'(t) \times \mathbf{u}(t)] = \langle -480t^3 - 160t^4, 24 + 75t^2, 12 + 4t \rangle$,
$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] = \langle 480t^3 + 160t^4, -24 - 75t^2, -12 - 4t \rangle$,
$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{u}'(t)] = 720t^8 - 9600t^3 + 6t^2 + 4$, unit tangent vector:
$\mathbf{T}(t) = \frac{2t}{\sqrt{400t^8+4t^2+4}}\mathbf{i} + \frac{2}{\sqrt{400t^8+4t^2+4}}\mathbf{j} + \frac{20t^4}{\sqrt{400t^8+4t^2+4}}\mathbf{k}$

Evaluate the following integrals.
Exercise:

Problem: $\int \left(\tan(t)\sec(t)\mathbf{i} - te^{3t}\mathbf{j}\right)dt$

Exercise:

Problem: $\int_1^4 \mathbf{u}(t)dt$, with $\mathbf{u}(t) = \left\langle \frac{\ln(t)}{t}, \frac{1}{\sqrt{t}}, \sin\left(\frac{t\pi}{4}\right) \right\rangle$

Solution:

$\frac{\ln(4)^2}{2}\mathbf{i} + 2\mathbf{j} + \frac{2\left(2+\sqrt{2}\right)}{\pi}\mathbf{k}$

Find the length for the following curves.
Exercise:

Problem: $\mathbf{r}(t) = \langle 3(t), 4\cos(t), 4\sin(t) \rangle$ for $1 \le t \le 4$

Exercise:

Problem: $\mathbf{r}(t) = 2\mathbf{i} + \text{t}\,\mathbf{j} + 3t^2\mathbf{k}$ for $0 \le t \le 1$

Solution:

$\frac{\sqrt{37}}{2} + \frac{1}{12}\sinh^{-1}(6)$

Reparameterize the following functions with respect to their arc length measured from $t = 0$ in direction of increasing t.
Exercise:

Problem: $\mathbf{r}(t) = 2t\,\mathbf{i} + (4t-5)\mathbf{j} + (1-3t)\mathbf{k}$

Exercise:

Problem: $\mathbf{r}(t) = \cos(2t)\mathbf{i} + 8t\mathbf{j} - \sin(2t)\mathbf{k}$

Solution:

$$\mathbf{r}(t(s)) = \cos\left(\frac{2s}{\sqrt{65}}\right)\mathbf{i} + \frac{8s}{\sqrt{65}}\mathbf{j} - \sin\left(\frac{2s}{\sqrt{65}}\right)\mathbf{k}$$

Find the curvature for the following vector functions.
Exercise:

Problem: $\mathbf{r}(t) = (2\sin t)\mathbf{i} - 4t\mathbf{j} + (2\cos t)\mathbf{k}$

Exercise:

Problem: $\mathbf{r}(t) = \sqrt{2}e^t\mathbf{i} + \sqrt{2}e^{-t}\mathbf{j} + 2t\mathbf{k}$

Solution:

$$\frac{e^{2t}}{\left(e^{2t}+1\right)^2}$$

Exercise:

Problem:

Find the unit tangent vector, the unit normal vector, and the binormal vector for $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin t\,\mathbf{k}$.

Exercise:

Problem:

Find the tangential and normal acceleration components with the position vector $\mathbf{r}(t) = \langle \cos t, \sin t, e^t \rangle$.

Solution:

$$a_T = \frac{e^{2t}}{\sqrt{1+e^{2t}}},\ a_N = \frac{\sqrt{2e^{2t}+4e^{2t}\sin t\cos t+1}}{\sqrt{1+e^{2t}}}$$

Exercise:

Problem:

A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.

Exercise:

Problem:

The position of a particle is given by $\mathbf{r}(t) = \langle t^2, \ln(t), \sin(\pi t) \rangle$, where t is measured in seconds and $\mathbf{r}$ is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at 1 sec?

Solution:

$\mathbf{v}(t) = \left\langle 2t, \frac{1}{t}, \cos(\pi t) \right\rangle$ m/sec, $\mathbf{a}(t) = \left\langle 2, -\frac{1}{t^2}, -\sin(\pi t) \right\rangle \text{m/sec}^2$,
speed $= \sqrt{4t^2 + \frac{1}{t^2} + \cos^2(\pi t)}$ m/sec; at $t = 1$, $\mathbf{r}(1) = \langle 1, 0, 0 \rangle$ m, $\mathbf{v}(1) = \langle 2, -1, 1 \rangle$ m/sec,
$\mathbf{a}(1) = \langle 2, -1, 0 \rangle$ m/sec^2, and speed $= \sqrt{6}$ m/sec

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity $\mathbf{v}_0$. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\mathbf{a}(t) = -g\mathbf{j}$.

Exercise:

Problem: Find the velocity vector function $\mathbf{v}(t)$.

Exercise:

Problem: Find the position vector $\mathbf{r}(t)$ and the parametric representation for the position.

Solution:

$\mathbf{r}(t) = \mathbf{v}_0 t - \frac{g}{2}t^2\mathbf{j}$, $\mathbf{r}(t) = \left\langle \mathbf{v}_0(\cos\theta)t, \mathbf{v}_0(\sin\theta)t, -\frac{g}{2}t^2 \right\rangle$

Exercise:

Problem:

At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

Glossary

acceleration vector
: the second derivative of the position vector

Kepler's laws of planetary motion
: three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

normal component of acceleration
: the coefficient of the unit normal vector $\mathbf{N}$ when the acceleration vector is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$

projectile motion
: motion of an object with an initial velocity but no force acting on it other than gravity

tangential component of acceleration
: the coefficient of the unit tangent vector $\mathbf{T}$ when the acceleration vector is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$

velocity vector
: the derivative of the position vector

Vector Fields

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

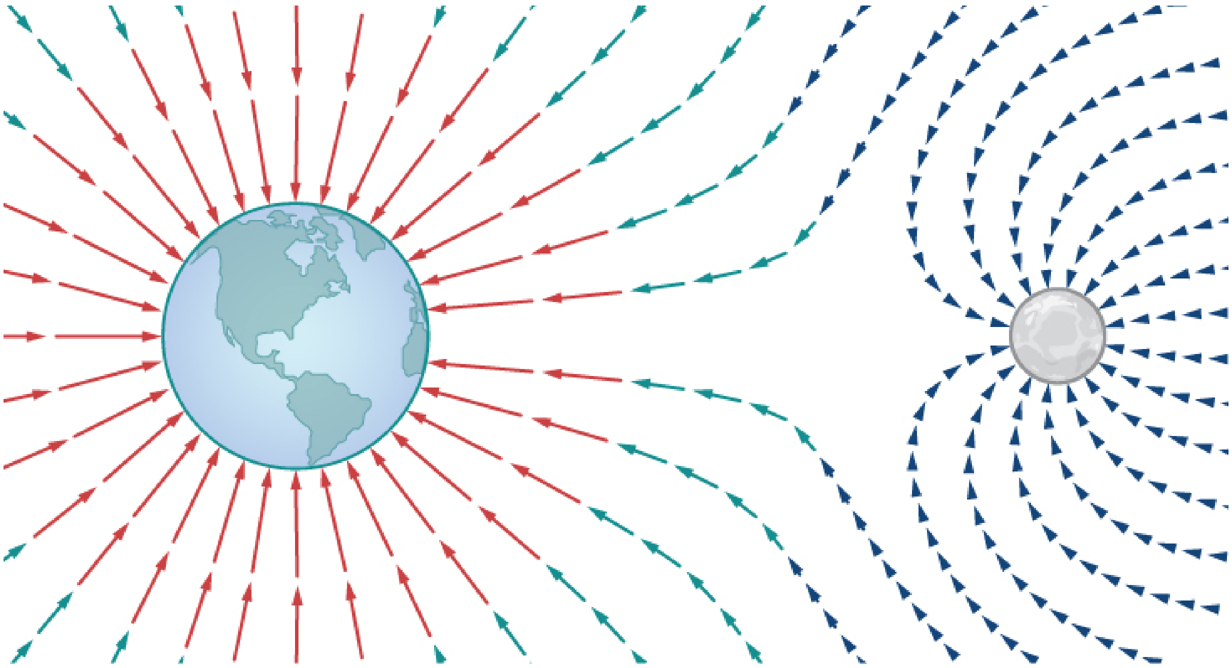
Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

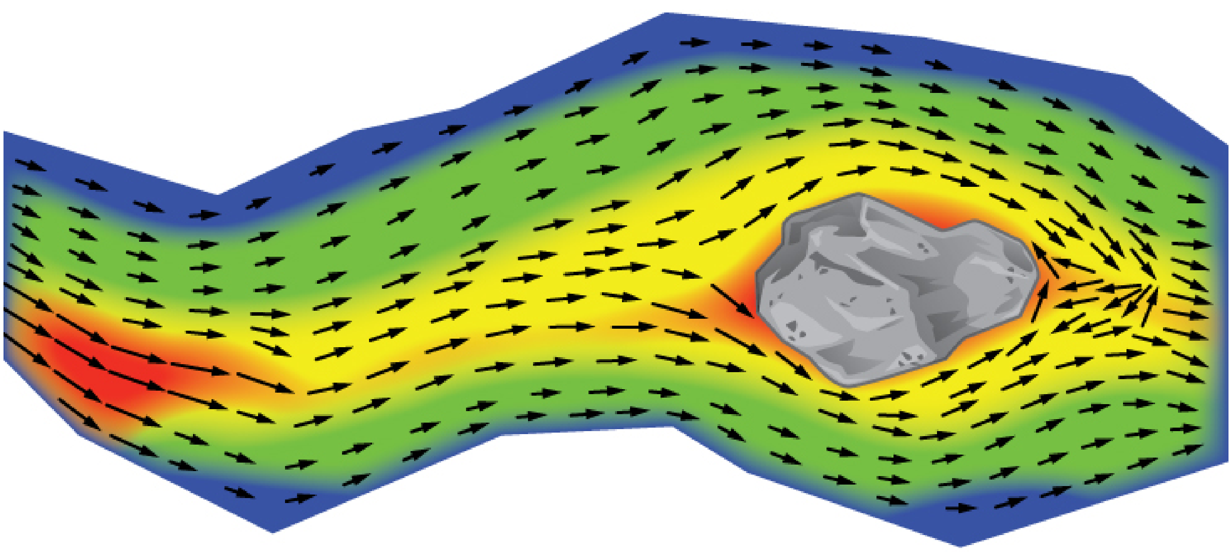
How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? [link] gives visual representations of such phenomena.

[link](a) shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

[link](b) shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.



(a)



(b)

(a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Note:
Definition
A **vector field F** in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}\,(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
A vector field **F** in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}\,(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:
Equation:

$$\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle.$$

The second way is to use the standard unit vectors:
Equation:

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}.$$

A vector field is said to be *continuous* if its component functions are continuous.

Example:
Exercise:

Problem:
Finding a Vector Associated with a Given Point

Let $\mathbf{F}(x,y) = (2y^2 + x - 4)\mathbf{i} + \cos(x)\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0, -1)$?

Solution:

Substitute the point values for *x* and *y*:
Equation:

$$\begin{aligned}\mathbf{F}(0,1) &= (2(-1)^2+0-4)\mathbf{i}+\cos(0)\mathbf{j}\\ &= -2\mathbf{i}+\mathbf{j}.\end{aligned}$$

Note:
Exercise:

Problem:

Let $\mathbf{G}\left(x,y\right)=x^2y\mathbf{i}-(x+y)\mathbf{j}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2,3)$?

Solution:

$12\mathbf{i}-\mathbf{j}$

Hint

Substitute the point values into the vector function.

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the “graph” of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4,-1)$ is $\langle 3,1\rangle$. Then, we would draw vector $\langle 3,1\rangle$ at point $(4,-1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a **radial field**, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example:
Exercise:

Problem:
Drawing a Radial Vector Field

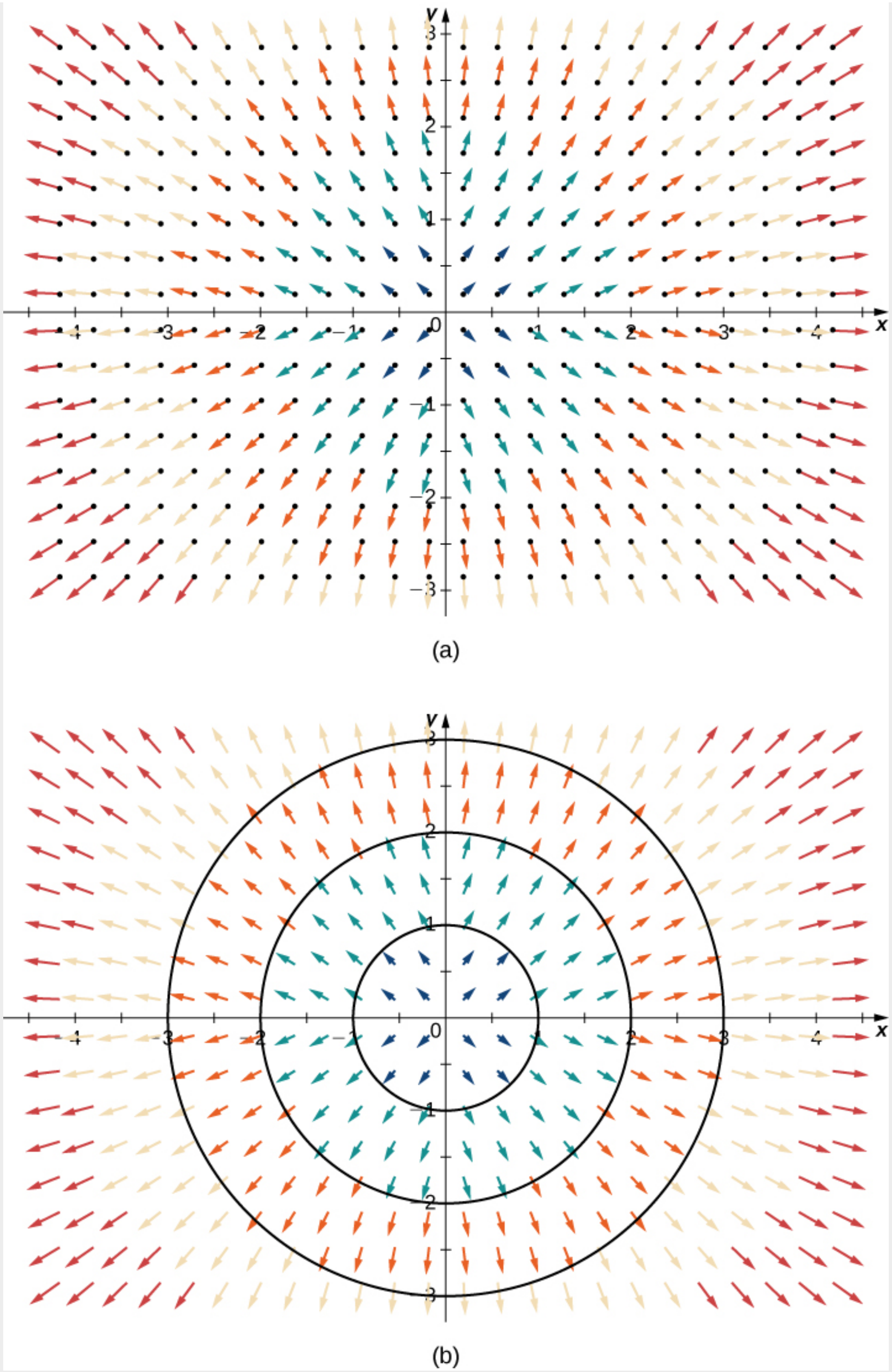
Sketch the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$.

Solution:

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

[link](a) shows the vector field. To see that each vector is perpendicular to the corresponding circle, [link](b) shows circles overlain on the vector field.

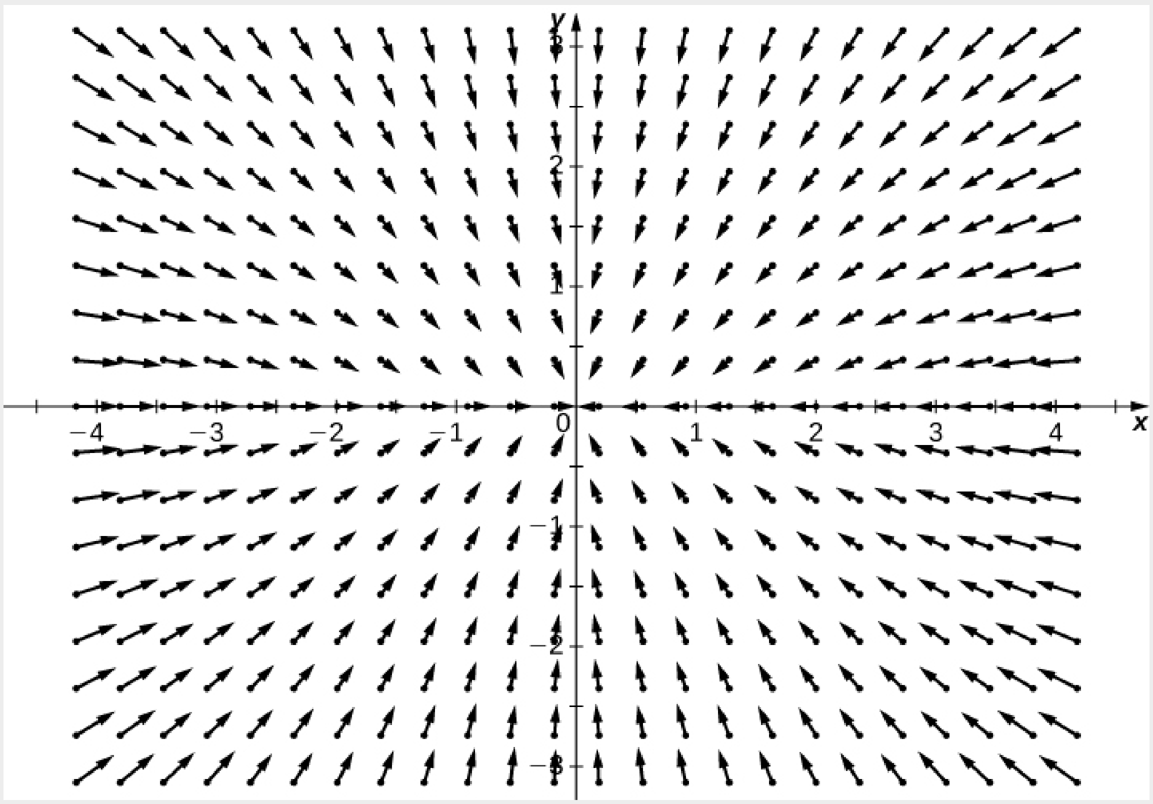


(a) A visual representation of the radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$. (b) The radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

Note:
Exercise:

Problem: Draw the radial field $\mathbf{F}(x,y) = -\frac{x}{3}\,\mathbf{i} - \frac{y}{3}\,\mathbf{j}$.

Solution:



Hint

Sketch enough vectors to get an idea of the shape.

In contrast to radial fields, in a **rotational field**, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all

vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example:
Exercise:

Problem:
Chapter Opener: Drawing a Rotational Vector Field

(credit: modification of work by NASA)

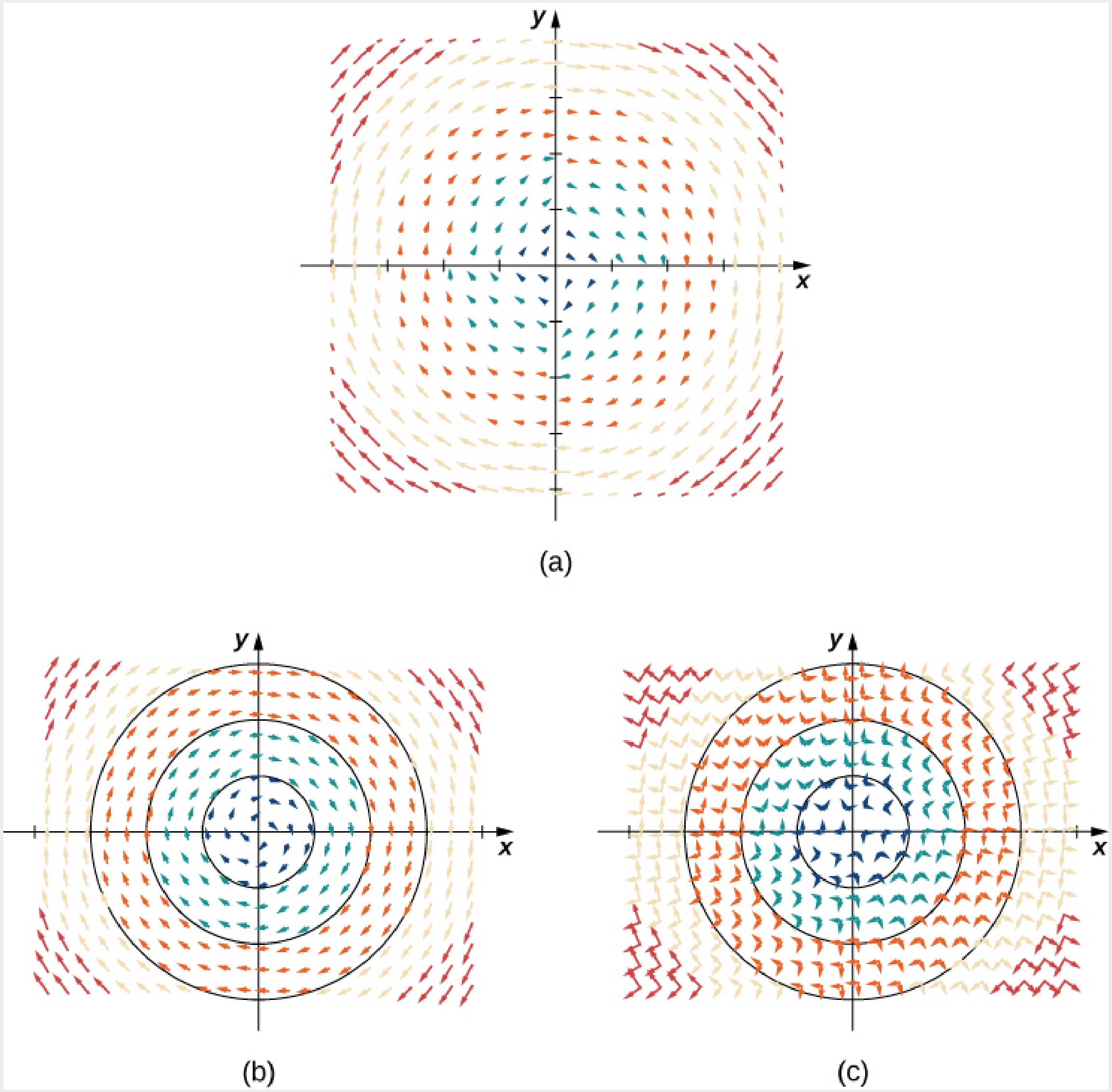
Sketch the vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$.

Solution:

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. [link] shows the resulting vector field.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$

$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,2\rangle$	$(-1,-1)$	$\langle -1,1\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -2,0\rangle$	$(1,-1)$	$\langle -1,-1\rangle$



(a) A visual representation of vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$. (b) Vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$ with circles centered at the origin. (c) Vector $\mathbf{F}(a,b)$ is perpendicular to radial vector $\langle a,b\rangle$ at point (a,b).

Analysis

Note that vector $\mathbf{F}(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0.$) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length *r* tangent to the circle with radius *r*, and it points in the clockwise direction.

Sketches such as that in [link] are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)

Example:
Exercise:

Problem:
Sketching a Vector Field

Sketch vector field $\mathbf{F}(x, y) = \frac{y}{x^2+y^2}\,\mathbf{i} - \frac{x}{x^2+y^2}\,\mathbf{j}$.

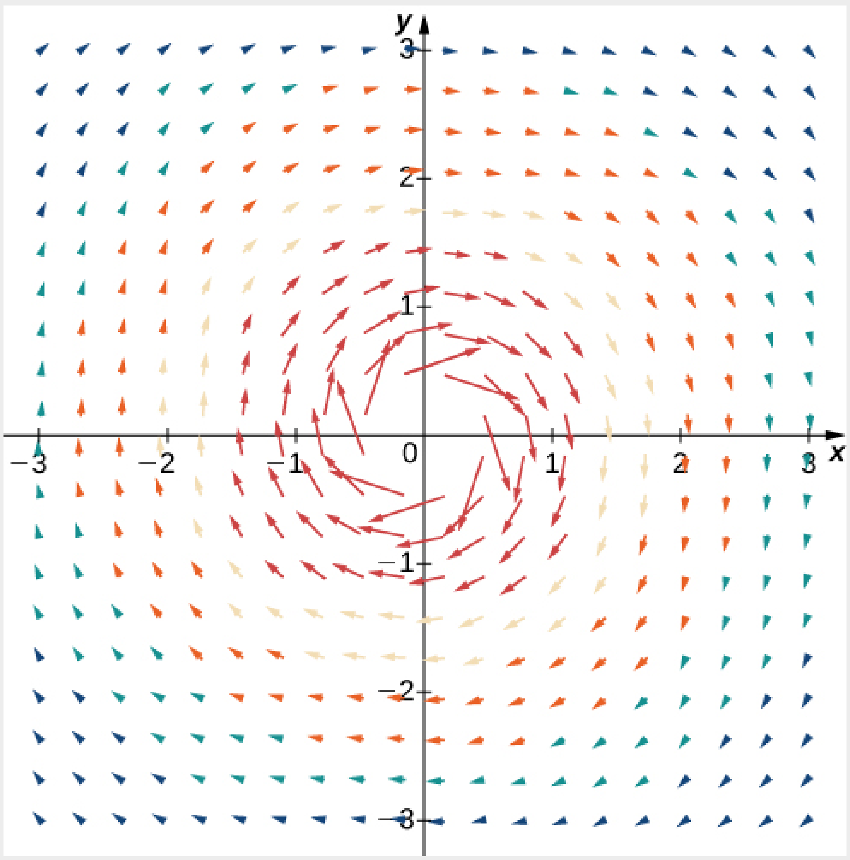
Solution:

To visualize this vector field, first note that the dot product $\mathbf{F}(a, b) \cdot (a\mathbf{i} + b\mathbf{j})$ is zero for any point (a, b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a, b) \to (0, 0)$, the magnitude of $\mathbf{F}(a, b)$ goes to infinity. To see this, note that
Equation:

$$||\mathbf{F}(a, b)|| = \sqrt{\frac{a^2 + b^2}{(a^2 + b^2)^2}} = \sqrt{\frac{1}{a^2 + b^2}}.$$

Since $\frac{1}{a^2+b^2} \to \infty$ as $(a, b) \to (0, 0)$, then $||\mathbf{F}(a, b)|| \to \infty$ as $(a, b) \to (0, 0)$. This vector field looks similar to the vector field in [link], but in this case the magnitudes of the vectors close to the origin are large. The table below shows a sample of points and the corresponding vectors, and [link] shows the vector field. Note that this vector field models the whirlpool motion of the river in [link](b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0, 0)$.

(x,y)	$\mathbf{F}(x,y)$	(x,y)	$\mathbf{F}(x,y)$	(x,y)	$\mathbf{F}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\left\langle 0,-\frac{1}{2}\right\rangle$	$(1,1)$	$\left\langle \frac{1}{2},-\frac{1}{2}\right\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\left\langle \frac{1}{2},0\right\rangle$	$(-1,1)$	$\left\langle \frac{1}{2},\frac{1}{2}\right\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\left\langle 0,\frac{1}{2}\right\rangle$	$(-1,-1)$	$\left\langle -\frac{1}{2},\frac{1}{2}\right\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\left\langle -\frac{1}{2},0\right\rangle$	$(1,-1)$	$\left\langle -\frac{1}{2},-\frac{1}{2}\right\rangle$



A visual representation of vector field $\mathbf{F}(x,y)=\frac{y}{x^2+y^2}\,\mathbf{i}-\frac{x}{x^2+y^2}\,\mathbf{j}$. This vector field could be used to model whirlpool motion of a fluid.

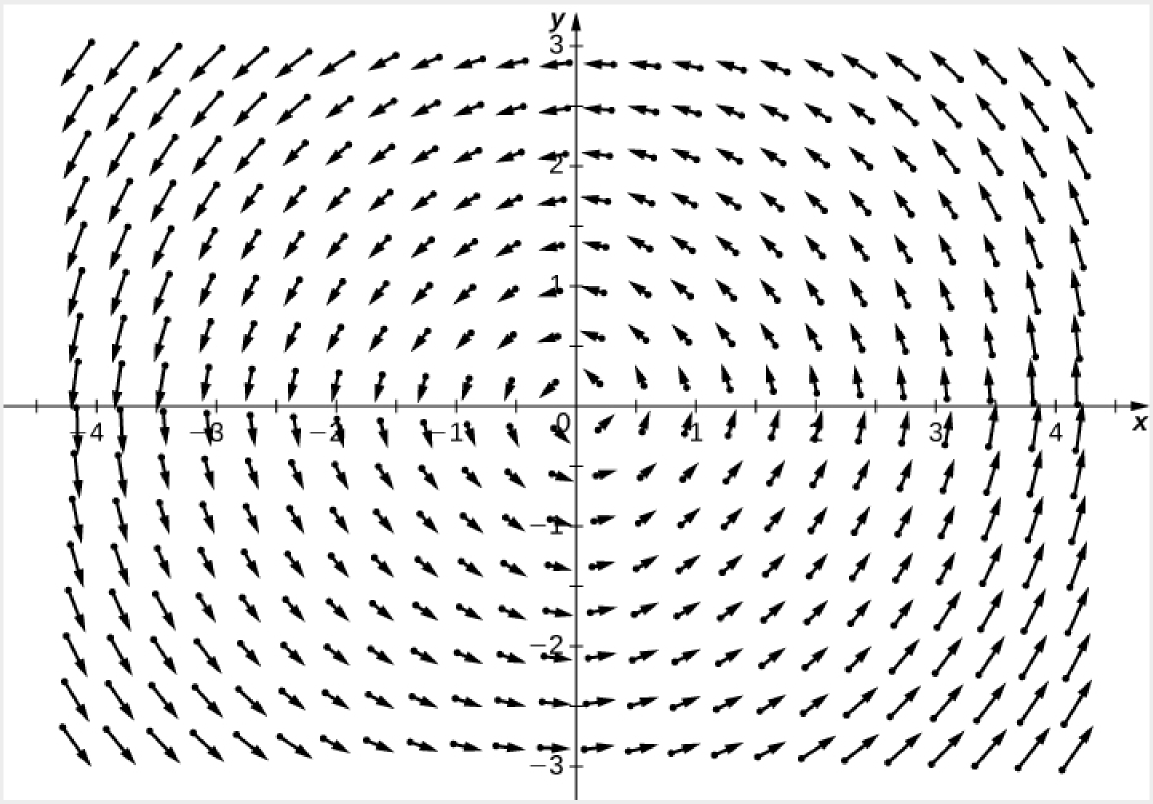
Note:
Exercise:

Problem:

Sketch vector field $\mathbf{F}(x, y) = \langle -2y, 2x \rangle$. Is the vector field radial, rotational, or neither?

Solution:

Rotational



Hint

Substitute enough points into $\mathbf{F}$ to get an idea of the shape.

Example:
Exercise:

Problem:
Velocity Field of a Fluid

Suppose that $\mathbf{v}\,(x,y) = -\frac{2y}{x^2+y^2}\,\mathbf{i} + \frac{2x}{x^2+y^2}\,\mathbf{j}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution:

To find the velocity of the fluid at point $(1,-1)$, substitute the point into **v**:
Equation:

$$\mathbf{v}\,(1,-1) = -\frac{2\,(-1)}{1+1}\,\mathbf{i} + \frac{2\,(1)}{1+1}\,\mathbf{j} = \mathbf{i} + \mathbf{j}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\mathbf{i}+\mathbf{j}|| = \sqrt{2}$ m/sec.

Note:
Exercise:

Problem:

Vector field $v(x,y) = \langle 4\,|x|, 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Solution:

$\sqrt{65}$ m/sec

Hint

Remember, speed is the magnitude of velocity.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field **F** is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example:
Exercise:

Problem:
A Unit Vector Field

Show that vector field $\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution:

To show that **F** is a unit field, we must show that the magnitude of each vector is 1. Note that

Equation:

$$\begin{aligned} \sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2}} \\ &= 1. \end{aligned}$$

Therefore, **F** is a unit vector field.

Note:
Exercise:

Problem: Is vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$ a unit vector field?

Solution:

No.

Hint

Calculate the magnitude of **F** at an arbitrary point (x, y).

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector)

is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{||\mathbf{F}||}, \frac{Q}{||\mathbf{F}||}, \frac{R}{||\mathbf{F}||} \right\rangle$. Notice that if $\mathbf{F}(x, y) = \langle y, -x \rangle$ is the vector field from [link], then the magnitude of $\mathbf{F}$ is $\sqrt{x^2 + y^2}$, and therefore the corresponding unit vector field is the field $\mathbf{G}$ from the previous example.

If $\mathbf{F}$ is a vector field, then the process of dividing $\mathbf{F}$ by its magnitude to form unit vector field $\mathbf{F}/\,||\mathbf{F}||$ is called *normalizing* the field $\mathbf{F}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

Equation:

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

or

Equation:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

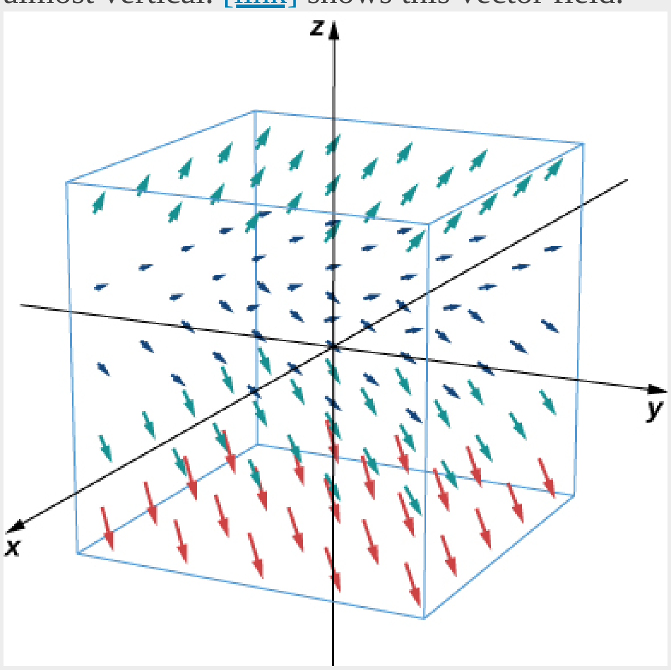
Example:
Exercise:

Problem:
Sketching a Vector Field in Three Dimensions

Describe vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

Solution:

For this vector field, the *x* and *y* components are constant, so every point in $\mathbb{R}^3$ has an associated vector with *x* and *y* components equal to one. To visualize **F**, we first consider what the field looks like in the *xy*-plane. In the *xy*-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0 \rangle$ associated with it. For points not in the *xy*-plane but slightly above it, the associated vector has a small but positive *z* component, and therefore the associated vector points slightly upward. For points that are far above the *xy*-plane, the *z* component is large, so the vector is almost vertical. [link] shows this vector field.



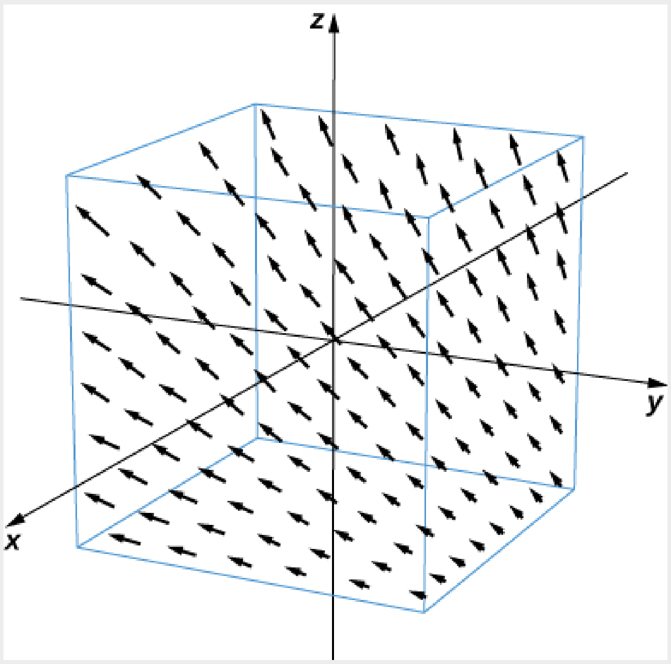
A visual representation of vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

Note:

Exercise:

Problem: Sketch vector field $\mathbf{G}\left(x,y,z\right)=\left\langle 2,\frac{z}{2},1\right\rangle$.

Solution:



Hint

Substitute enough points into the vector field to get an idea of the general shape.

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example:
Exercise:

Problem:
Describing a Gravitational Vector Field

Newton's law of gravitation states that $\mathbf{F} = -G\,\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field **F** denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution:

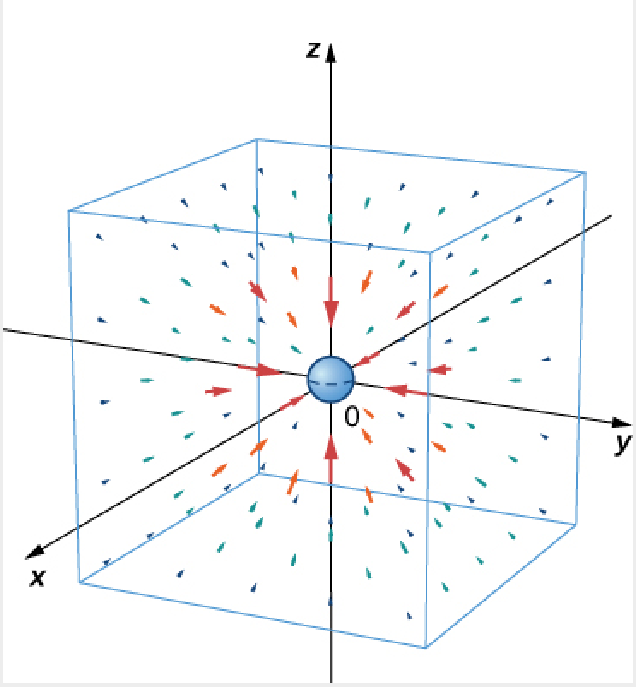
Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x,y,z\rangle}{||\langle x,y,z\rangle||}$, and hence $\hat{\mathbf{r}} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$. Therefore, gravitational vector field **F** exerted by object 1 on object 2 is

Equation:

$$\mathbf{F} = -Gm_1 m_2 \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

[link] shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.



A visual representation of gravitational vector field $\mathbf{F} = -Gm_1m_2\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle$ for a large mass at the origin.

Note:

Exercise:

Problem:

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Solution:

$1.49063 \times 10^{-18}, 4.96876 \times 10^{-19}, 9.93752 \times 10^{-19}\text{N}$

Hint

Follow [link] and first compute the distance between the asteroids.

Gradient Fields

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of *x* and *y*, then the gradient of f is
Equation:

$$\text{grad } f = \nabla f = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}.$$

We can see from the form in which the gradient is written that ∇f is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of *x*, *y*, and *z*, then the gradient of f is
Equation:

$$\text{grad } f = \nabla f = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}.$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$.

A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

Note:
Definition
A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a **gradient field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

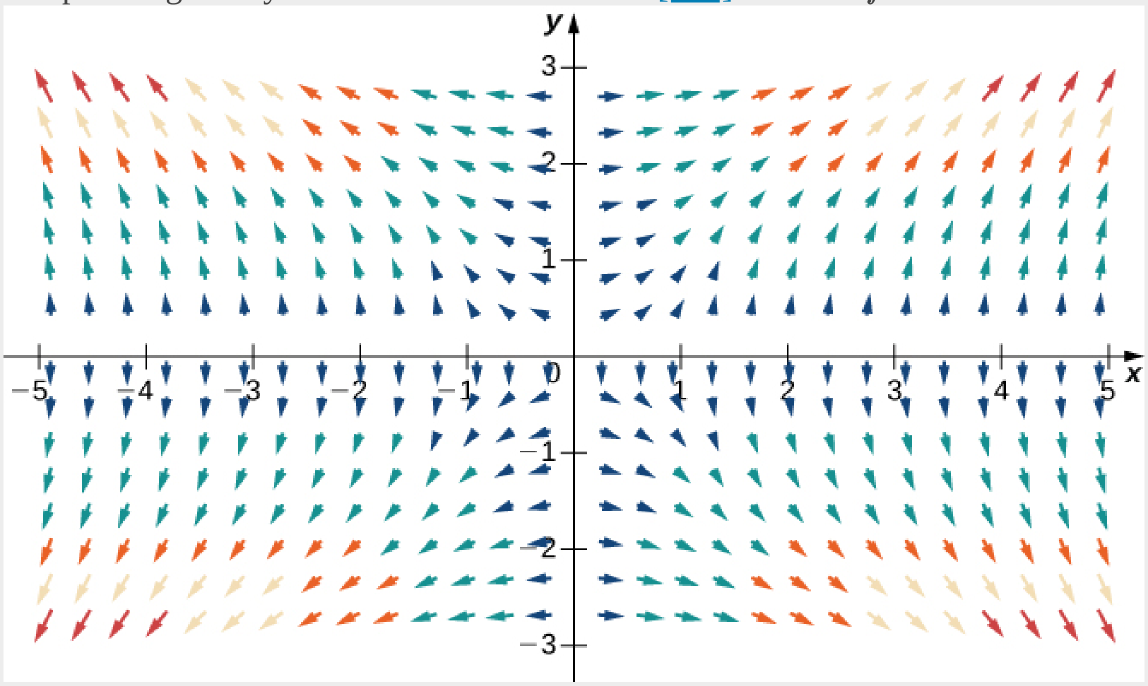
Example:
Exercise:

Problem:
Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x,y) = x^2y^2$.

Solution:

The gradient of f is $\nabla f = \langle 2xy^2, 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. [link] shows ∇f.



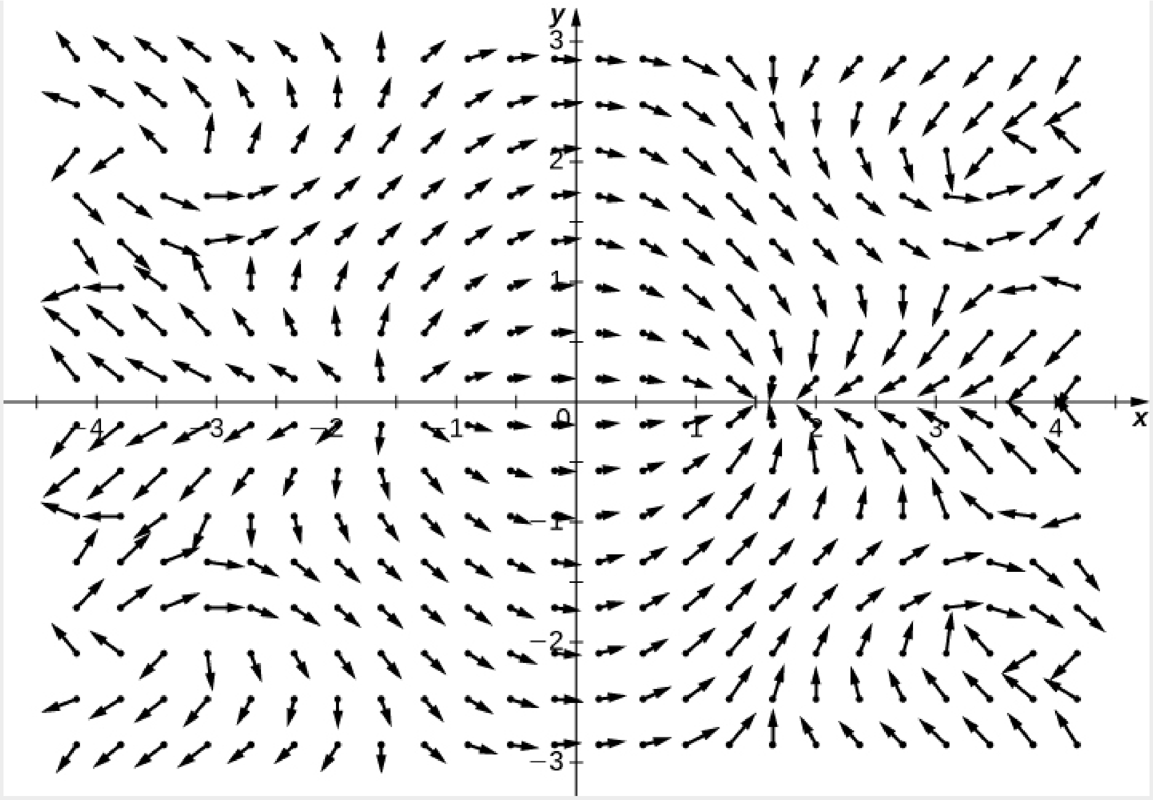
The gradient vector field is ∇f, where $f(x, y) = x^2y^2$.

Note:
Exercise:

Problem:

Use technology to plot the gradient vector field of $f(x, y) = \sin x \cos y$.

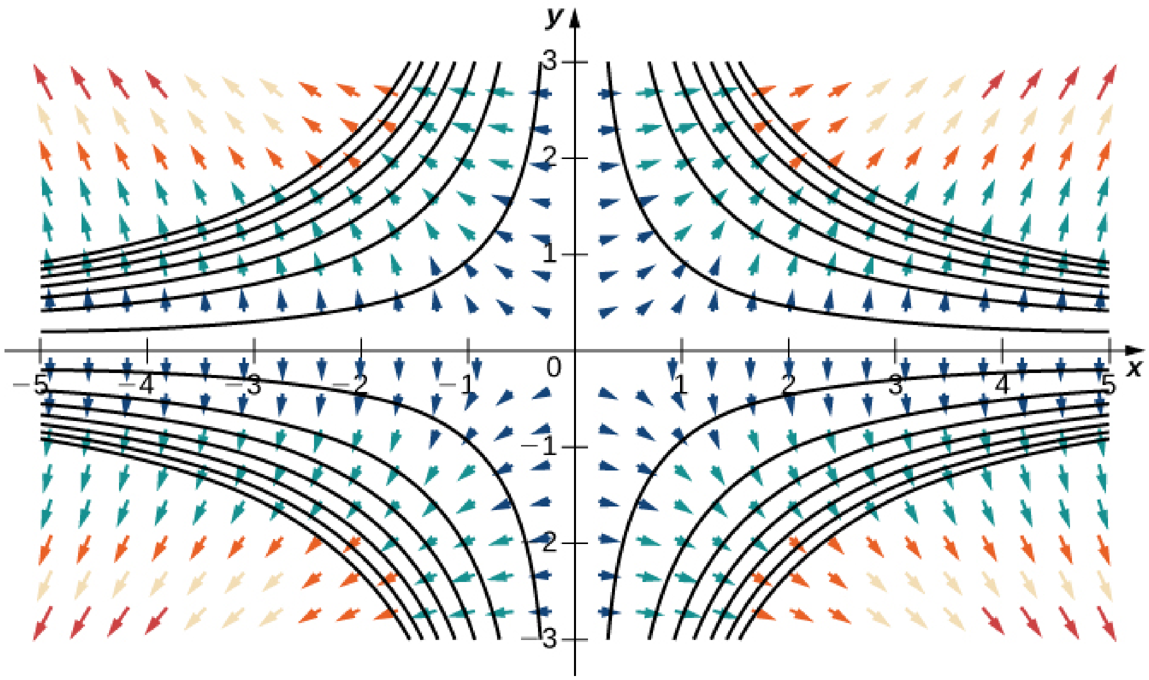
Solution:



Hint

Find the gradient of f.

Consider the function $f(x, y) = x^2y^2$ from [link]. [link] shows the level curves of this function overlaid on the function’s gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function’s gradient field.



The gradient field of $f(x, y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\mathbf{F}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\nabla f = \mathbf{F}$. In this situation, f is called a **potential function** for $\mathbf{F}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\mathbf{F}$ is defined instead as a function such that $-\nabla f = \mathbf{F}$. This is the case for certain contexts in physics, for example.

Example:
Exercise:

Problem:
Verifying a Potential Function

Is $f(x, y, z) = x^2yz - \sin(xy)$ a potential function for vector field

Equation:

$$\mathbf{F}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle ?$$

Solution:

We need to confirm whether $\nabla f = \mathbf{F}$. We have

Equation:

$$f_x = 2xyz - y\cos(xy), f_y = x^2z - x\cos(xy), \text{ and } f_z = x^2y.$$

Therefore, $\nabla f = \mathbf{F}$ and f is a potential function for $\mathbf{F}$.

Note:

Exercise:

Problem:

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\mathbf{F}(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Solution:

No

Hint

Compute the gradient of f.

Example:

Exercise:

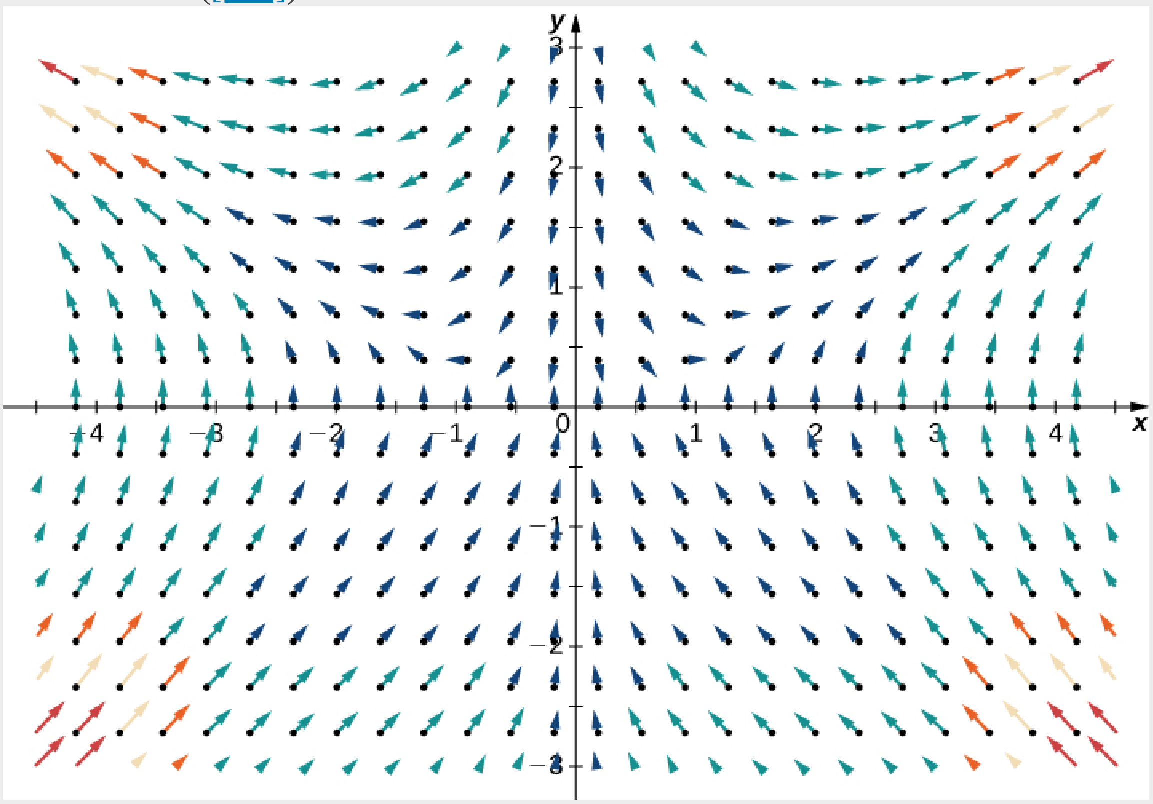
Problem:

Verifying a Potential Function

The velocity of a fluid is modeled by field $\mathbf{v}(x,y) = \left\langle xy, \frac{x^2}{2} - y\right\rangle$. Verify that $f(x,y) = \frac{x^2y}{2} - \frac{y^2}{2}$ is a potential function for $\mathbf{v}$.

Solution:

To show that f is a potential function, we must show that $\nabla f = \mathbf{v}$. Note that $f_x = xy$ and $f_x = \frac{x^2}{2} - y$. Therefore, $\nabla f = \left\langle xy, \frac{x^2}{2} - y \right\rangle$ and f is a potential function for v ([link]).



Velocity field $\mathbf{v}(x, y)$ has a potential function and is a conservative field.

Note:
Exercise:

Problem:

Verify that $f(x, y) = x^2y^2 + x$ is a potential function for velocity field $\mathbf{v}(x, y) = \langle 3x^2y^2 + 1, 2x^3y \rangle$.

Solution:

$\nabla f = \mathbf{v}$

Hint

Calculate the gradient.

If **F** is a conservative vector field, then there is at least one potential function f such that $\nabla f = \mathbf{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then *k* has infinitely many antiderivatives. Furthermore, if *F* and *G* are both antiderivatives of *k*, then *F* and *G* differ only by a constant. That is, there is some number *C* such that $F(x) = G(x) + C$.

Now let **F** be a conservative vector field and let f and *g* be potential functions for **F**. Since the gradient is like a derivative, **F** being conservative means that **F** is "integrable" with "antiderivatives" f and *g*. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant *C* such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

Note:
Uniqueness of Potential Functions
Let **F** be a conservative vector field on an open and connected domain and let f and *g* be functions such that $\nabla f = \mathbf{F}$ and $\nabla g = \mathbf{F}$. Then, there is a constant *C* such that $f = g + C$.

Proof

Since f and *g* are both potential functions for **F**, then $\nabla(f - g) = \nabla f - \nabla g = \mathbf{F} - \mathbf{F} = 0$. Let $h = f - g$, then we have $\nabla h = 0$. We would like to show that *h* is a constant function.

Assume *h* is a function of *x* and *y* (the logic of this proof extends to any number of independent variables). Since $\nabla h = 0$, we have $h_x = 0$ and $h_y = 0$. The expression $h_x = 0$ implies that *h* is a constant function with respect to *x*—that is, $h(x, y) = k_1(y)$ for some function k_1. Similarly, $h_y = 0$ implies $h(x, y) = k_2(x)$ for some function k_2. Therefore, function *h* depends only on *y* and also depends only on *x*. Thus, $h(x, y) = C$

for some constant *C* on the connected domain of **F**. Note that we really do need connectedness at this point; if the domain of **F** came in two separate pieces, then *k* could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the *cross-partial property*. This property helps test whether a given vector field is conservative.

Note:
The Cross-Partial Property of Conservative Vector Fields
Let **F** be a vector field in two or three dimensions such that the component functions of **F** have continuous second-order mixed-partial derivatives on the domain of **F**.
If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field in $\mathbb{R}^2$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If $\mathbf{F}\left(x,y,z\right) = \langle P\left(x,y,z\right), Q\left(x,y,z\right), R\left(x,y,z\right)\rangle$ is a conservative vector field in $\mathbb{R}^3$, then
Equation:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since **F** is conservative, there is a function $f(x,y)$ such that $\nabla f = \mathbf{F}$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut’s theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut’s theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

[link] shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won’t have equal cross-partials.

Example:

Exercise:

Problem:
Showing a Vector Field Is Not Conservative

Show that rotational vector field $\mathbf{F}\left(x,y\right)=\langle y,-x\rangle$ is not conservative.

Solution:

Let $P\left(x,y\right)=y$ and $Q\left(x,y\right)=-x$. If $\mathbf{F}$ is conservative, then the cross-partials would be equal—that is, P_y would equal $Q_{x.}$ Therefore, to show that $\mathbf{F}$ is not conservative, check that $P_y\ne Q_x$. Since $P_y=1$ and $Q_x=-1$, the vector field is not conservative.

Note:
Exercise:

Problem: Show that vector field $\mathbf{F}\left(x,y\right)xy\mathbf{i}-x^2y\mathbf{j}$ is not conservative.

Solution:

$P_y=x\ne Q_x=-2xy$

Hint

Check the cross-partials.

Example:
Exercise:

Problem:
Showing a Vector Field Is Not Conservative

Is vector field $\mathbf{F}\left(x,y,z\right)=\left\langle 7,-2,x^3\right\rangle$ conservative?

Solution:

Let $P(x,y,z)=7$, $Q(x,y,z)=-2$, and $R(x,y,z)=x^3$. If $\mathbf{F}$ is conservative, then all three cross-partial equations will be satisfied—that is, if $\mathbf{F}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$P_y = Q_x = R_y = Q_z = 0$, so the first two necessary equalities hold. However, $R_x = 3x^3$ and $P_z = 0$ so $R_x \neq P_z$. Therefore, **F** is not conservative.

Note:
Exercise:

Problem: Is vector field $G\left(x, y, z\right) = \langle y, x, xyz\rangle$ conservative?

Solution:

No

Hint

Check the cross-partials.

We conclude this section with a word of warning: [link] says that if **F** is conservative, then **F** has the cross-partial property. The theorem does *not* say that, if **F** has the cross-partial property, then **F** is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, [link] can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative. For example, consider vector field $\mathbf{F}(x, y) = \left\langle x^2 y, \frac{x^3}{3}\right\rangle$. This field has the cross-partial property, so it is natural to try to use [link] to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that **F** is conservative.

Key Concepts

- A vector field assigns a vector $\mathbf{F}(x, y)$ to each point (x, y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\mathbf{F}(x, y, z)$ to each point (x, y, z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field **F** is called conservative if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$
  or
  $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$
- **Vector field in $\mathbb{R}^3$**
  $\mathbf{F}\left(x,y,z\right) = \langle P\left(x,y,z\right), Q\left(x,y,z\right), R\left(x,y,z\right)\rangle$
  or
  $\mathbf{F}\left(x,y,z\right) = P\left(x,y,z\right)\mathbf{i} + Q\left(x,y,z\right)\mathbf{j} + R\left(x,y,z\right)\mathbf{k}$

Exercise:

Problem:

The domain of vector field $\mathbf{F} = \mathbf{F}\left(x,y\right)$ is a set of points $\left(x,y\right)$ in a plane, and the range of **F** is a set of *what* in the plane?

Solution:

Vectors

For the following exercises, determine whether the statement is *true or false*.

Exercise:

Problem:

Vector field $\mathbf{F} = \left\langle 3x^2, 1\right\rangle$ is a gradient field for both $\phi_1\left(x,y\right) = x^3 + y$ and $\phi_2\left(x,y\right) = y + x^3 + 100$.

Exercise:

Problem:

Vector field $\mathbf{F} = \frac{\langle y,x\rangle}{\sqrt{x^2+y^2}}$ is constant in direction and magnitude on a unit circle.

Solution:

False

Exercise:

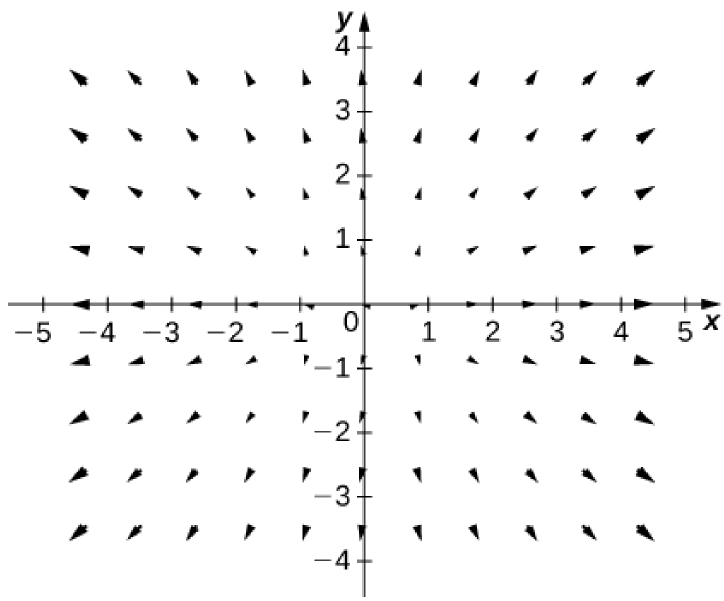
Problem: Vector field $\mathbf{F} = \frac{\langle y,x\rangle}{\sqrt{x^2+y^2}}$ is neither a radial field nor a rotation.

For the following exercises, describe each vector field by drawing some of its vectors.

Exercise:

Problem: [T] $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

Solution:



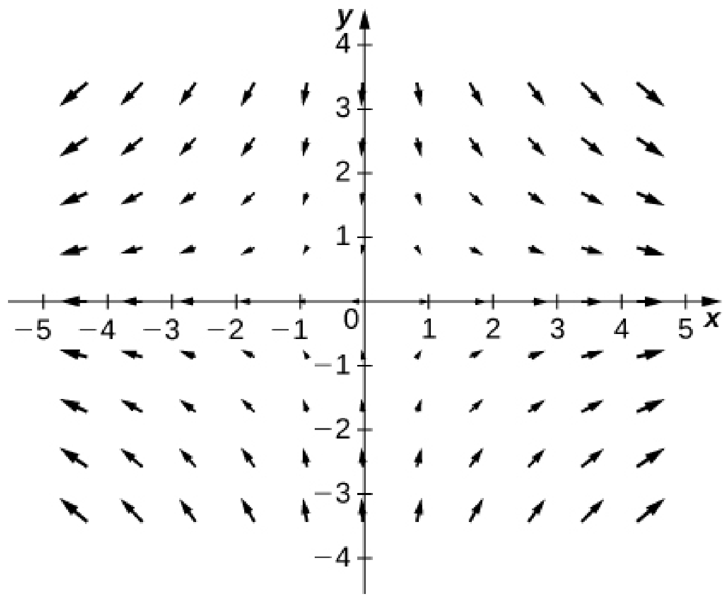
Exercise:

Problem: [T] $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

Exercise:

Problem: [T] $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

Solution:



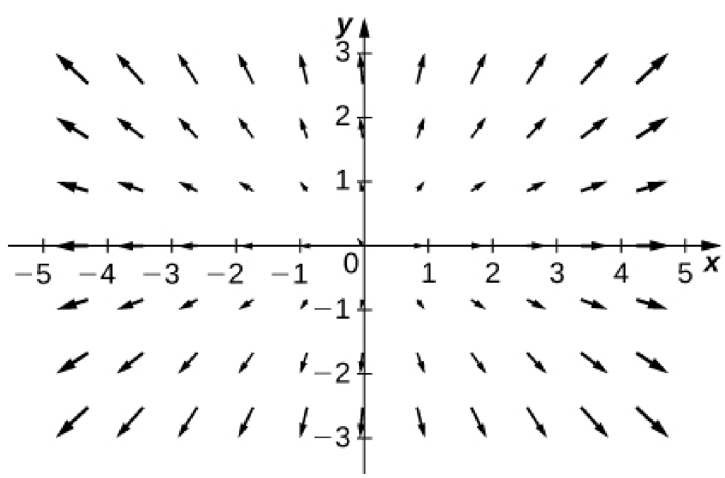
Exercise:

Problem: [T] $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

Exercise:

Problem: [T] $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y\mathbf{j}$

Solution:



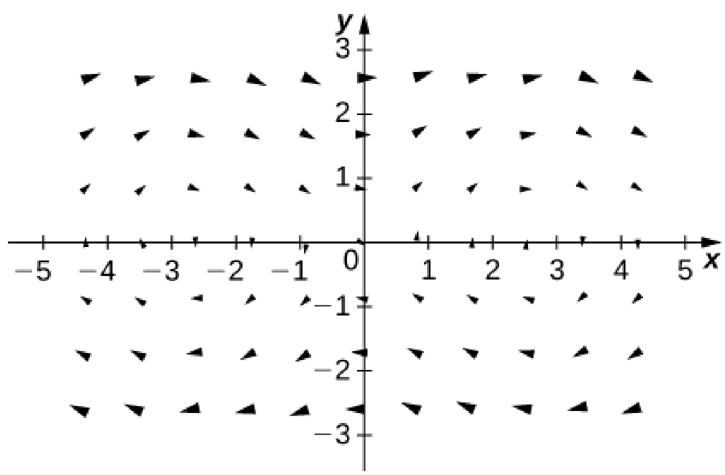
Exercise:

Problem: [T] $\mathbf{F}(x,y)=3\mathbf{i}+x\mathbf{j}$

Exercise:

Problem: [T] $\mathbf{F}(x,y)=y\mathbf{i}+\sin x\mathbf{j}$

Solution:



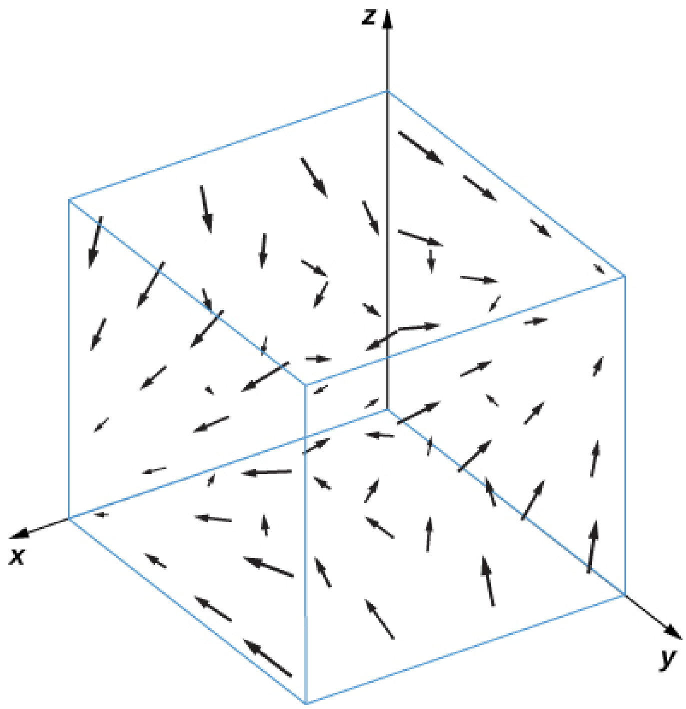
Exercise:

Problem: [T] $\mathbf{F}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Exercise:

Problem: [T] $\mathbf{F}(x,y,z)=2x\mathbf{i}-2y\mathbf{j}-2z\mathbf{k}$

Solution:



Exercise:

Problem: [T] $\mathbf{F}(x,y,z)=\frac{y}{z}\mathbf{i}-\frac{x}{z}\mathbf{j}$

For the following exercises, find the gradient vector field of each function f.

Exercise:

Problem: $f(x,y)=x\sin y+\cos y$

Solution:

$\mathbf{F}(x,y)=\sin(y)\mathbf{i}+(x\cos y-\sin y)\mathbf{j}$

Exercise:

Problem: $f(x,y,z)=ze^{-xy}$

Exercise:

Problem: $f(x,y,z)=x^2y+xy+y^2z$

Solution:

$\mathbf{F}(x,y,z)=(2xy+y)\mathbf{i}+(x^2+x+2yz)\mathbf{j}+y^2\mathbf{k}$

Exercise:

Problem: $f(x, y) = x^2 \sin(5y)$

Exercise:

Problem: $f(x, y) = \ln\left(1 + x^2 + 2y^2\right)$

Solution:

$$\mathbf{F}(x, y) = \left(\frac{2x}{1+x^2+2y^2}\right)\mathbf{i} + \left(\frac{4y}{1+x^2+2y^2}\right)\mathbf{j}$$

Exercise:

Problem: $f(x, y, z) = x \cos\left(\frac{y}{z}\right)$

Exercise:

Problem:

What is vector field $\mathbf{F}\,(x, y)$ with a value at (x, y) that is of unit length and points toward $(1, 0)$?

Solution:

$$\mathbf{F}(x, y) = \frac{(1-x)\mathbf{i} - y\mathbf{j}}{\sqrt{(1-x)^2 + y^2}}$$

For the following exercises, write formulas for the vector fields with the given properties.

Exercise:

Problem:

All vectors are parallel to the *x*-axis and all vectors on a vertical line have the same magnitude.

Exercise:

Problem: All vectors point toward the origin and have constant length.

Solution:

$$\mathbf{F}(x, y) = \frac{(y\mathbf{i} - x\mathbf{j})}{\sqrt{x^2 + y^2}}$$

Exercise:

Problem:

All vectors are of unit length and are perpendicular to the position vector at that point.

Exercise:

Problem:

Give a formula $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in a plane that has the properties that $\mathbf{F} = 0$ at $(0, 0)$ and that at any other point (a, b), $\mathbf{F}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Solution:

$\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

Exercise:

Problem:

Is vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y)) = (\sin x + y)\mathbf{i} + (\cos y + x)\mathbf{j}$ a gradient field?

Exercise:

Problem:

Find a formula for vector field $\mathbf{F}(x, y) = \mathrm{M}(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ given the fact that for all points (x, y), $\mathbf{F}$ points toward the origin and $|\mathbf{F}| = \frac{10}{x^2+y^2}$.

Solution:

$\mathbf{F}(x, y) = \frac{-10}{(x^2+y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$

For the following exercises, assume that an electric field in the *xy*-plane caused by an infinite line of charge along the *x*-axis is a gradient field with potential function $V(x, y) = c \ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

Exercise:

Problem:

Find the components of the electric field in the *x*- and *y*-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

Exercise:

Problem:

Show that the electric field at a point in the *xy*-plane is directed outward from the origin and has magnitude $\mathbf{E}| = \frac{c}{r}$, where $r = \sqrt{x^2 = y^2}$.

Solution:

$E = \frac{c}{|r|^2} r = \frac{c}{|r|} \frac{r}{|r|}$

A *flow line* (or *streamline*) of a vector field $\mathbf{F}$ is a curve $\mathbf{r}(t)$ such that $d\mathbf{r}/dt = \mathbf{F}(\mathbf{r}(t))$. If $\mathbf{F}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field. For the following exercises, show that the given curve $\mathbf{c}(t)$ is a flow line of the given velocity vector field $\mathbf{F}(x, y, z)$.

Exercise:

Problem: $\mathbf{c}(t) = \left(e^{2t}, \ln|t|, \frac{1}{t}\right), t \neq 0; \mathbf{F}(x, y, z) = \left\langle 2x, z, -z^2 \right\rangle$

Exercise:

Problem: $\mathbf{c}(t) = \left(\sin t, \cos t, e^t\right); \mathbf{F}(x, y, z) = \langle y, -x, z \rangle$

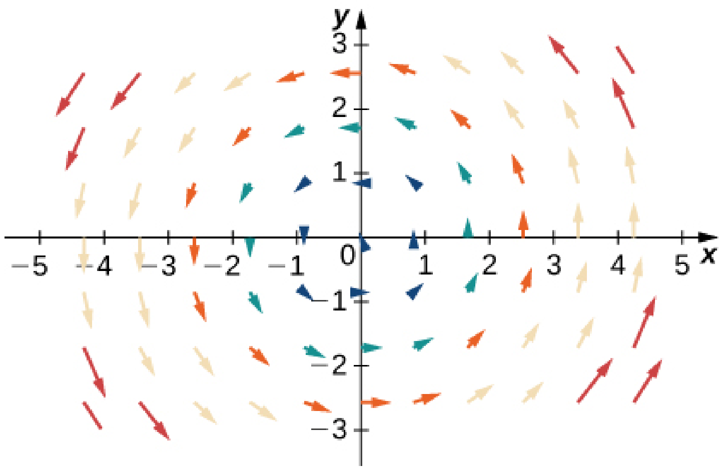
Solution:

$\mathbf{c}\prime(t) = \left(\cos t, -\sin t, e^{-t}\right) = \mathbf{F}(\mathbf{c}(t))$

For the following exercises, let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$, and $\mathbf{H} = x\mathbf{i} - y\mathbf{j}$. Match $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ with their graphs.

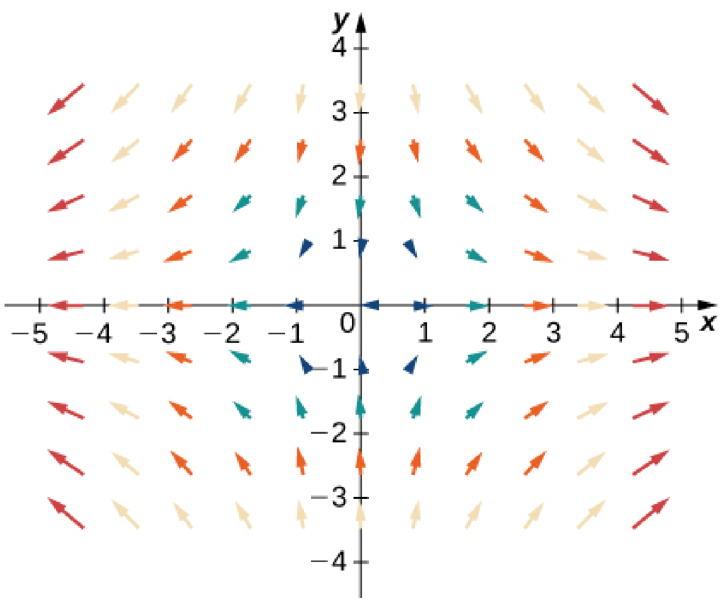
Exercise:

Problem:



Exercise:

Problem:

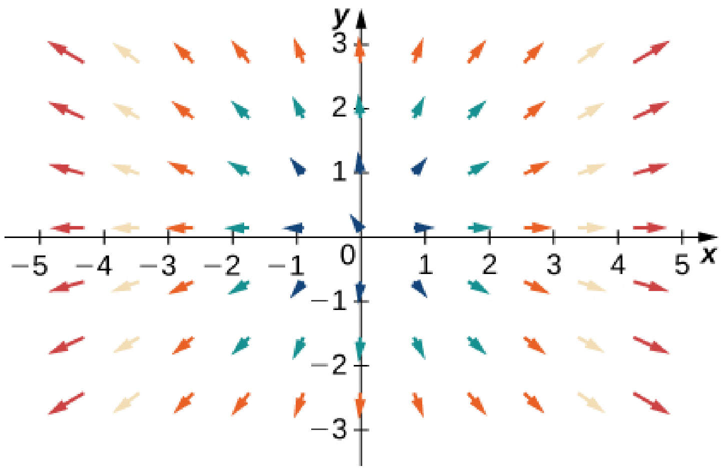


Solution:

H

Exercise:

Problem:

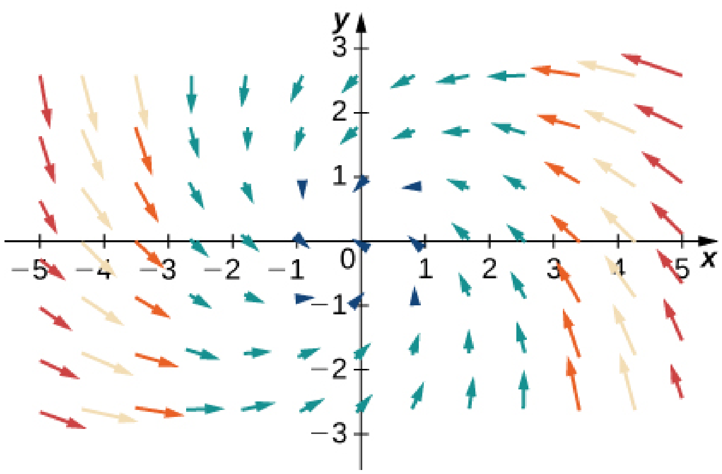


For the following exercises, let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$, and $\mathbf{H} = x\mathbf{i} - y\mathbf{j}$. Match the vector fields with their graphs in $(\mathrm{I}) - (\mathrm{IV})$.

a. $\mathbf{F} + \mathbf{G}$
b. $\mathbf{F} + \mathbf{H}$
c. $\mathbf{G} + \mathbf{H}$
d. $-\mathbf{F} + \mathbf{G}$

Exercise:

Problem:

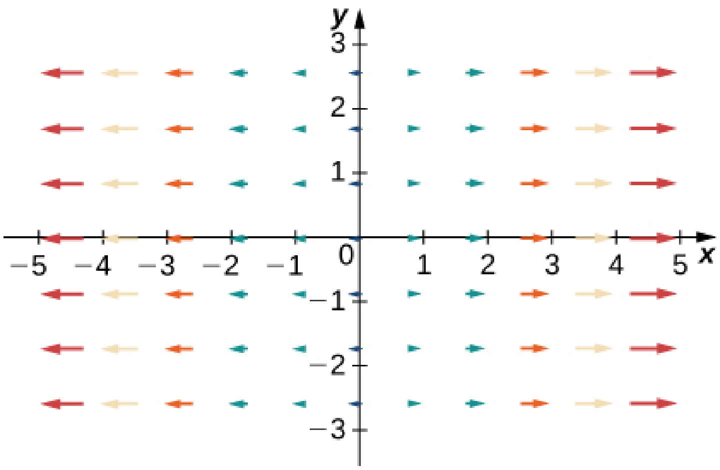


Solution:

d. $-\mathbf{F} + \mathbf{G}$

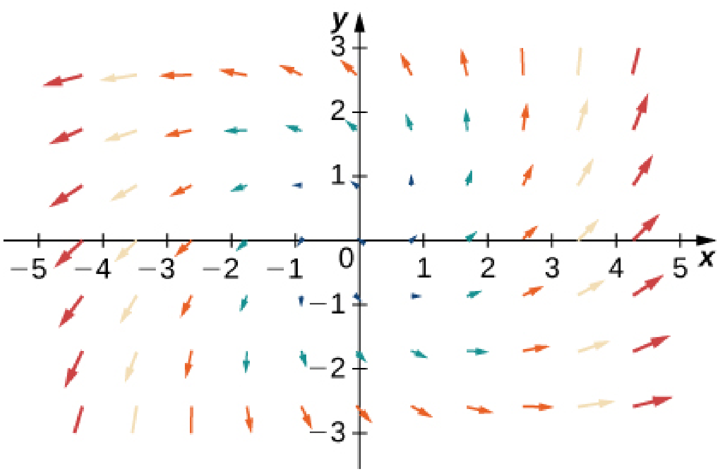
Exercise:

Problem:



Exercise:

Problem:

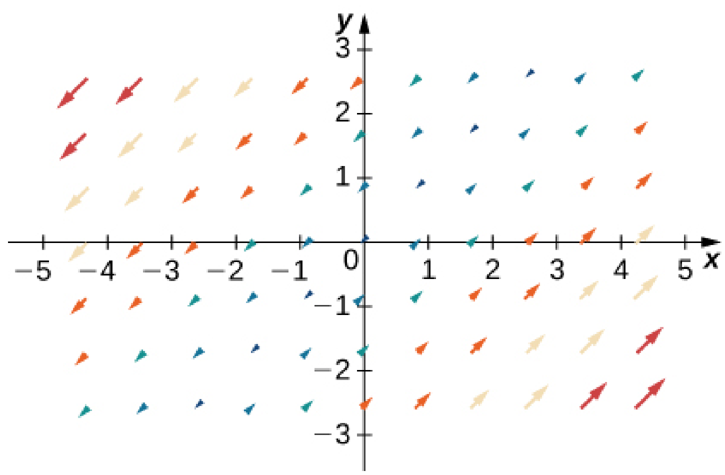


Solution:

a. $\mathbf{F} + \mathbf{G}$

Exercise:

Problem:



Glossary

conservative field
: a vector field for which there exists a scalar function f such that $\nabla f = \mathbf{F}$

gradient field
: a vector field $\mathbf{F}$ for which there exists a scalar function f such that $\nabla f = \mathbf{F}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

potential function
: a scalar function f such that $\nabla f = \mathbf{F}$

radial field
: a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

rotational field
: a vector field in which the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

unit vector field
: a vector field in which the magnitude of every vector is 1

vector field
: measured in $\mathbb{R}^2$, an assignment of a vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a

subset D of $\mathbb{R}^3$

Line Integrals

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)dx$, where the domain of integration is an interval $[a, b]$. Such an interval can be thought of as a curve in the *xy*-plane, since the interval defines a line segment with endpoints $(a, 0)$ and $(b, 0)$—in other words, a line segment located on the *x*-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the *x*-axis. Such a task requires a new kind of integral, called a *line integral*.

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

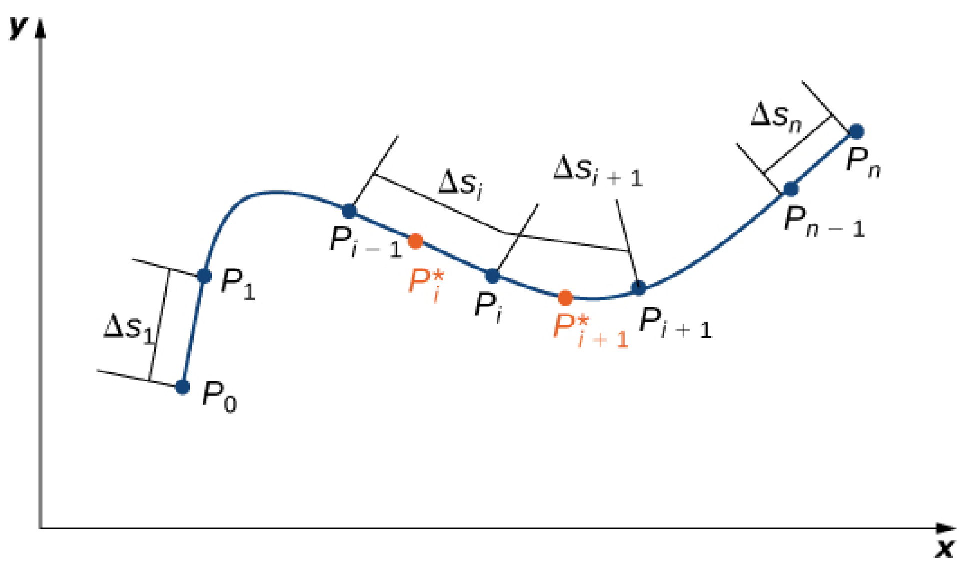
Scalar Line Integrals

A **line integral** gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the *x*-axis.

For a scalar line integral, we let *C* be a smooth curve in a plane or in space and let f be a function with a domain that includes *C*. We chop the curve into small pieces. For each piece, we choose point *P* in that piece and evaluate f at *P*. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x, y, z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into *n* subintervals $[t_{i-1}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ ([link]). Let t_i^* be a value in the *i*th interval $[t_{i-1}, t_i]$. Denote the endpoints of $\mathbf{r}(t_0), \mathbf{r}(t_1), \ldots, \mathbf{r}(t_n)$ by $P_0, \ldots, P_n$. Points P_i divide curve C into n pieces $C_1, C_2, \ldots, C_n,$ with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$, respectively. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f\left(P_i^*\right)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f\left(P_i^*\right)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f\left(P_i^*\right)\Delta s_i$. Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.s



Curve *C* has been divided into *n* pieces, and a point inside

each piece has been chosen.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ aren't necessarily the same; in the definition of a single-variable integral, the curve in the *x*-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

Note:
Definition
Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$. The **scalar line integral** of f along C is
Equation:

$$\int_C f(x,y,z)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(P_i^*\right) \Delta s_i$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t), y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as
Equation:

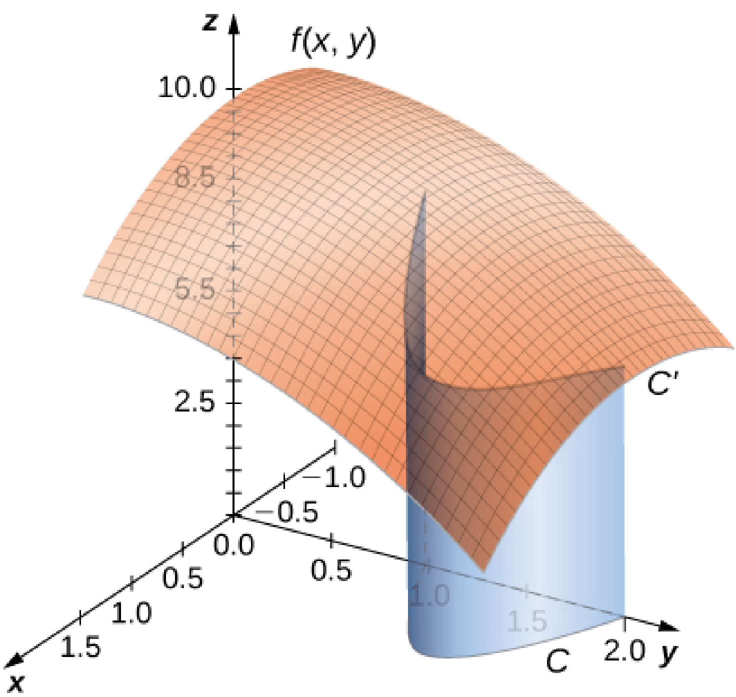
$$\int_C f(x,y)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(P_i^*\right) \Delta s_i,$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f ds$ always exists. Since $\int_C f ds$ is defined as a limit of Riemann sums, the continuity of f is enough to

guarantee the existence of the limit, just as the integral $\int_a^b g(x)dx$ exists if g is continuous over $[a, b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x, y) \geq 0$ for all points (x, y) on a smooth planar curve C. Imagine taking curve C and projecting it “up” to the surface defined by $f(x, y)$, thereby creating a new curve C' that lies in the graph of $f(x, y)$ ([link]). Now we drop a “sheet” from C' down to the xy-plane. The area of this sheet is $\int_C f(x, y)ds$. If $f(x, y) \leq 0$ for some points in C, then the value of $\int_C f(x, y)ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)dx.$)



The area of the blue sheet is $\int_C f(x, y)ds$.

From this geometry, we can see that line integral $\int_C f(x, y) ds$ does not depend on the parameterization $\mathbf{r}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

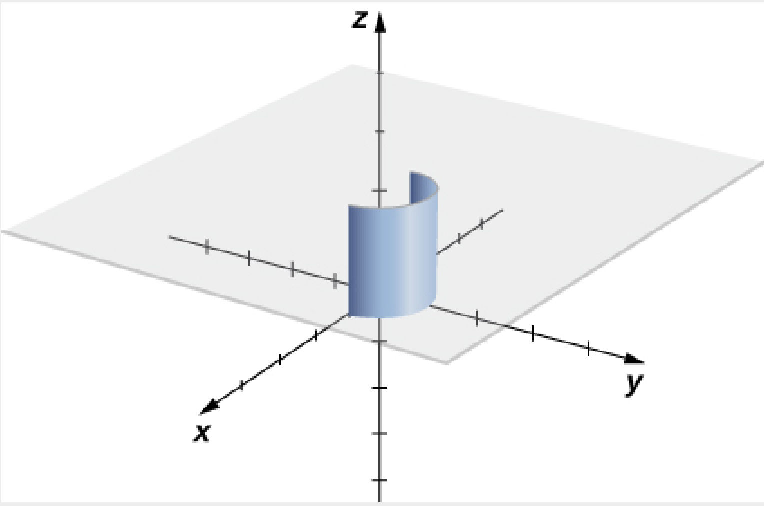
Example:
Exercise:

Problem:
Finding the Value of a Line Integral

Find the value of integral $\int_C 2ds$, where C is the upper half of the unit circle.

Solution:

The integrand is $f(x, y) = 2$. [link] shows the graph of $f(x, y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2ds = 2\pi$.



The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x, y) = 2$.

To see that $\int_C 2ds = 2\pi$ using the definition of line integral, we let $\mathbf{r}(t)$ be a parameterization of C. Then, $f(\mathbf{r}(t_i)) = 2$ for any number t_i in the domain of $\mathbf{r}$. Therefore,

Equation:

$$\begin{aligned} \int_C f ds &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\mathbf{r}\left(t_i^*\right)\right)\Delta s_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\Delta s_i \\ &= 2\lim_{n\to\infty} \sum_{i=1}^{n} 2\Delta s_i \\ &= 2\,(\text{length of C}) \\ &= 2\pi. \end{aligned}$$

Note:

Exercise:

Problem:

Find the value of $\int_C (x+y)ds$, where C is the curve parameterized by $x = t$, $y = t$, $0 \le t \le 1$.

Solution:

$\sqrt{2}$

Hint

Find the shape formed by C and the graph of function $f(x,y) = x + y$.

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f ds$ to an integral with a variable of integration that is t.

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ be a parameterization of C. Since we are assuming that C is smooth, $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is continuous for all t in $[a, b]$. In particular, $x\prime(t), y\prime(t)$, and $z\prime(t)$ exist for all t in $[a, b]$. According to the arc length formula, we have

Equation:

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \|\mathbf{r}'(t)\| dt.$$

If width $\Delta t_i = t_i - t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i} \|\mathbf{r}'(t)\| dt \approx \|r'\left(t_i^*\right)\Delta t_i$, $\|\mathbf{r}'(t)\|$ is almost constant over the interval $[t_{i-1}, t_i]$. Therefore,

Equation:

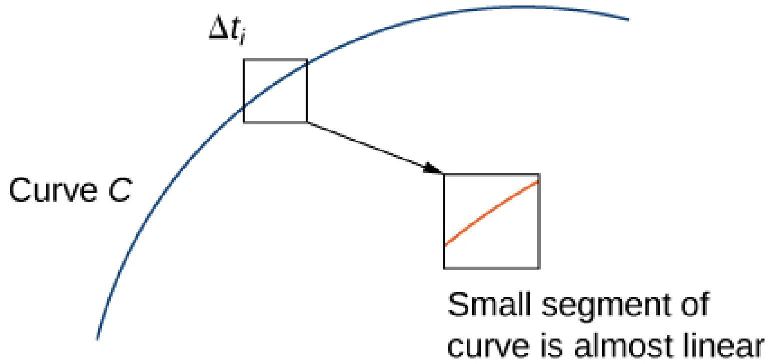
$$\int_{t_{i-1}}^{t_i} \|\mathbf{r}'(t)\| dt \approx \mathbf{r}'(t_i^*)\Delta t_i,$$

and we have

Equation:

$$\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\Delta s_i = \sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\mathbf{r}'(t_i^*)\Delta t_i.$$

See [link].



If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

Equation:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\mathbf{r}'(t_i^*)\Delta t_i = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|dt.$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\mathbf{r}'(t_i^*)\Delta t_i$ converges to the integral $\int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|dt$. Therefore, we have the following theorem.

Note:

Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}\,(t), a \le t \le b$. Then

Equation:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|dt.$$

Although we have labeled [link] as an equation, it is more accurately considered an approximation because we can show that the left-hand side of [link] approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

Equation:

$$\|\mathbf{r}'(t)\| = \sqrt{(x\prime(t))^2 + (y\prime(t))^2 + (z\prime(t))^2},$$

we obtain the following theorem, which we use to compute scalar line integrals.

Note:
Scalar Line Integral Calculation
Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \le t \le b$. Then

Equation:

$$\int_C f(x,y,z)ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

Similarly,

Equation:

$$\int_C f(x,y)ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\mathbf{r}'(t)\|dt$. In other words, the change in arc length can be viewed as a change in the t domain, scaled by the magnitude of vector $\mathbf{r}'(t)$.

Example:
Exercise:

Problem:

Evaluating a Line Integral

Find the value of integral $\int_C \left(x^2 + y^2 + z\right) ds$, where C is part of the helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

Solution:

To compute a scalar line integral, we start by converting the variable of integration from arc length *s* to *t*. Then, we can use [link] to compute the integral with respect to *t*. Note that $f(\mathbf{r}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$ and

Equation:

$$\begin{aligned} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2} &= \sqrt{\left(-\sin(t)\right)^2 + \cos^2(t) + 1} \\ &= \sqrt{2}. \end{aligned}$$

Therefore,

Equation:

$$\int_C \left(x^2 + y^2 + z\right) ds = \int_0^{2\pi} (1 + t)\sqrt{2}dt.$$

Notice that [link] translated the original difficult line integral into a manageable single-variable integral. Since

Equation:

$$\begin{aligned} \int_0^{2\pi} (1 + t)\sqrt{2}dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2, \end{aligned}$$

we have

Equation:

$$\int_C \left(x^2 + y^2 + z\right) ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Note:
Exercise:

Problem:

Evaluate $\int_C \left(x^2 + y^2 + z\right) ds$, where C is the curve with parameterization $\mathbf{r}\left(t\right) = \langle \sin\left(3t\right), \cos\left(3t\right) \rangle, 0 \le t \le \frac{\pi}{4}$.

Solution:

$\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4}$

Hint

Use the two-variable version of [link].

Example:
Exercise:

Problem:
Independence of Parameterization

Find the value of integral $\int_C \left(x^2 + y^2 + z\right) ds$, where C is part of the helix parameterized by $\mathbf{r}\left(t\right) = \langle \cos\left(2t\right), \sin\left(2t\right), 2t \rangle, 0 \le t \le \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution:

As with the previous example, we use [link] to compute the integral with respect to t. Note that $f\left(\mathbf{r}\left(t\right)\right) = \cos^2\left(2t\right) + \sin^2\left(2t\right) + 2t = 2t + 1$ and

Equation:

$$\begin{aligned} \sqrt{\left(x'\left(t\right)\right)^2 + \left(y'\left(t\right)\right)^2 + \left(z\prime(t)\right)^2} &= \sqrt{\left(-\sin t + \cos t + 4\right)} \\ &= 2\sqrt{2} \end{aligned}$$

so we have

Equation:

$$\begin{aligned}\int_C \left(x^2+y^2+z\right)ds &= 2\sqrt{2}\int_0^{\pi}(1+2t)dt \\ &= 2\sqrt{2}\left[t+t^2\right]_0^{\pi} \\ &= 2\sqrt{2}\left(\pi+\pi^2\right).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Note:

Exercise:

Problem:

Evaluate line integral $\int_C \left(x^2+yz\right)ds$, where C is the line with parameterization $\mathbf{r}(t)=\langle 2t, 5t, -t\rangle, 0 \le t \le 10$. Reparameterize C with parameterization $\mathbf{s}(t)=\langle 4t, 10t, -2t\rangle, 0 \le t \le 5$, recalculate line integral $\int_C \left(x^2+yz\right)ds$, and notice that the change of parameterization had no effect on the value of the integral.

Solution:

Both line integrals equal $-\frac{1000\sqrt{30}}{3}$.

Hint

Use [link].

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f\left(x,y,z\right)=1$, then

Equation:

$$\begin{aligned}\int_C f(x, y, z) ds &= \lim_{n \to \infty} \sum_{i=1}^{n} f\left(t_i^*\right) \Delta s_i \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i \\ &= \lim_{n \to \infty} \text{length}\,(C) \\ &= \text{length}\,(C).\end{aligned}$$

Therefore, $\int_C 1 ds$ is the arc length of C.

Example:
Exercise:

Problem:
Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\mathbf{r}(t) = \left\langle \cos t, \sin t, \sqrt{t} \right\rangle, 0 \le t \le 4\pi$. Find the length of the wire.

Solution:

The length of the wire is given by $\int_C 1 ds$, where C is the curve with parameterization $\mathbf{r}$. Therefore,
Equation:

$$\begin{aligned}\text{The length of the wire} &= \int_C 1 ds \\ &= \int_0^{4\pi} \|\mathbf{r}'(t)\| dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t} dt \\ &= \int_0^{4\pi} \sqrt{1+t} dt \\ &= \left[\frac{2(1+t)^{3/2}}{3}\right]_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

Note:
Exercise:

Problem:

Find the length of a wire with parameterization $\mathbf{r}(t) = \langle 3t+1, 4-2t, 5+2t \rangle, 0 \leq t \leq 4.$

Solution:

$4\sqrt{17}$

Hint

Find the line integral of one over the corresponding curve.

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

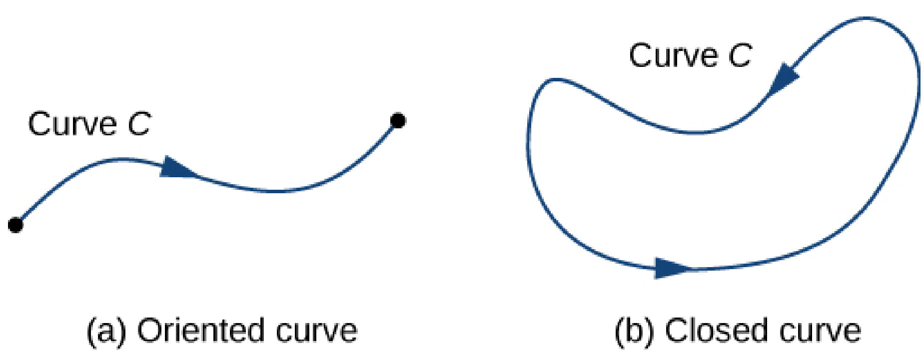
Equation:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\mathbf{F}$. How would we compute the work done by $\mathbf{F}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve *C*; such a specified direction is called an **orientation of a curve**. The specified direction is the *positive* direction along *C*; the opposite direction is the *negative* direction along *C*. When *C* has been given an orientation, *C* is called an *oriented curve* ([link]). The work done on the particle depends on the direction along the curve in which the particle is moving.

A **closed curve** is one for which there exists a parameterization $\mathbf{r}\,(t)$, $a \le t \le b$, such that $\mathbf{r}\,(a) = \mathbf{r}\,(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).



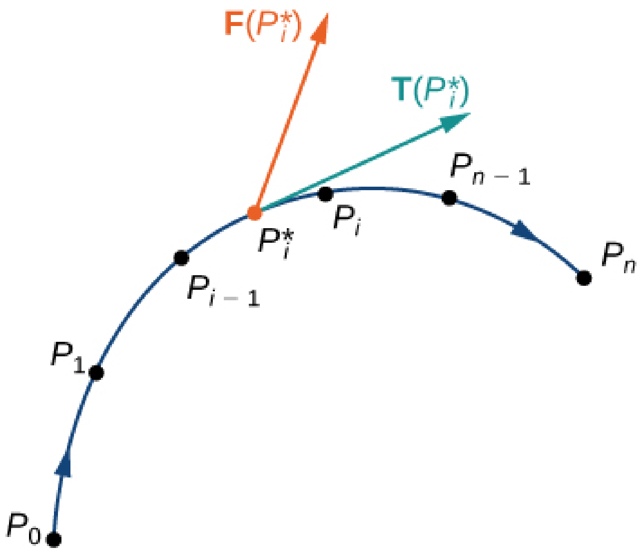
(a) An oriented curve between two points. (b) A closed oriented curve.

Let $\mathbf{r}\,(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into *n* subintervals $[t_{i-1}, t_i]$, $0 \le i \le n$, of equal width. Denote the endpoints of $\mathbf{r}\,(t_0)$, $\mathbf{r}\,(t_1)$,. . ., $\mathbf{r}\,(t_n)$ by P_0,. . ., P_n. Points P_i divide *C* into *n* pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each *i*, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\mathbf{r}(t_i^*)$ is a

point in the piece of C between P_{i-1} and P_i ([link]). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\mathbf{T}\left(P_i\right)$, the unit tangent vector at the endpoint of $\mathbf{r}(t_i^*)$. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\mathbf{F}\left(P_i^*\right) \cdot \left(\Delta s_i \mathbf{T}\left(P_i^*\right)\right)$, so the total work done along C is

Equation:

$$\sum_{i=1}^{n} \mathbf{F}\left(P_i^*\right) \cdot \left(\Delta s_i \mathbf{T}\left(P_i^*\right)\right) = \sum_{i=1}^{n} \mathbf{F}\left(P_i^*\right) \cdot \mathbf{T}\left(P_i^*\right) \Delta s_i.$$



Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\mathbf{F}$ is approximated by $\mathbf{F}\left(P_i^*\right) \cdot \mathbf{T}\left(P_i^*\right)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C.

Therefore, the work done by **F** in moving the particle in the positive direction along *C* is defined as
Equation:

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds,$$

which gives us the concept of a vector line integral.

> **Note:**
> Definition
> The **vector line integral** of vector field **F** along oriented smooth curve *C* is
> **Equation:**
>
> $$\int_C \mathbf{F} \cdot \mathbf{T} ds = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}\left(P_i^*\right) \cdot \mathbf{T}\left(P_i^*\right) \Delta s_i$$
>
> if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if *C* is an oriented curve, then we let −*C* represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function **r** and the variable *t*. To translate the integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ in terms of *t*, note that unit tangent vector **T** along *C* is given by $\mathbf{T} = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$ (assuming $||\mathbf{r}'(t)|| \neq 0$). Since $ds = ||\mathbf{r}'(t)||dt$, as we saw when discussing scalar line integrals, we have

Equation:

$$\mathbf{F}\cdot\mathbf{T}ds=\mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\frac{\mathbf{r}'\left(t\right)}{||\mathbf{r}'\left(t\right)||}||\mathbf{r}'\left(t\right)||dt=\mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\mathbf{r}'\left(t\right)dt.$$

Thus, we have the following formula for computing vector line integrals:
Equation:

$$\int_C \mathbf{F}\cdot\mathbf{T}ds=\int_a^b \mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\mathbf{r}'\left(t\right)dt.$$

Because of [link], we often use the notation $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the line integral $\int_C \mathbf{F}\cdot\mathbf{T}ds.$

If $\mathbf{r}\left(t\right)=\langle x\left(t\right),y\left(t\right),z\left(t\right)\rangle$, then $d\mathbf{r}$ denotes vector differential $\langle x'\left(t\right),y'\left(t\right),z'\left(t\right)\rangle dt.$

Example:
Exercise:

Problem:
Evaluating a Vector Line Integral

Find the value of integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t)=\langle\cos t,\sin t\rangle$, $0\le t\le\pi$ and $\mathbf{F}=\langle -y,x\rangle$.

Solution:

We can use [link] to convert the variable of integration from *s* to *t*. We then have
Equation:

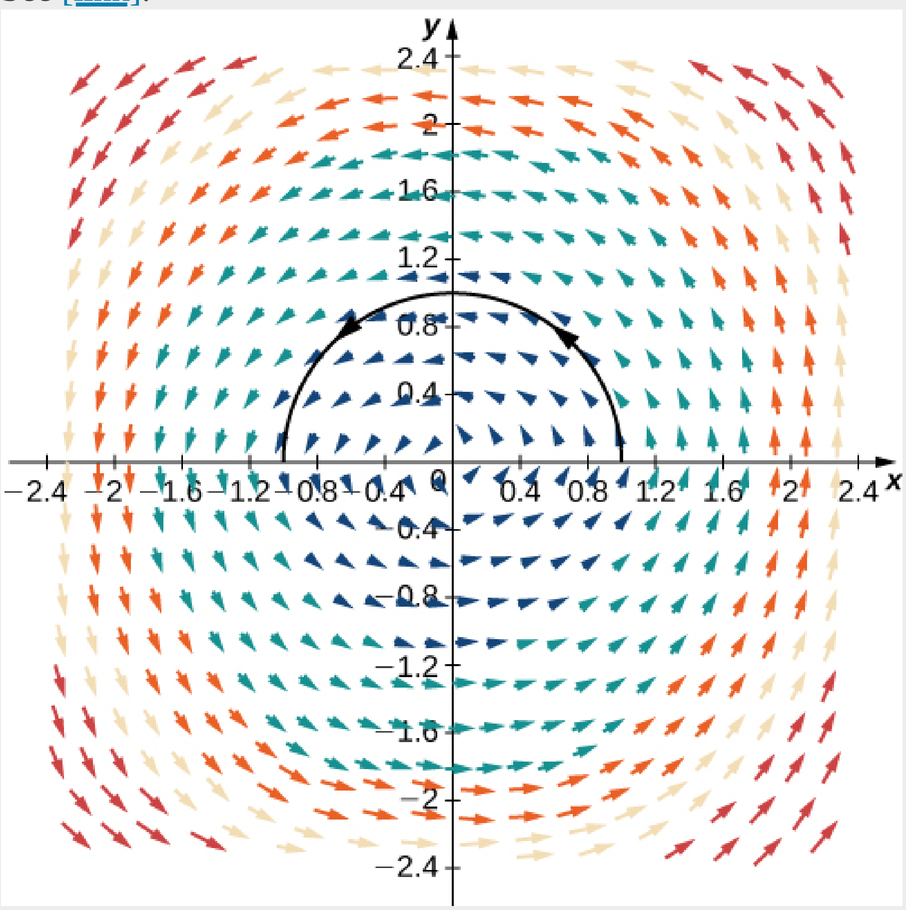
$$\mathbf{F}(\mathbf{r}(t))=\langle-\sin t,\cos t\rangle \text{ and } \mathbf{r}'(t)=\langle-\sin t,\cos t\rangle.$$

Therefore,

Equation:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle dt \\ &= \int_0^{\pi} \sin^2 t + \cos^2 t dt \\ &= \int_0^{\pi} 1 dt = \pi.\end{aligned}$$

See [link].



This figure shows curve $\mathbf{r}(t) = \langle \cos t, \sin t\rangle, 0 \le t \le \pi$ in vector field $\mathbf{F} = \langle -y, x\rangle$.

Example:
Exercise:

Problem:
Reversing Orientation

Find the value of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t) = \langle \cos t + \pi, \sin t \rangle, 0 \le t \le \pi$ and $\mathbf{F} = \langle -y, x \rangle$.

Solution:

Notice that this is the same problem as [link], except the orientation of the curve has been traversed. In this example, the parameterization starts at $\mathbf{r}(0) = \langle \pi, 0 \rangle$ and ends at $\mathbf{r}(\pi) = \langle 0, 0 \rangle$. By [link],
Equation:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi} \langle -\sin t, \cos t + \pi \rangle \cdot \langle -\sin t + \pi, \cos t \rangle dt \\ &= \int_0^{\pi} \langle -\sin t, -\cos t \rangle \cdot \langle \sin t, \cos t \rangle dt \\ &= \int_0^{\pi} \left(-\sin^2 t - \cos^2 t\right) dt \\ &= \int_0^{\pi} -1 dt \\ &= -\pi.\end{aligned}$$

Notice that this is the negative of the answer in [link]. It makes sense that this answer is negative because the orientation of the curve goes against the “flow” of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:
Equation:

$$\int_{-\mathrm{C}} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Note:
Exercise:

Problem:

Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ be a vector field and let C be the curve with parameterization $\langle t, t^2 \rangle$ for $0 \leq t \leq 2$. Which is greater: $\int_C \mathbf{F} \cdot \mathbf{T} ds$ or $\int_{-C} \mathbf{F} \cdot \mathbf{T} ds$?

Solution:

$\int_C \mathbf{F} \cdot \mathbf{T} ds$

Hint

Imagine moving along the path and computing the dot product $\mathbf{F} \cdot \mathbf{T}$ as you go.

Another standard notation for integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is $\int_C P dx + Q dy + R dz$. In this notation, P, Q, and R are functions, and we think of $d\mathbf{r}$ as vector $\langle dx, dy, dz \rangle$. To justify this convention, recall that $d\mathbf{r} = \mathbf{T} ds = \mathbf{r}'(t) dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore,

Equation:

$$\mathbf{F} \cdot d\mathbf{r} = \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = P dx + Q dy + R dz.$$

If $d\mathbf{r} = \langle dx, dy, dz \rangle$, then $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $d\mathbf{r} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore

Equation:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C Pdx + Qdy + Rdz \\ &= \int \left(P\left(\mathbf{r}(t)\right)\tfrac{dx}{dt} + Q\left(\mathbf{r}(t)\right)\tfrac{dy}{dt} + R\left(\mathbf{r}(t)\right)\tfrac{dz}{dt}\right)dt.\end{aligned}$$

Example:
Exercise:

Problem:

Finding the Value of an Integral of the Form $\int_C Pdx + Qdy + Rdz$

Find the value of integral $\int_C zdx + xdy + ydz$, where C is the curve parameterized by $\mathbf{r}\left(t\right) = \left\langle t^2, \sqrt{t}, t\right\rangle, 1 \le t \le 4$.

Solution:

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of *t*. In this case, [link] allows us to make this change:

Equation:

$$\begin{aligned}\int_C zdx + xdy + ydz &= \int_1^4 \left(t\left(2t\right) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}\right)dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t}\right)dt \\ &= \left[\tfrac{2t^3}{3} + \tfrac{t^{5/2}}{5} + \tfrac{2t^{3/2}}{3}\right]_{t=1}^{t=4} \\ &= \tfrac{793}{15}.\end{aligned}$$

Note:
Exercise:

Problem:

Find the value of $\int_C 4x\,dx + z\,dy + 4y^2\,dz$, where C is the curve parameterized by $\mathbf{r}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle, 0 \le t \le \frac{\pi}{4}$.

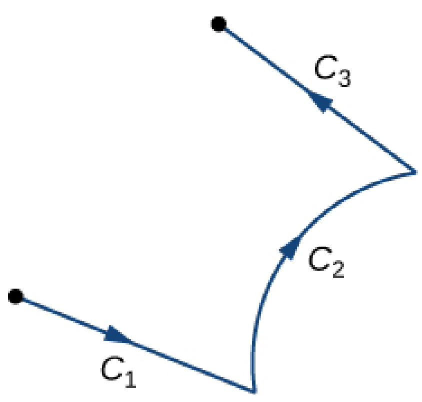
Solution:

-26

Hint

Write the integral in terms of *t* using [link].

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a **piecewise smooth curve**. To be precise, curve C is piecewise smooth if C can be written as a union of *n* smooth curves $C_1, C_2, \ldots, C_n$ such that the endpoint of C_i is the starting point of C_{i+1} ([link]). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.



The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Note:

Properties of Vector Line Integrals

Let **F** and **G** be continuous vector fields with domains that include the oriented smooth curve *C*. Then

i. $\int_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{G}\cdot d\mathbf{r}$

ii. $\int_C k\mathbf{F}\cdot d\mathbf{r} = k\int_C \mathbf{F}\cdot d\mathbf{r}$, where *k* is a constant

iii. $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$

iv. Suppose instead that *C* is a piecewise smooth curve in the domains of **F** and **G**, where $C = C_1 + C_2 + \cdots + C_n$ and C_1, C_2,. . ., C_n are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then
Equation:

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{C_2} \mathbf{F}\cdot d\mathbf{s} + \cdots + \int_{C_n} \mathbf{F}\cdot d\mathbf{s}.$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along *C*, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,. . ., $[a_{n-1}, a_n]$ are intervals, then
Equation:

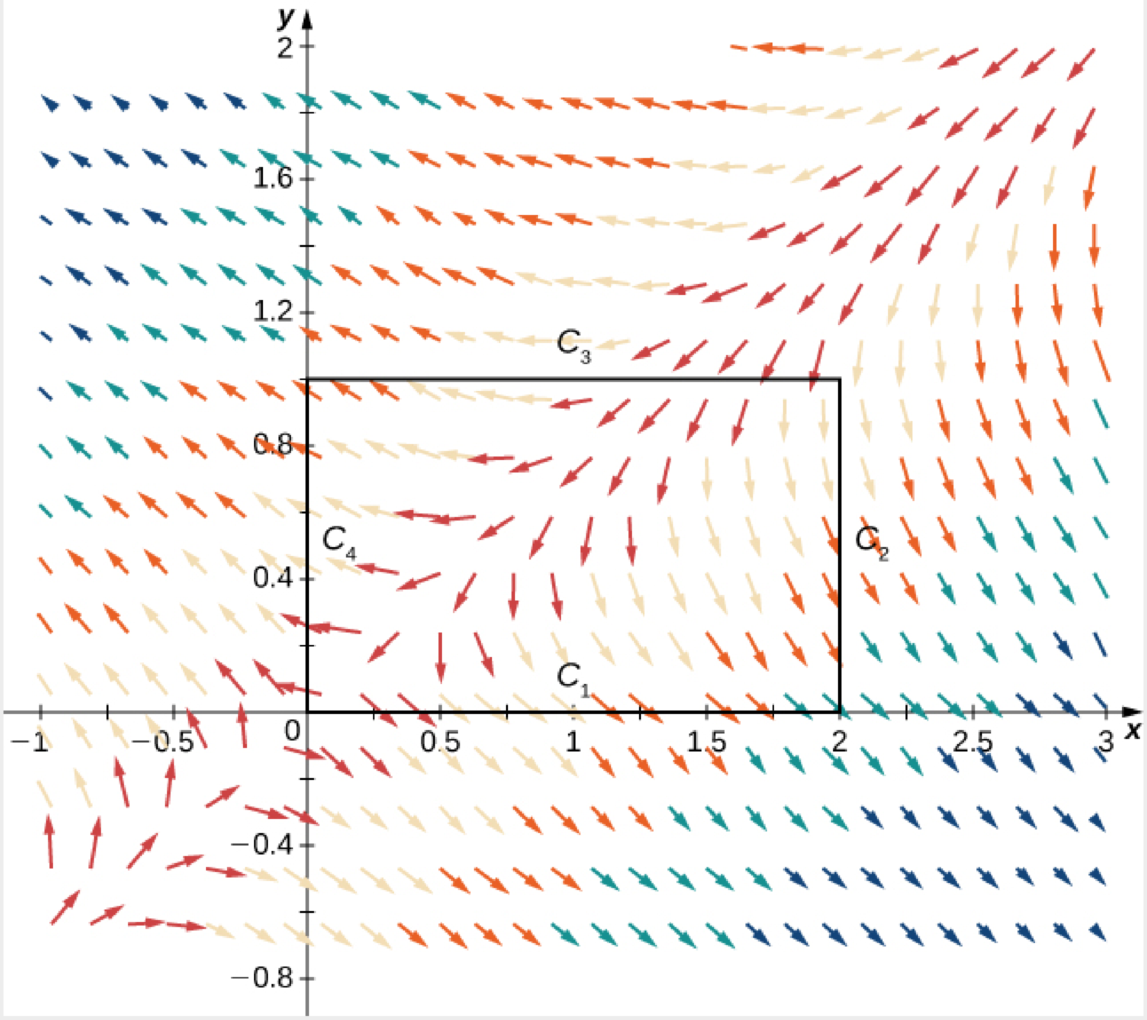
$$\int_{a_1}^{a_n} f(x)dx = \int_{a_1}^{a_2} f(x)dx + \int_{a_1}^{a_3} f(x)dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)dx,$$

which is analogous to property iv.

Example:
Exercise:

Problem:
Using Properties to Compute a Vector Line Integral

Find the value of integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0), (2,0), (2,1),$ and $(0,1)$, and where $\mathbf{F} = \langle x - 2y, y - x \rangle$ ([link]).



Rectangle and vector field for [link].

Solution:

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ ([link]). Then,

Equation:

$$\int_C \mathbf{F}\cdot\mathbf{T}d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot\mathbf{T}d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot\mathbf{T}d\mathbf{r} + \int_{C_4} \mathbf{F}\cdot\mathbf{T}d\mathbf{r}.$$

We want to compute each of the four integrals on the right-hand side using [link]. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

Equation:

$$\begin{aligned} &C_1 : \langle t, 0\rangle, 0 \le t \le 2 \\ &C_2 : \langle 2, t\rangle, 0 \le t \le 1 \\ &C_3 : \langle 2-t, 1\rangle, 0 \le t \le 2 \\ &C_4 : \langle 0, 1-t\rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

Equation:

$$\begin{aligned} \int_{C_1} \mathbf{F}\cdot\mathbf{T}d\mathbf{r} &= \int_0^2 \mathbf{F}\left(\mathbf{r}\left(t\right)\right)\cdot\mathbf{r}'\left(t\right)dt \\ &= \int_0^2 \langle t-2\left(0\right), 0-t\rangle\cdot\langle 1, 0\rangle dt = \int_0^1 t dt \\ &= \left[\frac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the

tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:
Equation:

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0,1\rangle dt \\ &= \int_0^1 (t-2)dt \\ &= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\tfrac{3}{2},\end{aligned}$$

Equation:

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot \mathbf{T}ds &= \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1,0\rangle dt \\ &= \int_0^2 tdt = 2,\end{aligned}$$

and
Equation:

$$\begin{aligned}\int_{C_4} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0,-1\rangle dt \\ &= \int_0^1 (t-1)dt \\ &= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\tfrac{1}{2}.\end{aligned}$$

Thus, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = 2$.

Note:
Exercise:

Problem:

Calculate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is vector field $\langle y^2, 2xy + 1 \rangle$ and C is a triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 5)$, oriented counterclockwise.

Solution:

0

Hint

Write the triangle as a union of its three sides, then calculate three separate line integrals.

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

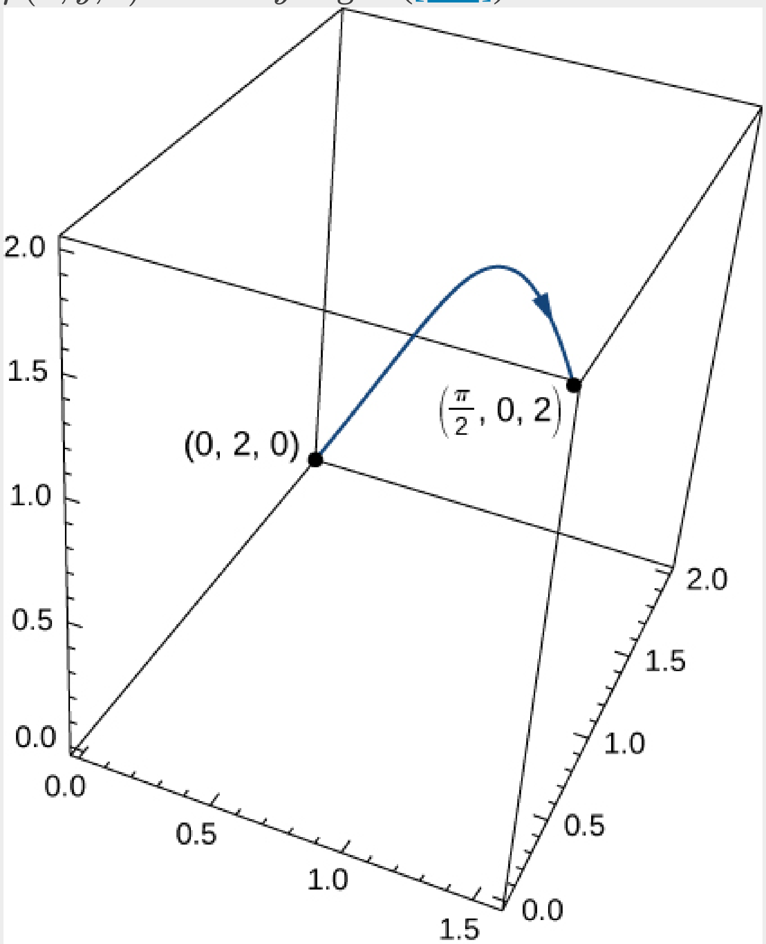
Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho\,(x, y, z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho\,(x, y, z) ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho\,(x^*, y^*, z^*) \Delta s$ for some point (x^*, y^*, z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho\,(x, y, z) ds$.

Example:

Exercise:

Problem:
Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \leq t \leq \frac{\pi}{2}$, with a density function given by $\rho(x, y, z) = e^x + yz$ kg/m ([link]).



The wire from [link].

Solution:

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)ds$, where C is the given helix. To calculate this integral, we write it in terms of t using [link]:

Equation:

$$\begin{aligned}\int_C e^x + yzds &= \int_0^{\pi/2} \left(\left(e^t + 4\cos t \sin t\right)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right) dt \\ &= \int_0^{\pi/2} \left(\left(e^t + 4\cos t \sin t\right)\sqrt{5}\right) dt \\ &= \sqrt{5}\left[e^t + 2\sin^2 t\right]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}\left(e^{\pi/2} + 1\right).\end{aligned}$$

Therefore, the mass is $\sqrt{5}\left(e^{\pi/2} + 1\right)$ kg.

Note:
Exercise:

Problem:

Calculate the mass of a spring in the shape of a helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, 0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Solution:

$18\sqrt{2}\pi^2$ kg

Hint

Calculate the line integral of ρ over the curve with parameterization $\mathbf{r}$.

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the work done by a vector field representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\mathbf{F}$, then the work required to move the object is given by $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Example:
Exercise:

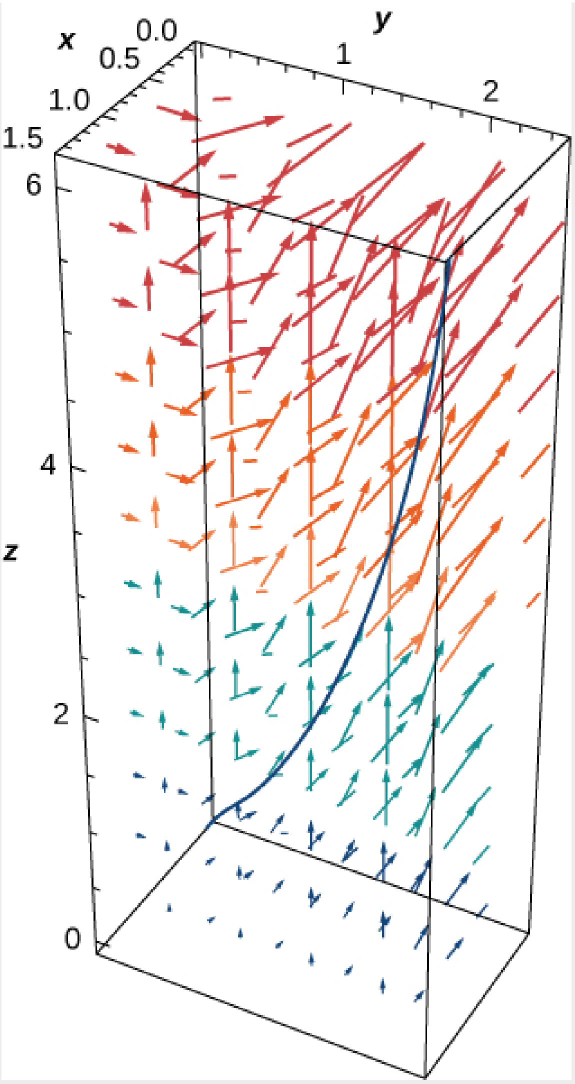
Problem:
Calculating Work

How much work is required to move an object in vector force field $\mathbf{F} = \langle yz, xy, xz \rangle$ along path $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle$, $0 \leq t \leq 1$? See [link].

Solution:

Let C denote the given path. We need to find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$. To do this, we use [link]:
Equation:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left(\left\langle t^5, t^3, t^6 \right\rangle \cdot \left\langle 2t, 1, 4t^3 \right\rangle \right) dt \\ &= \int_0^1 \left(2t^6 + t^3 + 4t^9 \right) dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5} \right]_{t=0}^{t=1} = \frac{131}{140}. \end{aligned}$$



The curve and vector field for [link].

Flux and Circulation

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is

especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\mathbf{F}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\mathbf{F}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\mathbf{F}$ along C is $\int_C \mathbf{F} \cdot \mathbf{T} ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\mathbf{N}(t)$ and instead compute integral $\int_C \mathbf{F} \cdot \mathbf{N} ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \mathbf{F} \cdot \mathbf{N} ds$ is the same as integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$, except the $\mathbf{T}$ in the Riemann sum is replaced with $\mathbf{N}$. Therefore, the flux across C is defined as

Equation:

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}\left(P_i^*\right) \cdot \mathbf{N}\left(P_i^*\right) \Delta s_i,$$

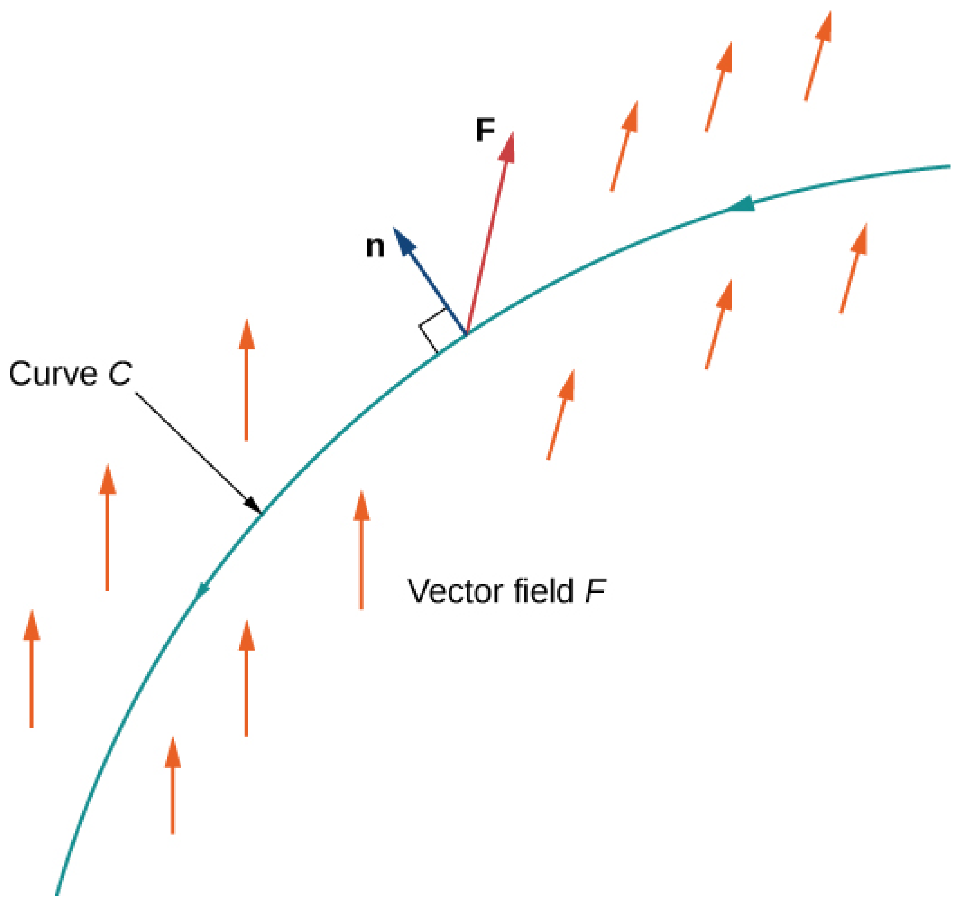
where P_i^* and Δs_i are defined as they were for integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\mathbf{N}$ and $\mathbf{T}$ are perpendicular vectors.

If $\mathbf{F}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\mathbf{F}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$ be the vector that is normal to C at the endpoint of $\mathbf{r}(t)$ and points to the right as we traverse C in the positive direction ([link]). Then, $\mathbf{N}(t) = \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}$ is the unit normal vector to C at the endpoint of $\mathbf{r}(t)$ that points to the right as we traverse C.

Note:
Definition

The **flux** of $\mathbf{F}$ across C is line integral $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|} ds.$



The flux of vector field $\mathbf{F}$ across curve C is computed by an integral similar to a vector line integral.

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see [link]).

Note:
Calculating Flux across a Curve

Let $\mathbf{F}$ be a vector field and let C be a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t)\rangle$. The flux of $\mathbf{F}$ across C is

Equation:

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{n}(t) dt$$

Proof

The proof of [link] is similar to the proof of [link]. Before deriving the formula, note that $||\mathbf{n}(t)|| = ||\langle y\prime(t), -x\prime(t)\rangle|| = \sqrt{\left(y\prime(t)\right)^2 + \left(x\prime(t)\right)^2} = ||\mathbf{r}'(t)||$. Therefore,

Equation:

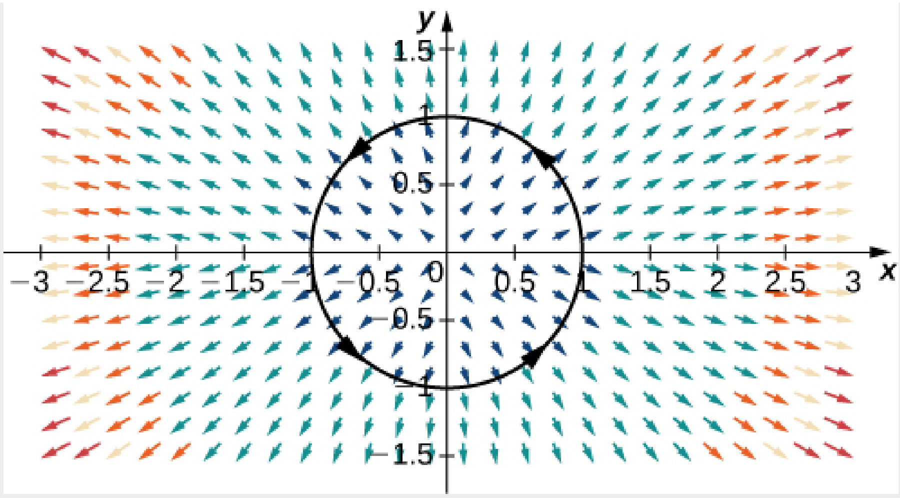
$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{N} ds &= \int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{||\mathbf{n}(t)||} ds \\ &= \int_a^b \mathbf{F} \cdot \frac{\mathbf{n}(t)}{||\mathbf{n}(t)||} ||\mathbf{r}'(t)|| dt \\ &= \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{n}(t) dt. \end{aligned}$$

□

Example:
Exercise:

Problem:
Flux across a Curve

Calculate the flux of $\mathbf{F} = \langle 2x, 2y\rangle$ across a unit circle oriented counterclockwise ([link]).



A unit circle in vector field $\mathbf{F} = \langle 2x, 2y \rangle$.

Solution:

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

Equation:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{N} ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t \rangle \cdot \langle \cos t, \sin t \rangle \, dt \\ &= \int_0^{2\pi} \left(2\cos^2 t + 2\sin^2 t\right) dt = 2\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt \\ &= 2\int_0^{2\pi} dt = 4\pi.\end{aligned}$$

Note:

Exercise:

Problem:

Calculate the flux of $\mathbf{F} = \langle x + y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Solution:

3/2

Hint

Use [link].

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a two-dimensional vector field. Recall that integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ is sometimes written as $\int_C P dx + Q dy$. Analogously, flux $\int_C \mathbf{F} \cdot \mathbf{N} ds$ is sometimes written in the notation $\int_C -Q dx + P dy$, because the unit normal vector $\mathbf{N}$ is perpendicular to the unit tangent $\mathbf{T}$. Rotating the vector $d\mathbf{r} = \langle dx, dy \rangle$ by 90° results in vector $\langle dy, -dx \rangle$. Therefore, the line integral in [link] can be written as $\int_C -2y dx + 2x dy$.

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field **F** along an oriented closed curve is called the **circulation** of **F** along *C*. Circulation line integrals have their own notation: $\oint_C \mathbf{F} \cdot \mathbf{T} ds$. The circle on the integral symbol denotes that *C* is "circular" in that it has no endpoints. [link] shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let **v** represent the velocity field of a fluid and let *C* be an oriented closed curve. At a particular point *P*, the closer the direction of **v**(*P*) is to the direction of **T**(*P*), the larger the value of the dot product $\mathbf{v}(P) \cdot \mathbf{T}(P)$. The maximum value of $\mathbf{v}(P) \cdot \mathbf{T}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\mathbf{v}(P) \cdot \mathbf{T}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \mathbf{v} \cdot \mathbf{T} ds$ measures the tendency of the fluid to move in the direction of *C*.

Example:
Exercise:

Problem:
Calculating Circulation

Let $\mathbf{F} = \langle -y, x \rangle$ be the vector field from [link] and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\mathbf{F}$ along C.

Solution:

We use the standard parameterization of the unit circle:
$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, 0 \le t \le 2\pi$. Then, $\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$. Therefore, the circulation of $\mathbf{F}$ along C is

Equation:

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\mathbf{F}$. At any point along the circle, the tangent vector and the vector from $\mathbf{F}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In [link], what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

Equation:

$$\oint_{-C} \mathbf{F} \cdot \mathbf{T} ds = -\oint_C \mathbf{F} \cdot \mathbf{T} ds = -2\pi.$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\mathbf{F}$.

Note:
Exercise:

Problem:

Calculate the circulation of $\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ along a unit circle oriented counterclockwise.

Solution:

2π

Hint

Use [link].

Example:
Exercise:

Problem:
Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\mathbf{F}(x,y) = \langle -2, y \rangle$. Assume the particle starts its movement at $(1,0)$.

Solution:

The work done by $\mathbf{F}$ on the particle is the circulation of $\mathbf{F}$ along C: $\oint_C \mathbf{F} \cdot \mathbf{T} ds$. We use the parameterization $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle, 0 \leq t \leq 2\pi$ for C. Then, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle$ and $\mathbf{F}\left(\mathbf{r}(t)\right) = \langle -2, 2\sin t \rangle$. Therefore, the circulation of $\mathbf{F}$ along C is
Equation:

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{T}ds &= \int_0^{2\pi} \langle -2, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle dt \\
&= \int_0^{2\pi} (4\sin t + 4\sin t\cos t)dt \\
&= \left[-4\cos t + 4\sin^2 t\right]_0^{2\pi} \\
&= \left(-4\cos(2\pi) + 2\sin^2(2\pi)\right) - \left(-4\cos(0) + 4\sin^2(0)\right) \\
&= -4 + 4 = 0.
\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of **F** along C is zero. Furthermore, notice that since **F** is the gradient of $f(x,y) = -2x + \frac{y^2}{2}$, **F** is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

Note:
Exercise:

Problem:

Calculate the work done by field $\mathbf{F}(x,y) = \langle 2x, 3y\rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,0)$.

Solution:

0

Hint

Use [link].

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line

segment along the *x*-axis, but the domain of integration in a line integral is a curve in a plane or in space.

- If *C* is a curve, then the length of *C* is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using [link]; vector line integrals can be calculated using [link].
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**
$$\int_C f\left(x, y, z\right) ds = \int_a^b f\left(\mathbf{r}(t)\right) \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2} dt$$
- **Calculating a vector line integral**
$$\int_C \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t) dt$$
or
$$\int_C P dx + Q dy + R dz = \int_a^b \left(P(\mathbf{r}(t)) \frac{dx}{dt} + Q(\mathbf{r}(t)) \frac{dy}{dt} + R(\mathbf{r}(t)) \frac{dz}{dt} \right) dt$$
- **Calculating flux**
$$\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{||\mathbf{n}(t)||} ds = \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{n}(t) dt$$

Exercise:

Problem:

True or False? Line integral $\int_C f(x, y) ds$ is equal to a definite integral if *C* is a smooth curve defined on $[a, b]$ and if function f is continuous on some region that contains curve *C*.

Solution:

True

Exercise:

Problem:

True or False? Vector functions $\mathbf{r}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{r}_2 = (1-t)\mathbf{i} + (1-t)^2\mathbf{j}$, $0 \le t \le 1$, define the same oriented curve.

Exercise:

Problem: *True or False?* $\int_{-C}(Pdx + Qdy) = \int_C(Pdx - Qdy)$

Solution:

False

Exercise:

Problem:

True or False? A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

Exercise:

Problem:

True or False? If C is given by $x(t) = t, y(t) = t$, $0 \le \text{t} \le 1$, then $\int_C xyds = \int_0^1 t^2 dt.$

Solution:

False

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

Exercise:

Problem: **[T]** $\int_C (x+y)ds$

$C\colon x = t, y = (1 - t), z = 0$ from (0, 1, 0) to (1, 0, 0)

Exercise:

Problem: [T] $\int_C (x - y)ds$

$C\colon \mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$ when $0 \le t \le 2$

Solution:

$\int_C (x - y)ds = 10$

Exercise:

Problem: [T] $\int_C (x^2 + y^2 + z^2)ds$

$C\colon \mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 8t\mathbf{k}$ when $0 \le t \le \frac{\pi}{2}$

Exercise:

Problem:

[T] Evaluate $\int_C xy^4 ds$, where C is the right half of circle $x^2 + y^2 = 16$ and is traversed in the clockwise direction.

Solution:

$\int_C xy^4 ds = \frac{8192}{5}$

Exercise:

Problem:

[T] Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2, -1)$ to (1, 2).

For the following exercises, find the work done.

Exercise:

Problem:

Find the work done by vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + 3xy\mathbf{j} - (x + z)\mathbf{k}$ on a particle moving along a line segment that goes from $(1, 4, 2)$ to $(0, 5, 1)$.

Solution:

$W = 8$

Exercise:

Problem:

Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

Exercise:

Problem:

Find the work done by force field $\mathbf{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k}$ on a particle as it moves along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from point $(1, 0, 0)$ to point $(-1, 0, 3\pi)$.

Solution:

$W = \frac{3\pi}{4}$

Exercise:

Problem:

Find the work done by vector field $\mathbf{F}(x, y) = y\mathbf{i} + 2x\mathbf{j}$ in moving an object along path *C*, which joins points (1, 0) and (0, 1).

Exercise:

Problem:

Find the work done by force $\mathbf{F}(x, y) = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$ in moving an object along curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{1}{6}\mathbf{k}$, where $0 \leq t \leq 2\pi$.

Solution:

$W = \pi$

Exercise:

Problem:

Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x, y) = y^2$.

For the following exercises, evaluate the line integrals.

Exercise:

Problem:

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = -1\mathbf{j}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4$$

Exercise:

Problem:

Evaluate $\int_\gamma \left(x^2 + y^2 + z^2\right)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t(0 \leq t \leq T)$.

Exercise:

Problem:

Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1, 1, 1)$ to $(3, 2, 0)$.

Solution:

$$\int_C yzdx + xzdy + xydz = -1$$

Exercise:

Problem:

Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C yds$.

Exercise:

Problem:

[T] Use a computer algebra system to evaluate the line integral $\int_C y^2dx + xdy$, where C is the arc of the parabola $x = 4 - y^2$ from (−5, −3) to (0, 2).

Solution:

$$\int_C \left(y^2\right)dx + (x)dy = \tfrac{245}{6}$$

Exercise:

Problem:

[T] Use a computer algebra system to evaluate the line integral $\int_C \left(x + 3y^2\right)dy$ over the path C given by $x = 2t, y = 10t$, where $0 \leq t \leq 1$.

Exercise:

Problem:

[T] Use a CAS to evaluate line integral $\int_C xydx + ydy$ over path C given by $x = 2t, y = 10t$, where $0 \leq t \leq 1$.

Solution:

$$\int_C xydx + ydy = \frac{190}{3}$$

Exercise:

Problem:

Evaluate line integral $\int_C (2x - y)dx + (x + 3y)dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

Exercise:

Problem:

[T] Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2} ds$, where C is $x = t, y = t, 1 \le t \le 5$.

Solution:

$\int_C \frac{y}{2x^2 - y^2} ds = \sqrt{2} \ln 5$

Exercise:

Problem:

[T] Use a CAS to evaluate $\int_C xyds$, where C is $x = t^2, y = 4t, 0 \le t \le 1$.

In the following exercises, find the work done by force field **F** on an object moving along the indicated path.

Exercise:

Problem: $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

C: $y = x^3$ from (0, 0) to (2, 8)

Solution:

$W = -66$

Exercise:

Problem: $\mathbf{F}(x, y) = 2xi + y\mathbf{j}$

C: counterclockwise around the triangle with vertices (0, 0), (1, 0), and (1, 1)

Exercise:

Problem: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le 2\pi$

Solution:

$W = -10\pi^2$

Exercise:

Problem:

Let $\mathbf{F}$ be vector field $\mathbf{F}(x, y) = \left(y^2 + 2xe^y + 1\right)\mathbf{i} + \left(2xy + x^2e^y + 2y\right)\mathbf{j}$. Compute the work of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j}, 0 \le t \le \frac{\pi}{2}$.

Exercise:

Problem:

Compute the work done by force $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} - z\mathbf{k}$ along path $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, where $0 \le t \le 1$.

Solution:

$W = 2$

Exercise:

Problem:

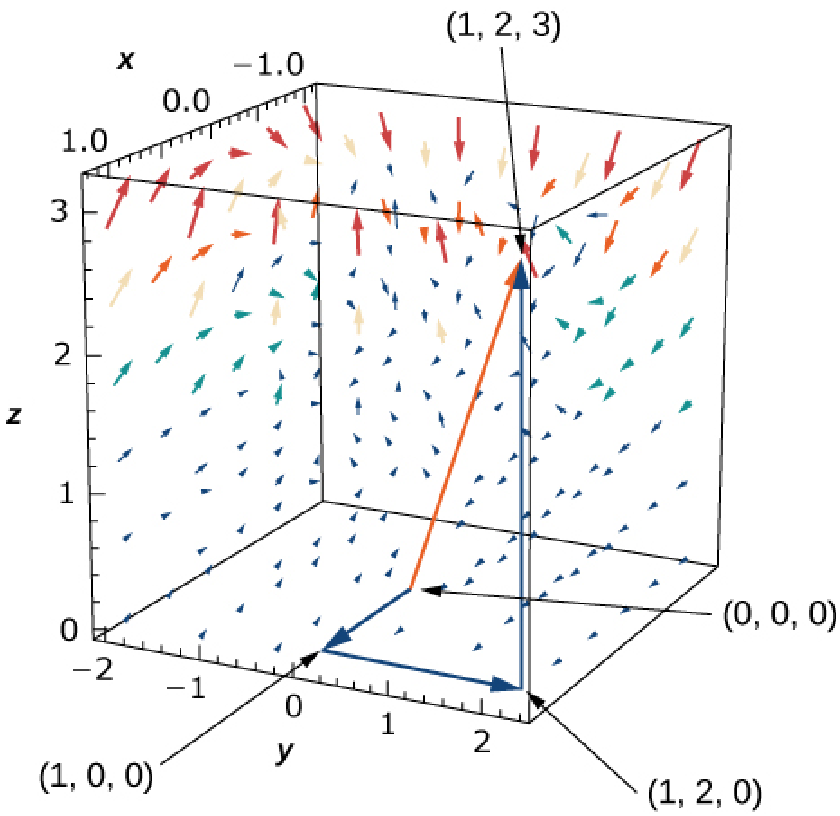
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \frac{1}{x+y}\mathbf{i} + \frac{1}{x+y}\mathbf{j}$ and C is the segment of the unit circle going counterclockwise from $(1, 0)$ to (0, 1).

Exercise:

Problem:

Force $\mathbf{F}(x, y, z) = zy\mathbf{i} + x\mathbf{j} + z^2x\mathbf{k}$ acts on a particle that travels from the origin to point (1, 2, 3). Calculate the work done if the particle travels:

a. along the path $(0, 0, 0) \to (1, 0, 0) \to (1, 2, 0) \to (1, 2, 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



Solution:

a. $W = 11$; b. $W = 11$; c. Yes

Exercise:

Problem:

Find the work done by vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + 3xy\mathbf{j} - (x + z)\mathbf{k}$ on a particle moving along a line segment that goes from (1, 4, 2) to (0, 5, 1).

Exercise:

Problem:

How much work is required to move an object in vector field $\mathbf{F}(x, y) = y\mathbf{i} + 3x\mathbf{j}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from (2, 0) to $(-2, 0)$?

Solution:

$W = 2\pi$

Exercise:

Problem:

A vector field is given by $\mathbf{F}(x, y) = (2x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

Exercise:

Problem:

Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point (1, 1).

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{25\sqrt{5} + 1}{120}$$

Exercise:

Problem: Find $\int_C y^2 dx + \left(xy - x^2\right) dy$ along C: $y = 3x$ from (0, 0) to (1, 3).

Exercise:

Problem:

Find $\int_C y^2 dx + \left(xy - x^2\right) dy$ along C: $y^2 = 9x$ from (0, 0) to (1, 3).

Solution:

$$\int_C y^2 dx + \left(xy - x^2\right) dy = 6.15$$

For the following exercises, use a CAS to evaluate the given line integrals.

Exercise:

Problem:

[T] Evaluate $\mathbf{F}(x, y, z) = x^2 z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$, where C is represented by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, 1 \leq t \leq 3.$

Exercise:

Problem:

[T] Evaluate line integral $\int_\gamma xe^y ds$ where, γ is the arc of curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Solution:

$$\int_\gamma xe^y ds \approx 7.157$$

Exercise:

Problem:

[T] Evaluate the integral $\int_\gamma xy^2 ds$, where γ is a triangle with vertices (0, 1, 2), (1, 0, 3), and $(0, -1, 0)$.

Exercise:

Problem:

[T] Evaluate line integral $\int_\gamma \left(y^2 - xy\right) dx$, where γ is curve $y = \ln x$ from (1, 0) toward $(e, 1)$.

Solution:

$$\int_\gamma \left(y^2 - xy\right) dx \approx -1.379$$

Exercise:

Problem:

[T] Evaluate line integral $\int_{\gamma} xy^4 ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

Exercise:

Problem:

[T] Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$ and

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, 0 \leq t \leq 1$.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx -1.133$$

Exercise:

Problem:

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = 2x \sin(y)\mathbf{i} + \left(x^2\cos(y) - 3y^2\right)\mathbf{j}$ and

C is any path from $(-1, 0)$ to (5, 1).

Exercise:

Problem:

Find the line integral of $\mathbf{F}(x, y, z) = 12x^2\mathbf{i} - 5xy\mathbf{j} + xz\mathbf{k}$ over path C defined by $y = x^2$, $z = x^3$ from point (0, 0, 0) to point (2, 4, 8).

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx 22.857$$

Exercise:

Problem:

Find the line integral of $\int_C \left(1 + x^2 y\right) ds$, where C is ellipse $\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$ from $0 \leq t \leq \pi$.

For the following exercises, find the flux.

Exercise:

Problem:

Compute the flux of $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$ across a line segment from (0, 0) to (1, 2).

Solution:

$\text{flux} = -\frac{1}{3}$

Exercise:

Problem:

Let $\mathbf{F} = 5\mathbf{i}$ and let C be curve $y = 0, 0 \leq x \leq 4$. Find the flux across C.

Exercise:

Problem:

Let $\mathbf{F} = 5\mathbf{j}$ and let C be curve $y = 0, 0 \leq x \leq 4$. Find the flux across C.

Solution:

$\text{flux} = -20$

Exercise:

Problem:

Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and let C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ $(0 \leq t \leq 2\pi)$. Calculate the flux across C.

Exercise:

Problem:

Let $\mathbf{F} = \left(x^2 + y^3\right)\mathbf{i} + (2xy)\mathbf{j}$. Calculate flux $\mathbf{F}$ orientated counterclockwise across curve C: $x^2 + y^2 = 9$.

Solution:

$\text{flux} = 0$

Exercise:

Problem:

Find the line integral of $\int_C z^2 dx + y dy + 2y dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from (0, 4, 3) to $(-4, 0, 3)$. C_2 is a line segment from $(-4, 0, 3)$ to (0, 1, 5).

Exercise:

Problem:

A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t, y = 2\sin t, z = t$. Find the mass of two turns of the spring if the wire has constant mass density.

Solution:

$m = 4\pi\rho\sqrt{5}$

Exercise:

Problem:

A thin wire is bent into the shape of a semicircle of radius *a*. If the linear mass density at point *P* is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

Exercise:

Problem:

An object moves in force field $\mathbf{F}(x, y, z) = y^2\mathbf{i} + 2(x+1)y\mathbf{j}$ counterclockwise from point (2, 0) along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point (2, 0) along the *x*-axis. How much work is done by the force field on the object?

Solution:

$W = 0$

Exercise:

Problem:

Find the work done when an object moves in force field $\mathbf{F}(x, y, z) = 2x\mathbf{i} - (x+z)\mathbf{j} + (y-x)\mathbf{k}$ along the path given by $\mathbf{r}(t) = t^2\mathbf{i} + \left(t^2 - t\right)\mathbf{j} + 3\mathbf{k}$, $0 \le t \le 1$.

Exercise:

Problem:

If an inverse force field $\mathbf{F}$ is given by $\mathbf{F}(x, y, z) = \frac{\mathbf{k}}{\|\mathbf{r}\|^3}\mathbf{r}$, where *k* is a constant, find the work done by $\mathbf{F}$ as its point of application moves along the *x*-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

Solution:

$W = \frac{k}{2}$

Exercise:

Problem:

David and Sandra plan to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path in the *xy*-plane from (0, 0) to (1, 1). The force field is $\mathbf{F}(x, y) = (x+2y)\mathbf{i} + (-x+y^2)\mathbf{j}$. David chooses the path that runs along the *x*-axis from (0, 0) to (1, 0) and then runs along the vertical line $x = 1$ from (1, 0) to the final point (1, 1). Sandra chooses the direct path along the diagonal line $y = x$ from (0, 0) to (1, 1). Whose line integral is larger and by how much?

Glossary

circulation

the tendency of a fluid to move in the direction of curve *C*. If *C* is a closed curve, then the circulation of $\mathbf{F}$ along *C* is line integral $\int_C \mathbf{F} \cdot \mathbf{T}ds$, which we also denote $\oint_C \mathbf{F} \cdot \mathbf{T}ds$

closed curve

a curve for which there exists a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$, such that $\mathbf{r}(a) = \mathbf{r}(b)$, and the curve is traversed exactly once

flux
: the rate of a fluid flowing across a curve in a vector field; the flux of vector field **F** across plane curve *C* is line integral $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{||\mathbf{n}(t)||}\, ds$

line integral
: the integral of a function along a curve in a plane or in space

orientation of a curve
: the orientation of a curve *C* is a specified direction of *C*

piecewise smooth curve
: an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral
: the scalar line integral of a function f along a curve *C* with respect to arc length is the integral $\int_C f ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral
: the vector line integral of vector field **F** along curve *C* is the integral of the dot product of **F** with unit tangent vector **T** of *C* with respect to arc length, $\int_C \mathbf{F} \cdot \mathbf{T} ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

Conservative Vector Fields

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

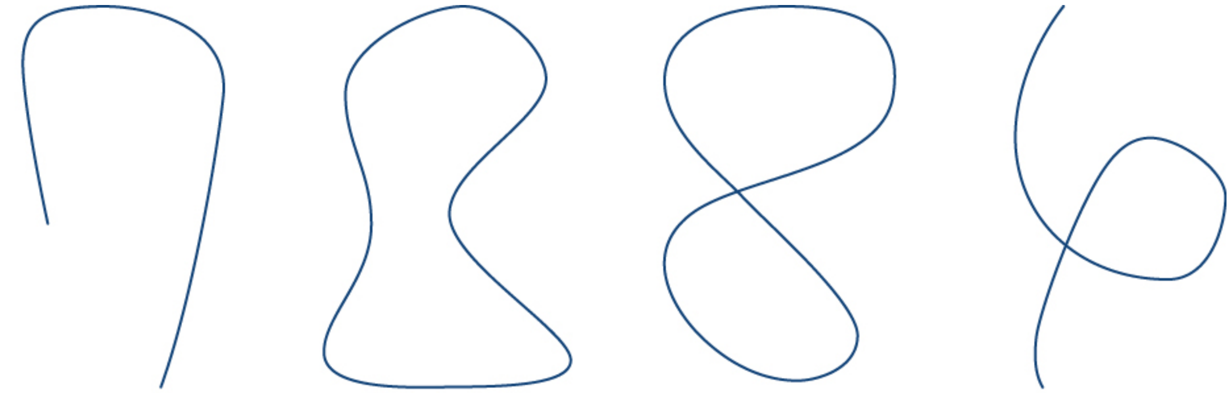
In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here.

We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve ([link]).

Note:
Definition
Curve C is a **closed curve** if there is a parameterization $\mathbf{r}\left(t\right), a \le t \le b$ of C such that the parameterization traverses the curve exactly once and $\mathbf{r}\left(a\right) = \mathbf{r}\left(b\right)$. Curve C is a **simple curve** if C does not cross itself. That is, C is simple if there exists a parameterization $\mathbf{r}\left(t\right), a \le t \le b$ of C such that $\mathbf{r}$ is one-to-one over $\left(a, b\right)$. It is possible for $\mathbf{r}\left(a\right) = \mathbf{r}\left(b\right)$, meaning that the simple curve is also closed.



Types of curves that are simple or not simple and closed or not closed.

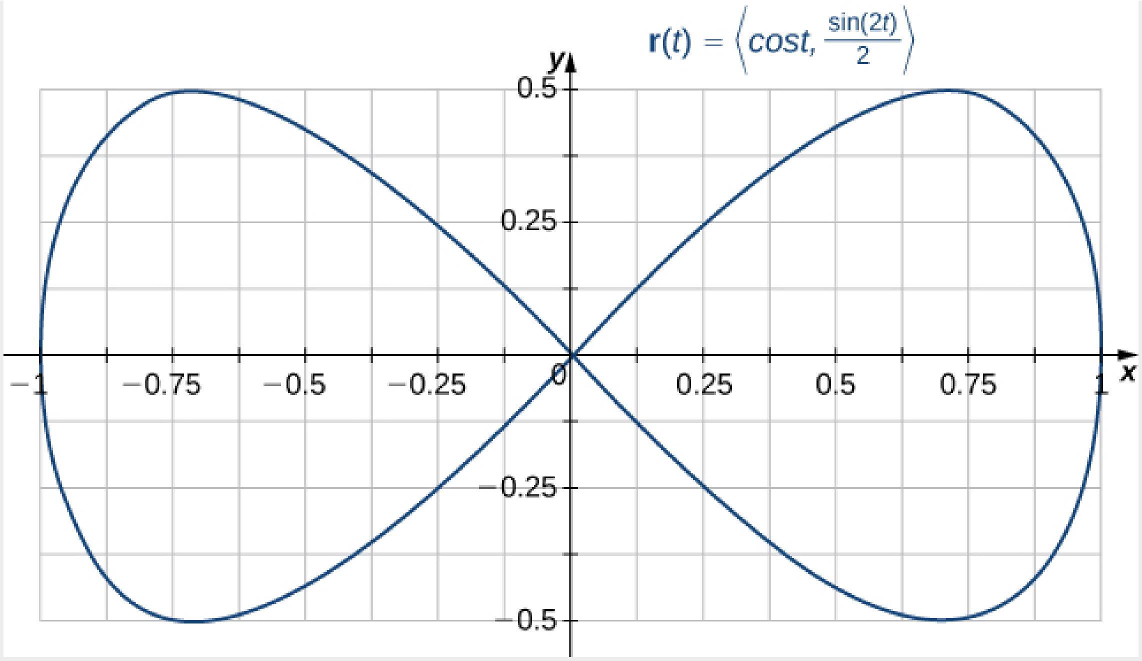
Example:
Exercise:

Problem:
Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\mathbf{r}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle, 0 \leq t \leq 2\pi$ a simple closed curve?

Solution:

Note that $\mathbf{r}\left(0\right) = \langle 1, 0 \rangle = \mathbf{r}\left(2\pi\right)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\mathbf{r}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \mathbf{r}\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin ([link]).



A curve that is closed but not simple.

Note:
Exercise:

Problem:

Is the curve given by parameterization $\mathbf{r}\left(t\right)=\langle 2\cos t,3\sin t\rangle ,0\le t\le 6\pi$, a simple closed curve?

Solution:

Yes

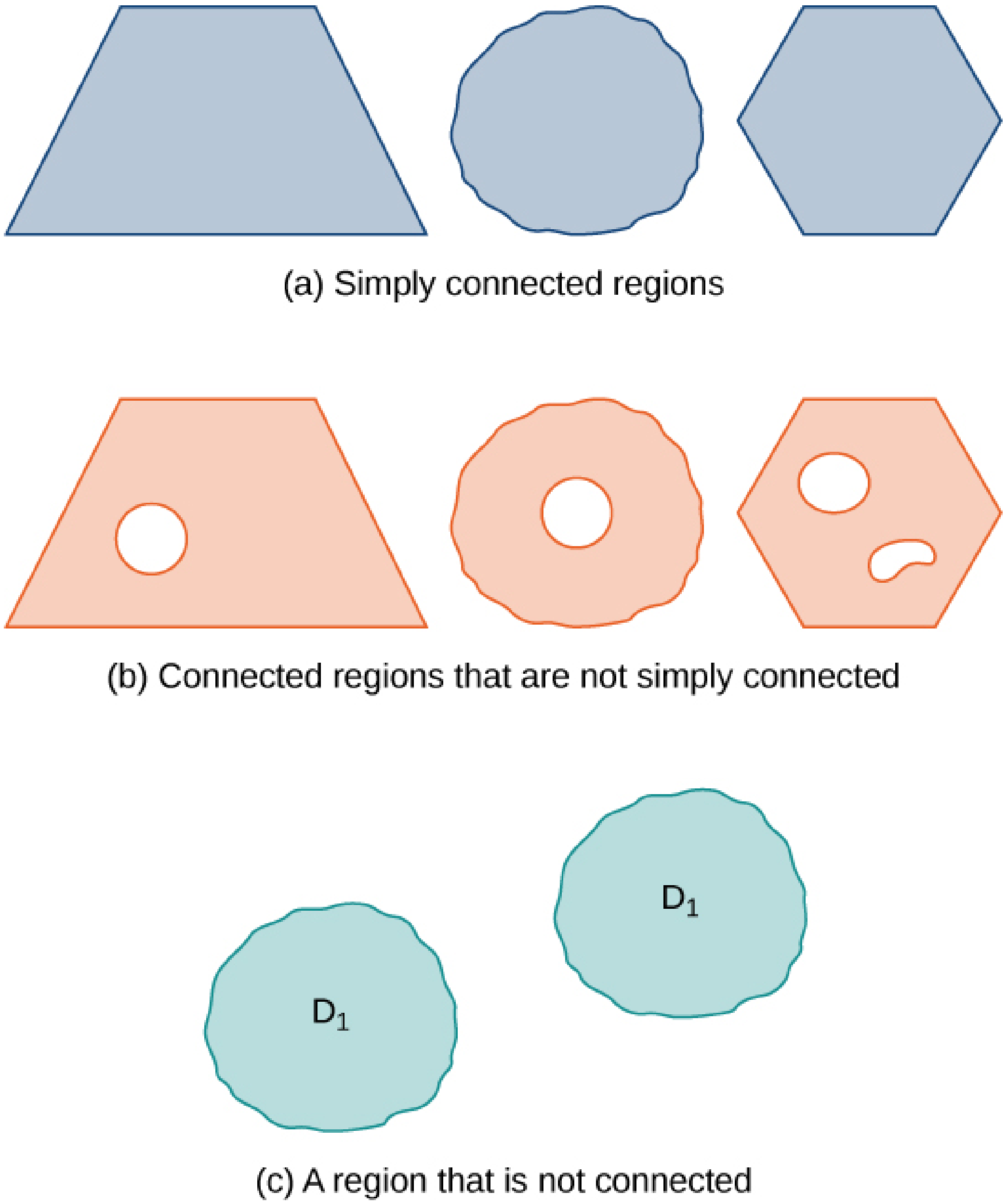
Hint

Sketch the curve.

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region’s boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

Note:
Definition
A region D is a **connected region** if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a **simply connected region** if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

All simply connected regions are connected, but not all connected regions are simply connected ([link]).

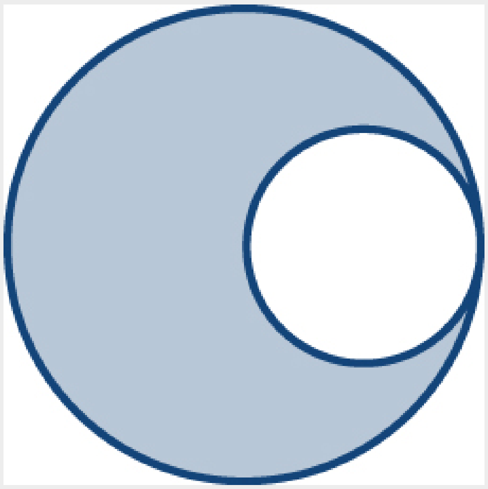


Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

Note:
Exercise:

Problem:

Is the region in the below image connected? Is the region simply connected?



Solution:

The region in the figure is connected. The region in the figure is not simply connected.

Hint

Consider the definitions.

Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,
Equation:

$$\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a).$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example:
Exercise:

Problem:
Evaluating a Line Integral and the Antiderivatives of the Endpoints

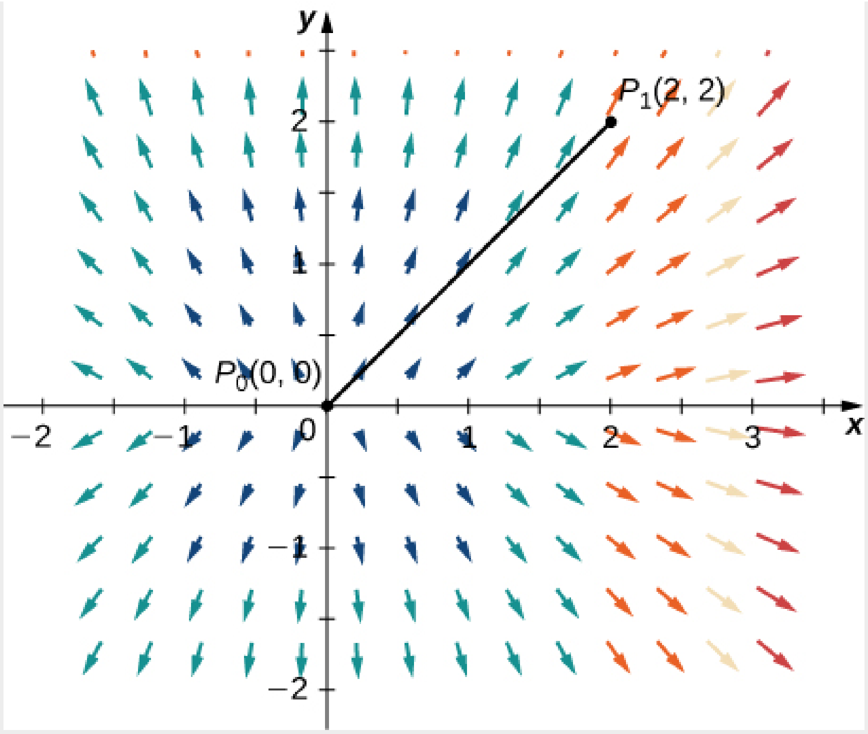
Let $\mathbf{F}(x,y) = \langle 2x, 4y \rangle$. Calculate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the line segment from (0,0) to (2,2)([link]).

Solution:

We use [link] to calculate $\int_C \mathbf{F} \bullet d\mathbf{r}$. Curve C can be parameterized by $\mathbf{r}(t) = \langle 2t, 2t \rangle, 0 \le t \le 1$. Then, $\mathbf{F}(\mathbf{r}(t)) = \langle 4t, 8t \rangle$ and $\mathbf{r}'(t) = \langle 2, 2 \rangle$, which implies that

Equation:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 4t, 8t \rangle \cdot \langle 2, 2 \rangle dt \\ &= \int_0^1 (8t + 16t)dt = \int_0^1 24t\,dt \\ &= \left[12t^2\right]_0^1 = 12. \end{aligned}$$



The value of line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ depends only on the value of the potential function of **F** at the endpoints of the curve.

Notice that $F = \nabla f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of **F**. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where *a* is the start point of the interval of integration and *b* is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral **F** to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

Equation:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = 12$$

and
Equation:

$$f(2,2) - f(0,0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a “derivative” can be calculated by evaluating an “antiderivative” at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals.**

Note:
The Fundamental Theorem for Line Integrals
Let C be a piecewise smooth curve with parameterization $\mathbf{r}(t), a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,
Equation:

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof

By [link],
Equation:

$$\int_C \nabla f \bullet d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt.$$

By the chain rule,
Equation:

$$\frac{d}{dt}\left(f\left(\mathbf{r}\left(t\right)\right) = \nabla f\left(\mathbf{r}\left(t\right)\right) \bullet \mathbf{r}'\left(t\right).$$

Therefore, by the Fundamental Theorem of Calculus,
Equation:

$$\begin{aligned} \int_C \nabla f \bullet d\mathbf{r} &= \int_a^b \nabla f\left(\mathbf{r}\left(t\right)\right) \bullet \mathbf{r}'\left(t\right) dt \\ &= \int_a^b \frac{d}{dt}\left(f\left(\mathbf{r}\left(t\right)\right) dt \\ &= \left[f\left(\mathbf{r}\left(t\right)\right)\right]_{t=a}^{t=b} \\ &= f\left(\mathbf{r}\left(b\right)\right) - f\left(\mathbf{r}\left(a\right)\right). \end{aligned}$$

□

We know that if **F** is a conservative vector field, there are potential functions f such that $\nabla f = \mathbf{F}$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right)$. In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where **F** is conservative, is a two-step process: (1) find a potential function (“antiderivative”) f for **F** and (2) compute the value of f at the endpoints of C and calculate their difference $f\left(\mathbf{r}\left(b\right)\right) - f\left(\mathbf{r}\left(a\right)\right)$. Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals. *A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.*

Example:
Exercise:

Problem:
Applying the Fundamental Theorem

Calculate integral $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \left\langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \right\rangle$ and C is a curve with parameterization $\mathbf{r}(t) = \langle t^2, t, t \rangle, 1 \le t \le e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution:

a. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use [link]:
Equation:

$$\begin{aligned}
\int_C \mathbf{F} \bullet dr &= \int_1^e \mathbf{F}(r(t)) \bullet r'(t) dt \\
&= \int_1^e \left\langle 2t^2 \ln t, \frac{t^4}{t} + t^2, 2t^2 \right\rangle \bullet \langle 2t, 1, 1 \rangle dt \\
&= \int_1^e \left(4t^3 \ln t + t^3 + 3t^2\right) dt \\
&= \int_1^e 4t^3 \ln t dt + \int_1^e \left(t^3 + 3t^2\right) dt \\
&= \int_1^e 4t^3 \ln t dt + \left[\frac{t^4}{4} + t^3\right]_1^e \\
&= r \int_1^e t^3 \ln t dt + \frac{e^4}{4} + e^3 - \frac{5}{4}.
\end{aligned}$$

Integral $\int_1^e t^3 \ln t dt$ requires integration by parts. Let $u = \ln t$ and $dv = t^3$. Then $u = \ln t, dv = t^3$
and

Equation:

$$du = \frac{1}{t}dt, v = \frac{t^4}{4}.$$

Therefore,
Equation:

$$\begin{aligned} \int_1^e t^3 \ln t dt &= \left[\tfrac{t^4}{4}\ln t\right]_1^e - \tfrac{1}{4}\int_1^e t^3 dt \\ &= \tfrac{e^4}{4} - \tfrac{1}{r}\left(\tfrac{e^4}{4} - \tfrac{1}{4}\right). \end{aligned}$$

Thus,
Equation:

$$\begin{aligned} \int_C \mathbf{F} \bullet dr &= 4\int_1^e t^3 \ln t dt + \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= 4\left(\tfrac{e^4}{4} - \tfrac{1}{4}\left(\tfrac{e^4}{4} - \tfrac{1}{4}\right)\right) + \tfrac{e^4}{4} + e^3 - \tfrac{5}{4} \\ &= e^4 - \tfrac{e^4}{4} + \tfrac{1}{4} + \tfrac{e^4}{4} + e^3 - \tfrac{5}{4} \\ &= e^4 + e^3 - 1. \end{aligned}$$

b. Given that $f(x, y, z) = x^2 \ln y + yz^2$ is a potential function for **F**, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that
Equation:

$$\begin{aligned} \int_C \mathbf{F} \bullet d\mathbf{r} &= \int_C \nabla f \bullet d\mathbf{r} \\ &= f(\mathbf{r}(e)) - f(\mathbf{r}(1)) \\ &= f\left(e^2, e, e\right) - f(1, 1, 1) \\ &= e^4 + e^3 - 1. \end{aligned}$$

This calculation is much more straightforward than the calculation we

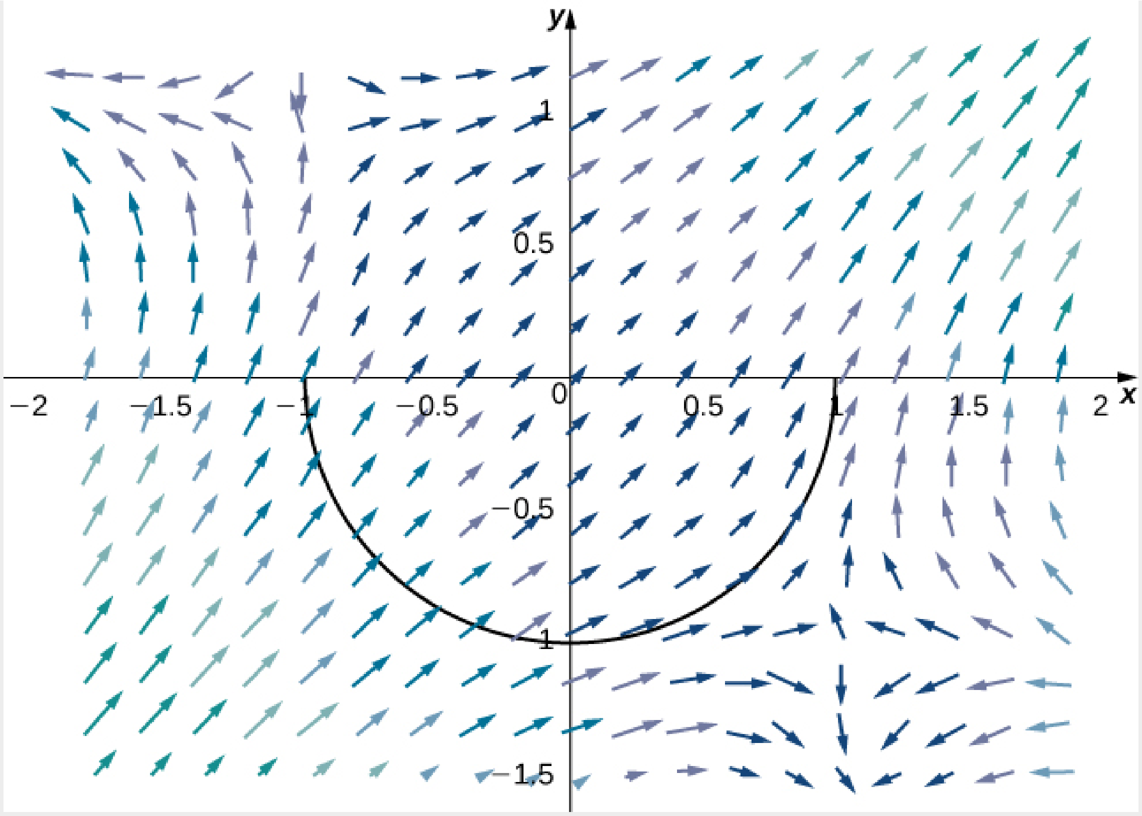
did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

[link] illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

Note:
Exercise:

Problem:

Given that $f\left(x, y\right) = {\left(x - 1\right)}^{2}y + {\left(y + 1\right)}^{2}x$ is a potential function for $\mathbf{F} = \left\langle 2xy - 2y + {\left(y + 1\right)}^{2}, {\left(x - 1\right)}^{2} + 2yx + 2x \right\rangle$, calculate integral $\int_{C} \mathbf{F} \cdot d\mathbf{r}$, where C is the lower half of the unit circle oriented counterclockwise.



Solution:

2

Hint

The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if **F** is conservative and C is a closed curve, then the circulation of **F** along C is zero—that is, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. To see why this is true, let f be a potential function for **F**. Since C is a closed curve, the terminal point **r**(b) of C is the same as the initial point **r**(a) of C—that is, $\mathbf{r}\,(a) = \mathbf{r}\,(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

Equation:

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C \nabla f\cdot d\mathbf{r}\\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(b))\\ &= 0.\end{aligned}$$

Recall that the reason a conservative vector field **F** is called “conservative” is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field **F** in integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ as a gravitational field, then the equation $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

Note:
Definition
Let **F** be a vector field with domain *D*. The vector field **F** is **independent of path** (or **path independent**) if $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any paths C_1 and C_2 in *D* with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Note:
Path Independence of Conservative Fields
If **F** is a conservative vector field, then **F** is independent of path.

Proof

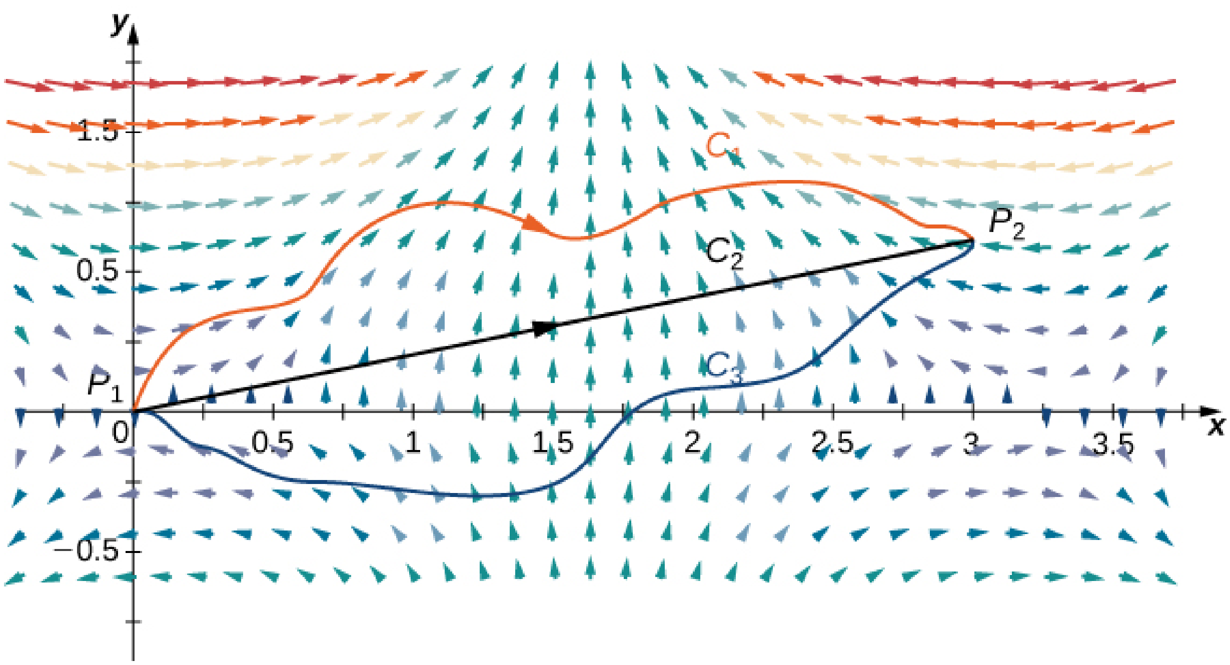
Let D denote the domain of **F** and let C_1 and C_2 be two paths in D with the same initial and terminal points ([link]). Call the initial point P_1 and the terminal point P_2. Since **F** is conservative, there is a potential function f for **F**. By the Fundamental Theorem for Line Integrals,

Equation:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and **F** is independent of path.

□



The vector field is conservative, and therefore independent of path.

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

We have shown that if **F** is conservative, then **F** is independent of path. It turns out that if the domain of **F** is open and connected, then the converse is also true. That is, if **F** is independent of path and the domain of **F** is open and connected, then **F** is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Note:
The Path Independence Test for Conservative Fields
If **F** is a continuous vector field that is independent of path and the domain *D* of **F** is open and connected, then **F** is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\mathbf{F} = \langle P, Q \rangle$ is conservative, we must find a potential function f for **F**. To that end, let *X* be a fixed point in *D*. For any point (x, y) in *D*, let *C* be a path from *X* to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$.

(Note that this definition of f makes sense only because **F** is independent of path. If **F** was not independent of path, then it might be possible to find another path C' from *X* to (x, y) such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_C \mathbf{F} \cdot d\mathbf{r}$, and in such a case f

(x, y) would not be a function.) We want to show that f has the property $\nabla f = \mathbf{F}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y) that stays inside D; C_2 is the horizontal line segment from (a, y) to (x, y) ([link]). Then

Equation:

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The first integral does not depend on *x*, so

Equation:

$$f_x = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \bullet d\mathbf{r}.$$

If we parameterize C_2 by $\mathbf{r}\,(t) = \langle t, y \rangle, a \leq t \leq x$, then

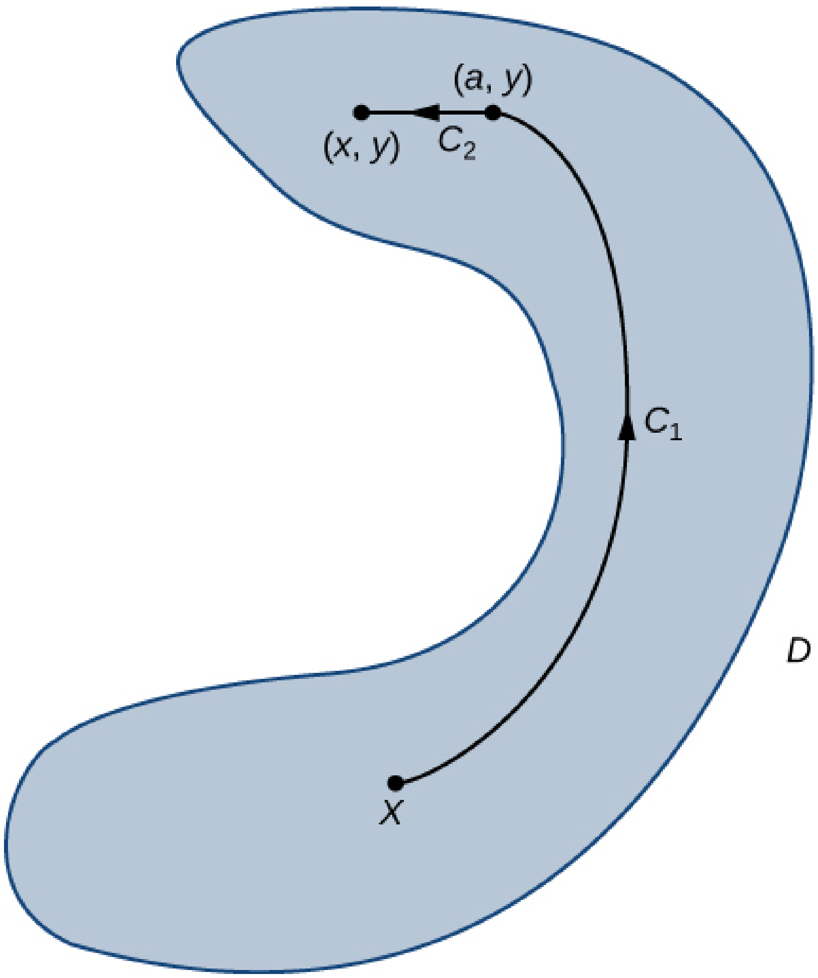
Equation:

$$\begin{aligned} f_x &= \tfrac{\partial}{\partial x} \int_{C^2} \mathbf{F} \bullet d\mathbf{r} \\ &= \tfrac{\partial}{\partial x} \int_a^x \mathbf{F}\,(\mathbf{r}\,(t)) \bullet \mathbf{r}'\,(t) dt \\ &= \tfrac{\partial}{\partial x} \int_a^x \mathbf{F}\,(\mathbf{r}\,(t)) \bullet \tfrac{d}{dt}(\langle t, y \rangle) dt \\ &= \tfrac{\partial}{\partial x} \int_a^x \mathbf{F}\,(r\,(t)) \bullet \langle 1, 0 \rangle dt \\ &= \tfrac{\partial}{\partial x} \int_a^x P\,(t, y) dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

Equation:

$$f_x = \frac{\partial}{\partial x} \int_a^x P(t, y) dt = P(x, y).$$



Here, C_1 is any path from C to (a, y) that stays inside D, and C_2 is the horizontal line segment from (a, y) to (x, y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y = Q(x, y)$.

Therefore $\nabla f = \mathbf{F}$ and $\mathbf{F}$ is conservative.

□

We have spent a lot of time discussing and proving [link] and [link], but we can summarize them simply: a vector field **F** on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example:
Exercise:

Problem:
Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\mathbf{F}(x,y)=\langle x^2y, y+5\rangle$ is not conservative.

Solution:

We can indicate that **F** is not conservative by showing that **F** is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1}\mathbf{F}\bullet dr \neq \int_{C_2}\mathbf{F}\bullet d\mathbf{r}$.

Let C_1 be the curve with parameterization $r_1(t)=\langle t,t\rangle, 0\le t\le 1$ and let C_2 be the curve with parameterization $r_2(t)=\langle t,t^2\rangle, 0\le t\le 1$ ([link]). Then

Equation:

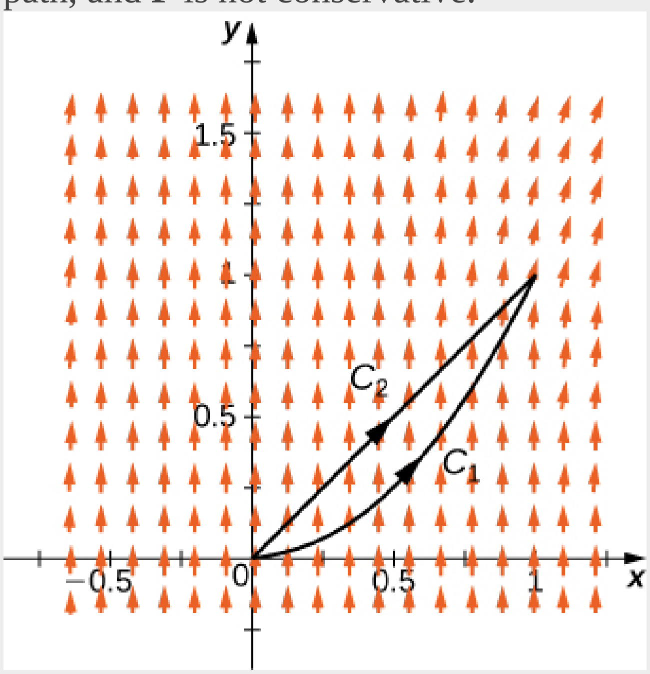
$$\begin{aligned}\int_{C_1}\mathbf{F}\cdot dr &= \int_0^1 \mathbf{F}\left(r_1(t)\right)\cdot r_1{}'(t)dt \\ &= \int_0^1 \langle t^3, t+5\rangle\cdot\langle 1,1\rangle dt = \int_0^1 \left(t^3+t+5\right)dt \\ &= \left[\frac{t^4}{4}+\frac{t^2}{2}+5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

Equation:

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot dr &= \int_0^1 \mathbf{F}\left(r_2(t)\right) \cdot r_2\prime(t)dt \\ &= \int_0^1 \left\langle t^4, t^2+5\right\rangle \cdot \langle 1, 2t\rangle dt = \int_0^1 \left(t^4+2t^3+10t\right)dt \\ &= \left[\frac{t^5}{5}+\frac{t^4}{2}+5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} \neq \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$, the value of a line integral of **F** depends on the path between two given points. Therefore, **F** is not independent of path, and **F** is not conservative.



Curves C_1 and C_2 are both oriented from left to right.

Note:
Exercise:

Problem:

Show that $\mathbf{F}(x, y) = \langle xy, x^2y^2 \rangle$ is not path independent by considering the line segment from $(0, 0)$ to $(0, 2)$ and the piece of the graph of $y = \frac{x^2}{2}$ that goes from $(0, 0)$ to $(0, 2)$.

Solution:

If C_1 and C_2 represent the two curves, then $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} \neq \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$.

Hint

Calculate the corresponding line integrals.

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if **F** is conservative, then calculating $\int_C \mathbf{F} \cdot dr$ has two steps: first, find a potential function f for **F** and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field **F**, we must be able to find a potential function f for **F**. Therefore, we must answer the following question: Given a conservative vector field **F**, how do we find a function f such that $\nabla f = \mathbf{F}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example:
Exercise:

Problem:
Finding a Potential Function

Find a potential function for $\mathbf{F}(x, y) = \langle 2xy^3, 3x^2y^2 + \cos(y) \rangle$, thereby showing that $\mathbf{F}$ is conservative.

Solution:

Suppose that $f(x, y)$ is a potential function for $\mathbf{F}$. Then, $\nabla f = \mathbf{F}$, and therefore

Equation:

$$f_x = 2xy^3 \text{ and } f_y = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x = 2xy^3$ with respect to *x* yields the equation

Equation:

$$f(x, y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to *x*, we must add a constant of integration that is a constant with respect to *x*, but may still be a function of *y*. The equation $f(x, y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to *x*:

Equation:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^3\right) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\mathbf{F}$,

Equation:

$$f_y = 3x^2y^2 + \cos(y),$$

and therefore

Equation:

$$3x^2y^2 + h'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h\prime(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x, y) = x^2 y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x, y) = x^2 y^3 + \sin(y)$.

To verify that f is a potential function, note that $\nabla f = \left\langle 2xy^3, 3x^2y^2 + \cos y \right\rangle = \mathbf{F}$.

Note:
Exercise:

Problem:

Find a potential function for $\mathbf{F}(x, y) = \left\langle e^x y^3 + y, 3e^x y^2 + x \right\rangle$.

Solution:

$f(x, y) = e^x y^3 + xy$

Hint

Follow the steps in [link].

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

Note:
Problem-Solving Stragegy: Finding a Potential Function for a Conservative Vector Field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x, y) + h(y)$, where $h(y)$ is unknown.

2. Take the partial derivative of $g(x,y)+h(y)$ with respect to *y*, which results in the function $g_y(x,y)+h'(y)$.
3. Use the equation $g_y(x,y)+h'(y)=Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y)=g(x,y)+h(y)+C$, where C is a constant, is a potential function for **F**.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example:
Exercise:

Problem:
Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $\mathbf{F}(x,y)=\langle 2xy, x^2+2yz^3, 3y^2z^2+2z\rangle$, thereby showing that **F** is conservative.

Solution:

Suppose that f is a potential function. Then, $\nabla f=\mathbf{F}$ and therefore $f_x=2xy$. Integrating this equation with respect to *x* yields the equation $f(x,y,z)=x^2y+g(y,z)$ for some function *g*. Notice that, in this case, the constant of integration with respect to *x* is a function of *y* and *z*.

Since f is a potential function,
Equation:

$$x^2+2yz^3=f_y=x^2+g_y.$$

Therefore,
Equation:

$$g_y = 2yz^3.$$

Integrating this function with respect to *y* yields
Equation:

$$g(y,z) = y^2z^3 + h(z)$$

for some function $h(z)$ of *z* alone. (Notice that, because we know that g is a function of only *y* and *z*, we do not need to write $g(y,z) = y^2z^3 + h(x,z)$.) Therefore,
Equation:

$$f(x,y,z) = x^2y + g(y,z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find *h*. Since f is a potential function,
Equation:

$$3y^2z^2 + 2z = g_z = 3y^2z^2 + h'(z).$$

This implies that $h'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function
Equation:

$$f(x,y,z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\nabla f = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z \rangle = \mathbf{F}$.

Note:
Exercise:

Problem:

Find a potential function for
$\mathbf{F}(x, y, z) = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z \rangle$.

Solution:

$f(x, y, z) = 4x^3 + \sin y \cos z + z$

Hint

Following [link], begin by integrating with respect to *x*.

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x, y) is given by vector field

Equation:

$$\mathbf{F}(x, y) = -G \left\langle \frac{x}{\left(x^2 + y^2\right)^{3/2}}, \frac{y}{\left(x^2 + y^2\right)^{3/2}} \right\rangle,$$

where *G* is the universal gravitational constant. In the next example, we build a potential function for **F**, thus confirming what we already know: that gravity is conservative.

Example:

Exercise:

Problem:

Finding a Potential Function

Find a potential function f for $\mathbf{F}(x, y) = -G \left\langle \frac{x}{\left(x^2+y^2\right)^{3/2}}, \frac{y}{\left(x^2+y^2\right)^{3/2}} \right\rangle$.

Solution:

Suppose that f is a potential function. Then, $\nabla f = \mathbf{F}$ and therefore
Equation:

$$f_x = \frac{-Gx}{\left(x^2+y^2\right)^{3/2}}.$$

To integrate this function with respect to *x*, we can use *u*-substitution. If $u = x^2 + y^2$, then $\frac{du}{2} = xdx$, so
Equation:

$$\begin{aligned} \int \frac{-Gx}{\left(x^2+y^2\right)^{3/2}} dx &= \int \frac{-G}{2u^{3/2}} du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2+y^2}} + h(y) \end{aligned}$$

for some function $h(y)$. Therefore,
Equation:

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\mathbf{F}$,
Equation:

$$f_y = \frac{-Gy}{\left(x^2+y^2\right)^{3/2}}.$$

Since $f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y)$, f_y also equals $\frac{-Gy}{\left(x^2+y^2\right)^{3/2}} + h'(y)$.

Therefore,

Equation:

$$\frac{-Gy}{\left(x^2+y^2\right)^{3/2}} + h\prime(y) = \frac{-Gy}{\left(x^2+y^2\right)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

Equation:

$$f\left(x,y\right) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field **F**. To confirm that f is a potential function, note that

Equation:

$$\begin{aligned} \nabla f &= \left\langle -\frac{1}{2}\ \frac{G}{\left(x^2+y^2\right)^{3/2}}(2x), -\frac{1}{2}\ \frac{G}{\left(x^2+y^2\right)^{3/2}}(2y)\right\rangle \\ &= \left\langle \frac{-Gx}{\left(x^2+y^2\right)^{3/2}}, \frac{-Gy}{\left(x^2+y^2\right)^{3/2}}\right\rangle \\ &= \mathbf{F}. \end{aligned}$$

Note:
Exercise:

Problem:

Find a potential function f for the three-dimensional gravitational force $\mathbf{F}(x,y,z) = \left\langle \frac{-Gx}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-Gy}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-Gz}{\left(x^2+y^2+z^2\right)^{3/2}}\right\rangle$.

Solution:

$f(x,y,z) = \frac{G}{\sqrt{x^2+y^2+z^2}}$

Hint

Use the Problem-Solving Strategy.

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if **F** is conservative, then **F** has the cross-partial property (see [link]). That is, if $\mathbf{F} = \langle P, Q, R \rangle$ is conservative, then $P_y = Q_x, P_z = R_x$, and $Q_z = R_y$. So, if **F** has the cross-partial property, then is **F** conservative? If the domain of **F** is open and simply connected, then the answer is yes.

Note:
The Cross-Partial Test for Conservative Fields
If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x, P_z = R_x$, and $Q_z = R_y$ throughout D, then **F** is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Note:
Cross-Partial Property of Conservative Fields
Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x, P_z = R_x$, and $Q_z = R_y$ throughout D if and only if **F** is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\mathbf{F} = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then **F** is conservative if and only if $P_y = Q_x$.

Example:
Exercise:

Problem:
Determining Whether a Vector Field Is Conservative

Determine whether vector field $\mathbf{F}(x, y, z) = \langle xy^2 z, x^2 yz, z^2 \rangle$ is conservative.

Solution:

Note that the domain of **F** is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use [link] to determine whether **F** is conservative. Let
Equation:

$$P(x, y, z) = xy^2 z, Q(x, y, z) = x^2 yz, \text{ and } R(x, y, z) = z^2.$$

Since $Q_z = x^2 y$ and $R_y = 0$, the vector field is not conservative.

Example:
Exercise:

Problem:
Determining Whether a Vector Field Is Conservative

Determine vector field $\mathbf{F}(x, y) = \left\langle x \ln(y), \frac{x^2}{2y} \right\rangle$ is conservative.

Solution:

Note that the domain of **F** is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of **F** is part of a plane above the *x*-axis, and this domain is simply

connected (there are no holes in this region and this region is connected). Therefore, we can use [link] to determine whether **F** is conservative. Let

Equation:

$$P(x,y) = x\ln(y) \text{ and } Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y = \frac{x}{y} = Q_x$ and thus **F** is conservative.

Note:

Exercise:

Problem:

Determine whether $\mathbf{F}(x,y) = \langle \sin x \cos y, \cos x \sin y \rangle$ is conservative.

Solution:

It is conservative.

Hint

Use [link].

When using [link], it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of [link], the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from [link]:

Equation:

$$\mathbf{F}(x,y) = \frac{y}{x^2+y^2}\mathbf{i} + \frac{-x}{x^2+y^2}\mathbf{j}.$$

This vector field satisfies the cross-partial property, since
Equation:

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{\left(x^2+y^2\right)-y\left(2y\right)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

and
Equation:

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{-\left(x^2+y^2\right)+x\left(2x\right)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}.$$

Since **F** satisfies the cross-partial property, we might be tempted to conclude that **F** is conservative. However, **F** is not conservative. To see this, let
Equation:

$$\mathbf{r}\left(t\right)=\langle\cos t,\sin t\rangle,0\le t\le\pi$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let
Equation:

$$\mathrm{s}\left(t\right)=\langle\cos t,-\sin t\rangle,0\le t\le\pi$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t+\cos^2 t=1$,
Equation:

$$\mathbf{F}\left(\mathbf{r}\left(t\right)\right)\bullet\mathbf{r}'\left(t\right)=\langle\sin\left(t\right),-\cos\left(t\right)\rangle\bullet\langle-\sin\left(t\right),\cos\left(t\right)\rangle=-1$$

and
Equation:

$$\begin{aligned}\mathbf{F}\left(s(t)\right)\cdot \mathbf{s\prime}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\sin t, -\cos t\rangle \\ &= \sin^2 t + \cos^2 t \\ &= 1.\end{aligned}$$

Therefore,
Equation:

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} -1dt = -\pi \text{ and } \int_{C_2} \mathbf{F}\cdot dr = \int_0^{\pi} 1dt = \pi.$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. Therefore, **F** is not independent of path and **F** is not conservative.

To summarize: **F** satisfies the cross-partial property and yet **F** is not conservative. What went wrong? Does this contradict [link]? The issue is that the domain of **F** is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of **F** has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, [link] does not apply to **F**.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$, then our first question should be: Is **F** conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals can't help us and we have to use other methods, such as using [link].

Example:
Exercise:

Problem:

Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y, z) = \left\langle 2xe^y z + e^x z, x^2 e^y z, x^2 e^y + e^x \right\rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution:

Before trying to compute the integral, we need to determine whether **F** is conservative and whether the domain of **F** is simply connected. The domain of **F** is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of **F** is simply connected. Let

Equation:

$$P(x, y, z) = 2xe^y z + e^x z, Q(x, y, z) = x^2 e^y z, \text{ and } R(x, y, z) = x^2 e^y + e^x$$

so that $\mathbf{F} = \langle P, Q, R \rangle$. Since the domain of **F** is simply connected, we can check the cross partials to determine whether **F** is conservative. Note that

Equation:

$$\begin{aligned} P_y &= 2xe^y z = Q_x \\ P_z &= 2xe^y + e^x = R_x \\ Q_z &= x^2 e^y = R_y. \end{aligned}$$

Therefore, **F** is conservative.

To evaluate $\int_C \mathbf{F} \cdot dr$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for **F**. Let f be a potential function for **F**. Then, $\nabla f = \mathbf{F}$, and therefore $f_x = 2xe^y z + e^x z$. Integrating this equation with respect to *x* gives $f\left(x, y, z\right) = x^2 e^y z + e^x z + h\left(y, z\right)$ for some function *h*. Differentiating this equation with respect to *y* gives $x^2 e^y z + h_y = Q = x^2 e^y z$, which implies that $h_y = 0$. Therefore, *h* is a function of *z* only, and $f(x, y, z) = x^2 e^y z + e^x z + h(z)$. To find *h*, note that $f_z = x^2 e^y + e^x + h\prime(z) = R = x^2 e^y + e^x$. Therefore, $h\prime(z) = 0$

and we can take $h(z) = 0$. A potential function for **F** is $f(x, y, z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,
Equation:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(1,1,1) - f(0,0,0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that **F** is conservative, we would have had to parameterize C and use [link]. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

Note:
Exercise:

Problem:

Calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0, \pi)$ and endpoint $(0, -\pi)$.

Solution:

-10π

Hint

Use the Fundamental Theorem for Line Integrals.

Example:
Exercise:

Problem:
Work Done on a Particle

Let $\mathbf{F}\left(x,y\right)=\left\langle 2xy^2,2x^2y\right\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the *x*-axis or the *y*-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that **F** does positive work on the particle.

Solution:

We show that **F** does positive work on the particle by showing that **F** is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that **F** is conservative, suppose $f\left(x,y\right)$ were a potential function for **F**. Then, $\nabla f=\mathbf{F}=\left\langle 2xy^2,2x^2y\right\rangle$ and therefore $f_x=2xy^2$ and $f_y=2x^2y$. Equation $f_x=2xy^2$ implies that $f\left(x,y\right)=x^2y^2+h\left(y\right)$. Deriving both sides with respect to *y* yields $f_y=2x^2y+h'\left(y\right)$. Therefore, $h'(y)=0$ and we can take $h\left(y\right)=0$.

If $f(x,y)=x^2y^2$, then note that $\nabla f=\left\langle 2xy^2,2x^2y\right\rangle=\mathbf{F}$, and therefore f is a potential function for **F**.

Let (a,b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by **F** on the particle is $\int_C\mathbf{F}\cdot d\mathbf{r}$. By the Fundamental Theorem for Line Integrals,

Equation:

$$\begin{aligned}\int_C\mathbf{F}\cdot d\mathbf{r}&=\int_C\nabla f\cdot d\mathbf{r}\\&=f(a,b)-f(0,0)\\&=a^2b^2.\end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$, and $\mathbf{F}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use [link]. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

Note:
Exercise:

Problem:

Let $\mathbf{F}(x, y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4, 4)$ to $(1, 1)$ along any smooth curve. Is the work done by $\mathbf{F}$ on the particle positive, negative, or zero?

Solution:

Negative

Hint

Use the Fundamental Theorem for Line Integrals.

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.

- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field **F**, we can test whether **F** is conservative by using the cross-partial property. If **F** has the cross-partial property and the domain is simply connected, then **F** is conservative (and thus has a potential function). If **F** is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $$\int_C \nabla f \cdot d\mathbf{r} = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right)$$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $$\oint_C \nabla f \cdot d\mathbf{r} = 0$$

Exercise:

Problem:

True or *False?* If vector field **F** is conservative on the open and connected region *D*, then line integrals of **F** are path independent on *D*, regardless of the shape of *D*.

Solution:

True

Exercise:

Problem:

True or *False?* Function $\mathbf{r}\left(t\right) = \mathbf{a} + t\left(\mathbf{b} - \mathbf{a}\right)$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\mathbf{a}$ to $\mathbf{b}$.

Exercise:

Problem:

True or *False?* Vector field
$\mathbf{F}(x, y, z) = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ is conservative.

Solution:

True

Exercise:

Problem:

True or *False?* Vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$ is conservative.

Exercise:

Problem:

Justify the Fundamental Theorem of Line Integrals for $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the case when $\mathbf{F}(x, y) = (2x + 2y)\mathbf{i} + (2x + 2y)\mathbf{j}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from (5, 0) to (3, 4).

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 24$$

Exercise:

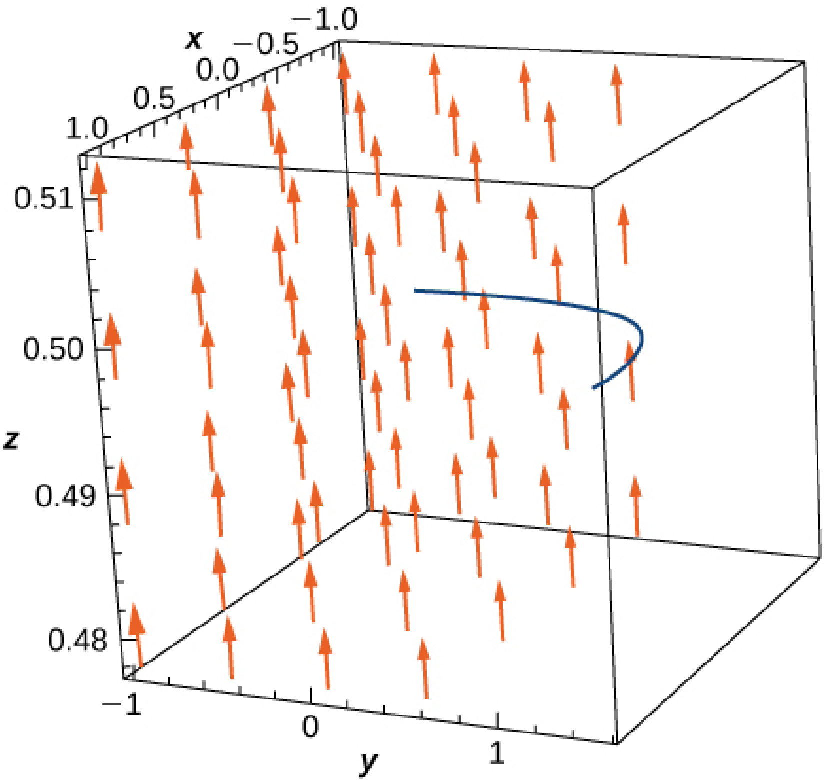
Problem:

[T] Find $\int_C \mathbf{F} \cdot d\mathbf{r}, ,]$ where
$\mathbf{F}(x, y) = (ye^{xy} + \cos x)\mathbf{i} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\mathbf{j}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

Exercise:

Problem:

[T] Evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (e^x \sin y - y)\mathbf{i} + (e^x \cos y - x - 2)\mathbf{j}$, and C is the path given by $r(t) = \left[t^3 \sin \frac{\pi t}{2}\right]\mathbf{i} - \left[\frac{\pi}{2} \cos\left(\frac{\pi t}{2} + \frac{\pi}{2}\right)\right]\mathbf{j}$ for $0 \le t \le 1$.



Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = e - \frac{3\pi}{2}$$

For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

Exercise:

Problem: $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3y^2x^2\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y) = (-y + e^x \sin y)\mathbf{i} + [(x+2)e^x \cos y]\mathbf{j}$

Solution:

Not conservative

Exercise:

Problem: $\mathbf{F}(x, y) = \left(e^{2x} \sin y\right)\mathbf{i} + \left[e^{2x} \cos y\right]\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y) = (6x + 5y)\mathbf{i} + (5x + 4y)\mathbf{j}$

Solution:

Conservative, $f(x, y) = 3x^2 + 5xy + 2y^2$

Exercise:

Problem: $\mathbf{F}(x, y) = [2x\cos(y) - y\cos(x)]\mathbf{i} + \left[-x^2 \sin(y) - \sin(x)\right]\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y) = [ye^x + \sin(y)]\mathbf{i} + [e^x + x\cos(y)]\mathbf{j}$

Solution:

Conservative, $f(x, y) = ye^x + x\sin(y)$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

Exercise:

Problem: $\oint_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r}$, where C is any path from (0, 0) to (2, 4)

Exercise:

Problem:

$\oint_C (2y dx + 2x dy)$, where C is the line segment from (0, 0) to (4, 4)

Solution:

$$\oint_C (2y dx + 2x dy) = 32$$

Exercise:

Problem:

[T] $\oint_C \left[\arctan \frac{y}{x} - \frac{xy}{x^2+y^2}\right] dx + \left[\frac{x^2}{x^2+y^2} + e^{-y}(1-y)\right] dy,$ where C is any smooth curve from (1, 1) to $(-1, 2)$

Exercise:

Problem: Find the conservative vector field for the potential function
Equation:

$$f(x, y) = 5x^2 + 3xy + 10y^2.$$

Solution:

$$\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 10y)\mathbf{j}$$

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.
Exercise:

Problem: $\mathbf{F}(x, y) = (12xy)\mathbf{i} + 6\left(x^2 + y^2\right)\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y) = (e^x \cos y)\mathbf{i} + 6\left(e^x \sin y\right)\mathbf{j}$

Solution:

F is not conservative.

Exercise:

Problem: $\mathbf{F}(x,y) = \left(2xye^{x^2y}\right)\mathbf{i} + 6\left(x^2e^{x^2y}\right)\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x,y,z) = (ye^z)\mathbf{i} + (xe^z)\mathbf{j} + (xye^z)\mathbf{k}$

Solution:

F is conservative and a potential function is $f(x,y,z) = xye^z$.

Exercise:

Problem: $\mathbf{F}(x,y,z) = (\sin y)\mathbf{i} - (x\cos y)\mathbf{j} + \mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x,y,z) = \left(-\frac{1}{y}\right)\mathbf{i} + \left(\frac{x}{y^2}\right)\mathbf{j} + (2z-1)\mathbf{k}$

Solution:

F is conservative and a potential function is $f(x,y,z) = z^2 - z - \frac{x}{y}$.

Exercise:

Problem: $\mathbf{F}(x,y,z) = 3z^2\mathbf{i} - \cos y\mathbf{j} + 2xz\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x,y,z) = (2xy)\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + y^2\mathbf{k}$

Solution:

F is conservative and a potential function is $f(x,y,z) = x^2y + y^2z$.

For the following exercises, determine whether the given vector field is conservative and find a potential function.

Exercise:

Problem: $\mathbf{F}\left(x,y\right)=\left(e^{x}\cos y\right)\mathbf{i}+6\left(e^{x}\sin y\right)\mathbf{j}$

Exercise:

Problem: $\mathbf{F}\left(x,y\right)=\left(2xye^{x^2y}\right)\mathbf{i}+6\left(x^2e^{x^2y}\right)\mathbf{j}$

Solution:

$\mathbf{F}$ is conservative and a potential function is $f\left(x,y\right)=e^{x^2y}$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

Exercise:

Problem:

Evaluate $\int_C \nabla f\cdot d\mathbf{r}$, where $f(x,y,z)=\cos(\pi x)+\sin(\pi y)-xyz$ and C is any path that starts at $\left(1,\frac{1}{2},2\right)$ and ends at $(2,1,-1)$.

Exercise:

Problem:

[T] Evaluate $\int_C \nabla f\cdot d\mathbf{r}$, where $f(x,y)=xy+e^{x}$ and C is a straight line from $(0,0)$ to $(2,1)$.

Solution:

$$\int_C \mathbf{F}\cdot dr=e^2+1$$

Exercise:

Problem:

[T] Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

Exercise:

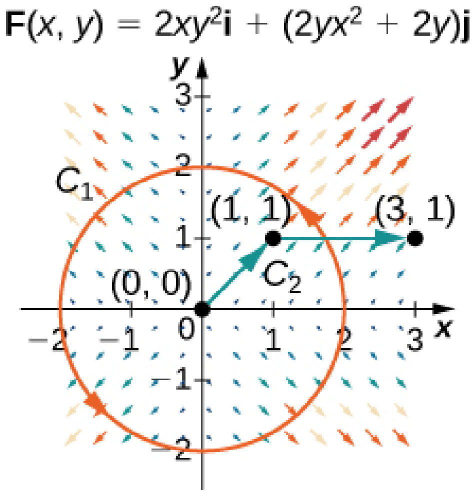
Problem:

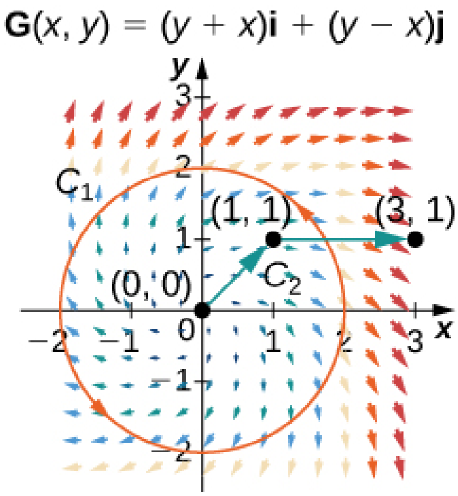
Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = xyz^2 - yz$ and C has initial point (1, 2) and terminal point (3, 5).

Solution:

$$\int_C \mathbf{F} \cdot dr = 41$$

For the following exercises, let $\mathbf{F}(x, y) = 2xy^2\mathbf{i} + \left(2yx^2 + 2y\right)\mathbf{j}$ and $G(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from (0, 0) to (1, 1) followed by a line segment from (1, 1) to (3, 1).





Exercise:

Problem: Calculate the line integral of **F** over C_1.

Exercise:

Problem: Calculate the line integral of **G** over C_1.

Solution:

$$\oint_{C_1} \mathbf{G} \cdot d\mathbf{r} = -8\pi$$

Exercise:

Problem: Calculate the line integral of **F** over C_2.

Exercise:

Problem: Calculate the line integral of **G** over C_2.

Solution:

$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 7$$

Exercise:

Problem:

[T] Let $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z\sin(yz)\mathbf{j} + y\sin(yz)\mathbf{k}$. Calculate $\oint_C \mathbf{F} \cdot dr$, where C is a path from $A = (0, 0, 1)$ to $B = (3, 1, 2)$.

Exercise:

Problem:

[T] Find line integral $\oint_C \mathbf{F} \cdot dr$ of vector field $\mathbf{F}(x, y, z) = 3x^2z\mathbf{i} + z^2\mathbf{j} + \left(x^3 + 2yz\right)\mathbf{k}$ along curve C parameterized by $r(t) = \left(\frac{\ln t}{\ln 2}\right)\mathbf{i} + t^{3/2}\mathbf{j} + t\cos\left(\pi t\right), 1 \le t \le 4$.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 150$$

For the following exercises, show that the following vector fields are conservative by using a computer. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given curve.

Exercise:

Problem:

$\mathbf{F} = \left(xy^2 + 3x^2y\right)\mathbf{i} + (x + y)x^2\mathbf{j}$; C is the curve consisting of line segments from $(1, 1)$ to $(0, 2)$ to $(3, 0)$.

Exercise:

Problem:

$\mathbf{F} = \frac{2x}{y^2+1}\mathbf{i} - \frac{2y\left(x^2+1\right)}{\left(y^2+1\right)^2}\mathbf{j}$; C is parameterized by $x = t^3 - 1, y = t^6 - t, 0 \le t \le 1$.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -1$$

Exercise:

Problem:

[T] $\mathbf{F} = \left[\cos\left(xy^2\right) - xy^2 \sin\left(xy^2\right)\right]\mathbf{i} - 2x^2 y \sin\left(xy^2\right)\mathbf{j}$; C is curve $\left(e^t, e^{t+1}\right), -1 \le t \le 0$.

Exercise:

Problem:

The mass of Earth is approximately $6 \times 10^{27}\text{g}$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8}\text{cm}^3/\text{s}^2 \cdot \text{g}$. The distance of Earth from the Sun is about $1.5 \times 10^{12}\text{cm}$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by 1 cm.

Solution:

$4 \times 10^{31}\text{erg}$

Exercise:

Problem:

[T] Let $\mathbf{F} = (x, y, z) = (e^x \sin y)\mathbf{i} + (e^x \cos y)\mathbf{j} + z^2\mathbf{k}$. Evaluate the integral $\int_C \mathbf{F} \cdot ds$, where $\mathbf{c}(t) = \left(\sqrt{t}, t^3, e^{\sqrt{t}}\right), 0 \le t \le 1$.

Exercise:

Problem:

[T] Let $\mathbf{c} : [1, 2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C 2x \cos y dx - x^2 \sin y dy$, where $\mathbf{F} = (2x \cos y)\mathbf{i} - \left(x^2 \sin y\right)\mathbf{j}$.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 0.4687$$

Exercise:

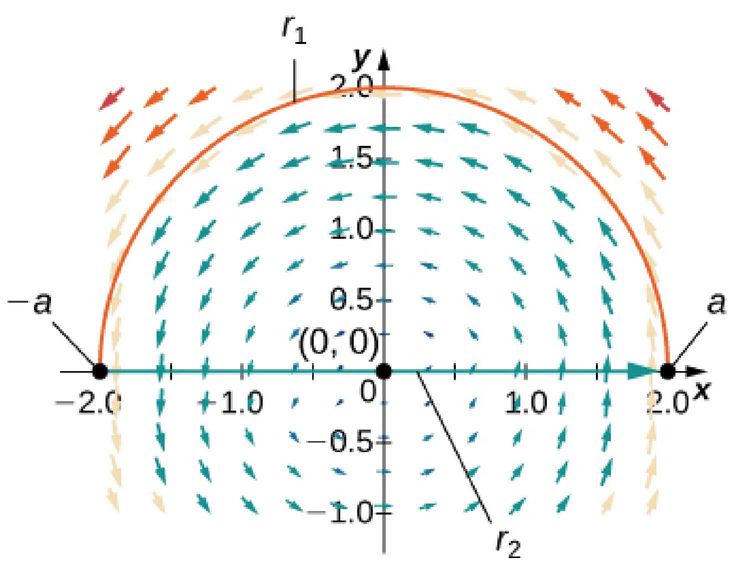
Problem:

[T] Use a computer algebra system to find the mass of a wire that lies along curve $\mathbf{r}(t) = \left(t^2 - 1\right)\mathbf{j} + 2t\mathbf{k}, 0 \le t \le 1$, if the density is $\frac{3}{2}t$.

Exercise:

Problem:

Find the circulation and flux of field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around and across the closed semicircular path that consists of semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, 0 \le t \le \pi$, followed by line segment $\mathbf{r}_2(t) = t\mathbf{i}, -a \le t \le a$.



Solution:

circulation $= \pi a^2$ and flux $= 0$

Exercise:

Problem:

Compute $\int_C \cos x \cos y dx - \sin x \sin y dy$, where $\mathbf{c}(t) = \left(t, t^2\right), 0 \leq t \leq 1.$

Exercise:

Problem: Complete the proof of [link] by showing that $f_y = Q\left(x, y\right)$.

Glossary

closed curve
: a curve that begins and ends at the same point

connected region
: a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals
: the value of line integral $\int_C \nabla f \cdot d\mathbf{r}$ depends only on the value of f at the endpoints of C: $\int_C \nabla f \cdot d\mathbf{r} = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right)$

independence of path
: a vector field $\mathbf{F}$ has path independence if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any curves C_1 and C_2 in the domain of $\mathbf{F}$ with the same initial points and terminal points

simple curve
: a curve that does not cross itself

simply connected region
: a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

Green’s Theorem

- Apply the circulation form of Green’s theorem.
- Apply the flux form of Green’s theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green’s theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green’s theorem has two forms: a circulation form and a flux form, both of which require region *D* in the double integral to be simply connected. However, we will extend Green’s theorem to regions that are not simply connected.

Put simply, Green’s theorem relates a line integral around a simply closed plane curve *C* and a double integral over the region enclosed by *C*. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

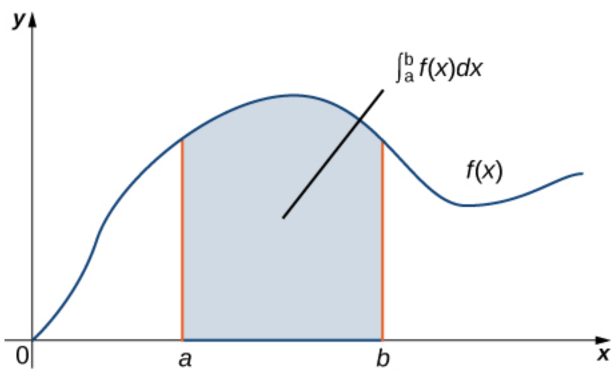
Recall that the Fundamental Theorem of Calculus says that
Equation:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of *F* at the endpoints *a* and *b* of that segment. Since the numbers *a* and *b* are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F\prime(x)dx$ based on information about the boundary of line segment $[a, b]$ ([link]). The same idea is true of the Fundamental Theorem for Line Integrals:
Equation:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

When we have a potential function (an “antiderivative”), we can calculate the line integral based solely on information about the boundary of curve *C*.



The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only

on the values of the antiderivative at the endpoints of $[a, b]$.

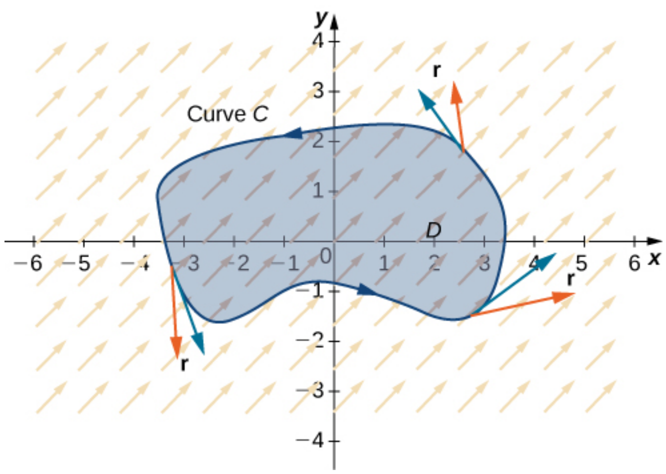
Green’s theorem takes this idea and extends it to calculating double integrals. Green’s theorem says that we can calculate a double integral over region *D* based solely on information about the boundary of *D*. Green’s theorem also says we can calculate a line integral over a simple closed curve *C* based solely on information about the region that *C* encloses. In particular, Green’s theorem connects a double integral over region *D* to a line integral around the boundary of *D*.

Circulation Form of Green’s Theorem

The first form of Green’s theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve *C* to a double integral over the region enclosed by *C*. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

Note:
Green’s Theorem, Circulation Form
Let *D* be an open, simply connected region with a boundary curve *C* that is a piecewise smooth, simple closed curve oriented counterclockwise ([link]). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on *D*. Then,
Equation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA.$$



The circulation form of Green’s theorem relates a line integral over curve *C* to a double integral over region *D*.

Notice that Green’s theorem can be used only for a two-dimensional vector field **F**. If **F** is a three-dimensional field, then Green’s theorem does not apply. Since

Equation:

$$\int_C P dx + Q dy = \int_C \mathbf{F} \cdot \mathbf{T} ds,$$

this version of Green's theorem is sometimes referred to as the *tangential form* of Green's theorem.

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\mathbf{F} = \langle P, Q \rangle$ is conservative. In this case,

Equation:

$$\oint_C P dx + Q dy = 0$$

because the circulation is zero in conservative vector fields. By [link], $\mathbf{F}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

Equation:

$$\iint_D (Q_x - P_y) dA = \iint_D 0 dA = 0 = \oint_C P dx + Q dy,$$

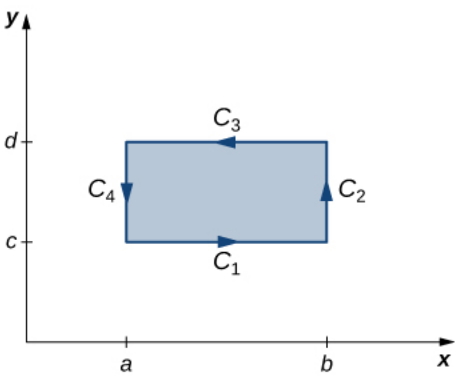
which confirms Green's theorem in the case of conservative vector fields.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 ([link]). We parameterize each side of D as follows:

Equation:

$$\begin{array}{rcl} C_1 \colon \mathbf{r}_1(t) & = & \langle t, c \rangle, a \le t \le b \\ C_2 \colon \mathbf{r}_2(t) & = & \langle b, t \rangle, c \le t \le d \\ -C_3 \colon \mathbf{r}_3(t) & = & \langle t, d \rangle, a \le t \le b \\ -C_4 \colon \mathbf{r}_4(t) & = & \langle a, t \rangle, c \le t \le d. \end{array}$$



Rectangle D is oriented counterclockwise.

Then,

Equation:

$$\begin{aligned}\int_C \mathbf{F}\bullet d\mathbf{r} &= \int_{C_1}\mathbf{F}\bullet d\mathbf{r}+\int_{C_2}\mathbf{F}\bullet d\mathbf{r}+\int_{C_3}\mathbf{F}\bullet d\mathbf{r}+\int_{C_4}\mathbf{F}\bullet d\mathbf{r}\\ &= \int_{C_1}\mathbf{F}\bullet d\mathbf{r}+\int_{C_2}\mathbf{F}\bullet d\mathbf{r}-\int_{-C_3}\mathbf{F}\bullet d\mathbf{r}-\int_{-C_4}\mathbf{F}\bullet d\mathbf{r}\\ &= \int_a^b \mathbf{F}(\mathbf{r}_1(t))\bullet\mathbf{r}_1(t)dt+\int_c^d \mathbf{F}(\mathbf{r}_2(t))\bullet\mathbf{r}_2(t)dt\\ &\quad -\int_a^b \mathbf{F}(\mathbf{r}_3(t))\bullet\mathbf{r}_3(t)dt-\int_c^d \mathbf{F}(\mathbf{r}_4(t))\bullet\mathbf{r}_4(t)dt\\ &= \int_a^b P(t,c)dt+\int_c^d Q(b,t)dt-\int_a^b P(t,d)dt-\int_c^d Q(a,t)dt\\ &= \int_a^b (P(t,c)-P(t,d))dt+\int_c^d (Q(b,t)-Q(a,t))dt\\ &= -\int_a^b (P(t,d)-P(t,c))dt+\int_c^d (Q(b,t)-Q(a,t))dt.\end{aligned}$$

By the Fundamental Theorem of Calculus,

Equation:

$$P(t,d)-P(t,c)=\int_c^d \frac{\partial}{\partial y}P(t,y)dy \text{ and } Q(b,t)-Q(a,t)=\int_a^b \frac{\partial}{\partial x}Q(x,t)dx.$$

Therefore,

Equation:

$$\begin{aligned}&-\int_a^b (P(t,d)-P(t,c))dt+\int_c^d (Q(b,t)-Q(a,t))dt\\ &= -\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)dydt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)dxdt.\end{aligned}$$

But,

Equation:

$$\begin{aligned}-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)dydt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)dxdt &= -\int_a^b\int_c^d \frac{\partial}{\partial y}P(x,y)dydx+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,y)dx\\ &= \int_a^b\int_c^d (Q_x-P_y)dydx\\ &= \int\int_D (Q_x-P_y)dA.\end{aligned}$$

Therefore, $\int_C \mathbf{F} \bullet d\mathbf{r} = \int\int_D (Q_x - P_y) dA$ and we have proved Green's theorem in the case of a rectangle.

To prove Green's theorem over a general region *D*, we can decompose *D* into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

□

Example:
Exercise:

Problem:
Applying Green's Theorem over a Rectangle

Calculate the line integral
Equation:

$$\oint_C x^2 y dx + (y-3) dy,$$

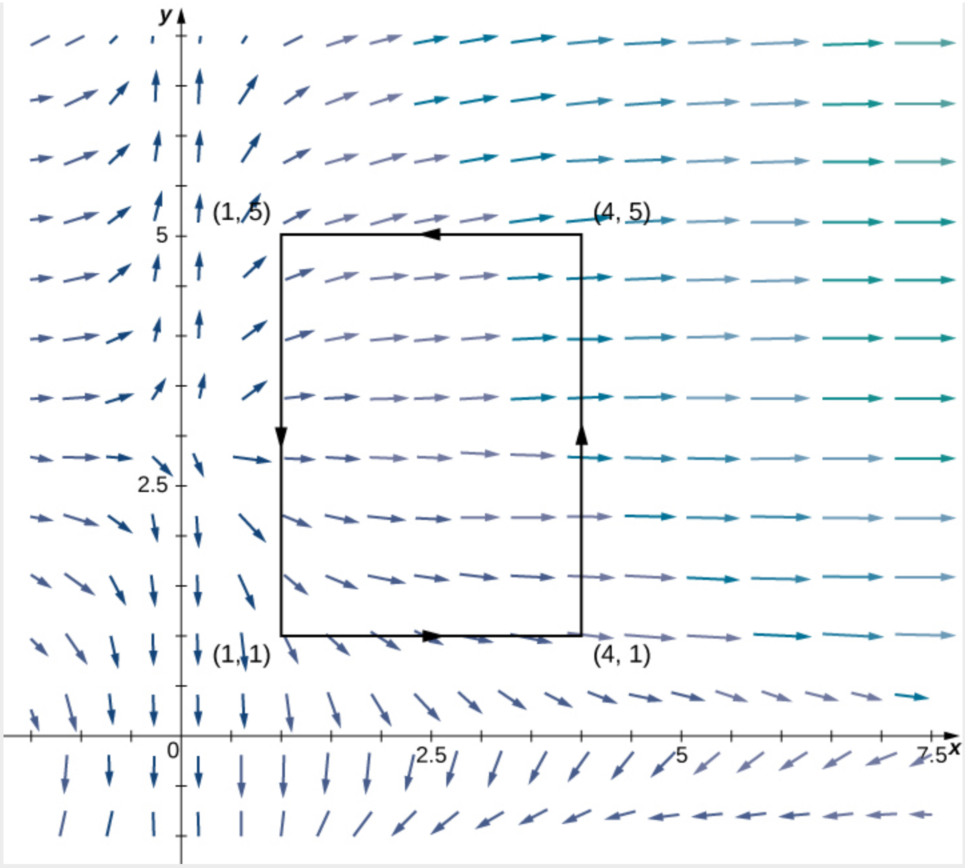
where *C* is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution:

Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle x^2 y, y-3 \rangle$. Then, $Q_x = 0$ and $P_y = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let *D* be the rectangular region enclosed by *C* ([link]). By Green's theorem,
Equation:

$$\begin{aligned} \oint_C x^2 y dx + (y-3) dy &= \iint_D (Q_x - P_y) dA \\ &= \int\int_D -x^2 dA = \int_1^5 \int_1^4 -x^2 dx dy \\ &= \int_1^5 -21 dy = -84. \end{aligned}$$



The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from Line Integrals to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example:
Exercise:

Problem:
Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field
Equation:

$$\mathbf{F}(x, y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution:

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

Equation:

$$W = \oint_C (y + \sin x)dx + (e^y - x)dy.$$

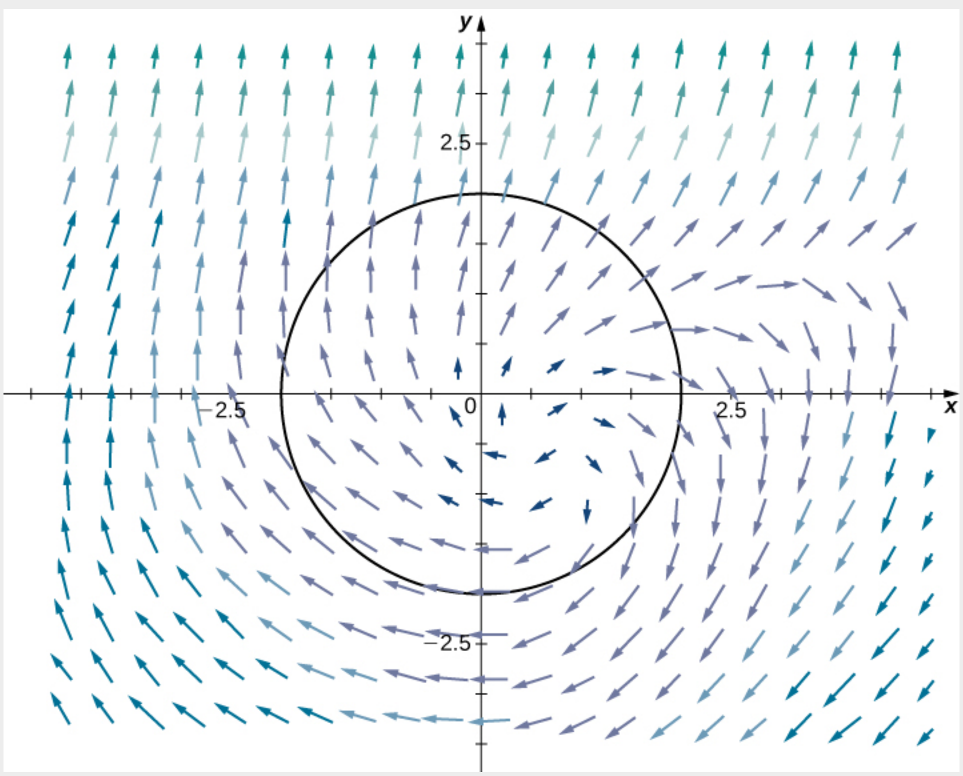
As with [link], this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem ([link]).

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

Equation:

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)dy \\ &= \iint_D (Q_x - P_y)dA = \iint_D -2dA \\ &= -2\,(\text{area}\,(D)) = -2\pi\left(2^2\right) = -8\pi. \end{aligned}$$



The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

Note:
Exercise:

Problem: Use Green’s theorem to calculate line integral
Equation:

$$\oint_C \sin(x^2)dx + (3x - y)dy,$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

Solution:

$\frac{45}{2}$

Hint

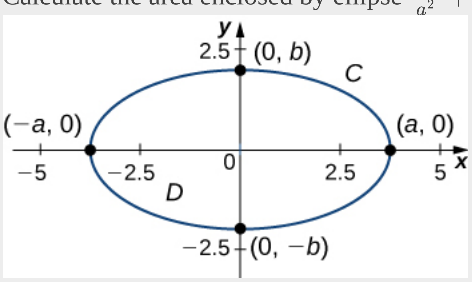
Transform the line integral into a double integral.

In the preceding two examples, the double integral in Green’s theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green’s theorem to translate a double integral into a line integral.

Example:
Exercise:

Problem:
Applying Green’s Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ([link]).



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution:

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by
Equation:

$$x = a\cos t, y = b\sin t, 0 \le t \le 2\pi.$$

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the x-axis and the region below. The area of the ellipse is

Equation:

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dydx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dydx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

Equation:

$$\mathbf{F}(x,y) = \langle P, Q\rangle = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\mathbf{F}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\mathbf{F}$ over C into the double integral of 1 over D.

By Green's theorem,

Equation:

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)dA \\ &= \int_C \mathbf{F} \bullet d\mathbf{r} = \tfrac{1}{2}\int_C -ydx + xdy \\ &= \tfrac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t dt \\ &= \tfrac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t dt = \tfrac{1}{2}\int_0^{2\pi} ab dt = \pi ab.\end{aligned}$$

Therefore, the area of the ellipse is πab.

In [link], we used vector field $\mathbf{F}(x,y) = \langle P, Q\rangle = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $\mathbf{F}(x,y) = \langle P, Q\rangle = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in [link],

Equation:

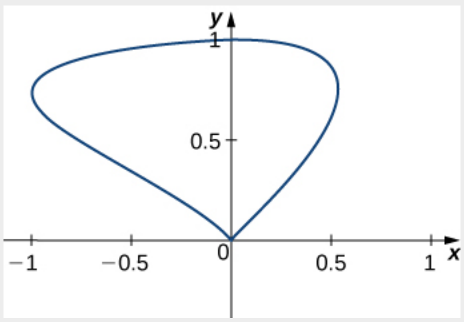
$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -ydx + xdy.$$

It’s worth noting that if $\mathbf{F} = \langle P, Q \rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. [link] is not the only equation that uses a vector field’s mixed partials to get the area of a region.

Note:
Exercise:

Problem:

Find the area of the region enclosed by the curve with parameterization $\mathbf{r}(t) = \langle \sin t \cos t, \sin t \rangle, 0 \le t \le \pi$.



Solution:

$\frac{4}{3}$

Hint

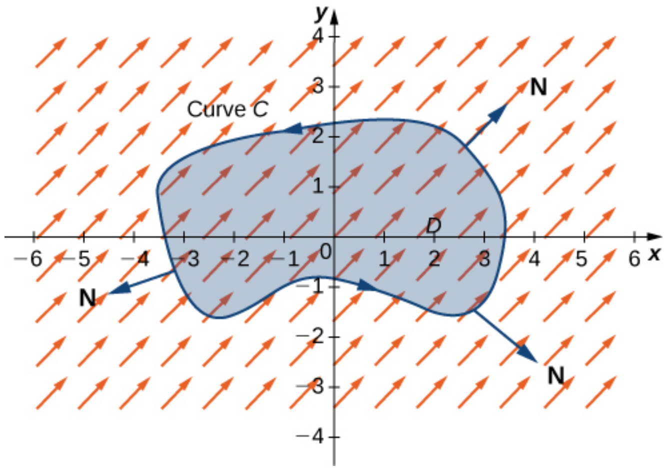
Use [link].

Flux Form of Green’s Theorem

The circulation form of Green’s theorem relates a double integral over region D to line integral $\oint_C \mathbf{F} \cdot \mathbf{T} ds$, where C is the boundary of D. The flux form of Green’s theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green’s theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

Note:
Green’s Theorem, Flux Form
Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise ([link]). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,
Equation:

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D P_x + Q_y dA.$$



The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector **N**, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \mathbf{F} \cdot \mathbf{N} ds = \oint_C -Q dx + P dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

Equation:

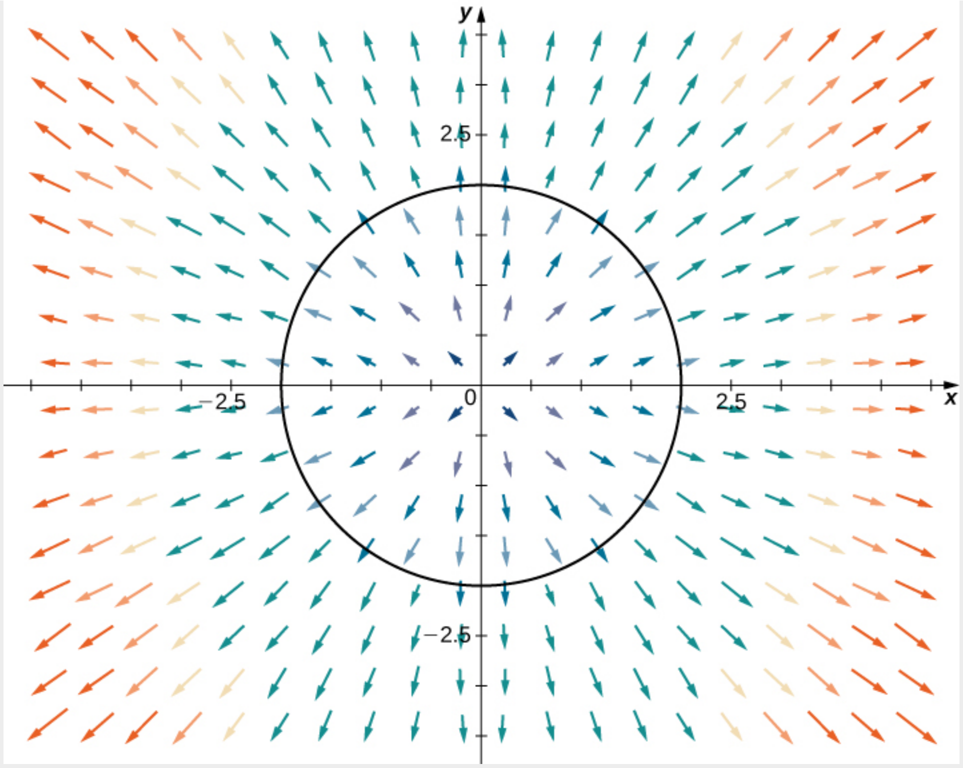
$$\begin{aligned} \oint_C -Q dx + P dy &= \oint_C M dx + N dy \\ &= \iint_D N_x - M_y dA \\ &= \iint_D P_x - (-Q)_y dA \\ &= \iint_D P_x + Q_y dA. \end{aligned}$$

□

Example:
Exercise:

Problem:
Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin ([link]) and let $\mathbf{F}(x, y) = \langle x, y \rangle$. Calculate the flux across C.



Curve C is a circle of radius r centered at the origin.

Solution:

Let D be the disk enclosed by C. The flux across C is $\oint_C \mathbf{F} \cdot \mathbf{N} ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x, y) = x$ and $Q(x, y) = y$ so that $\mathbf{F} = \langle P, Q \rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

Equation:

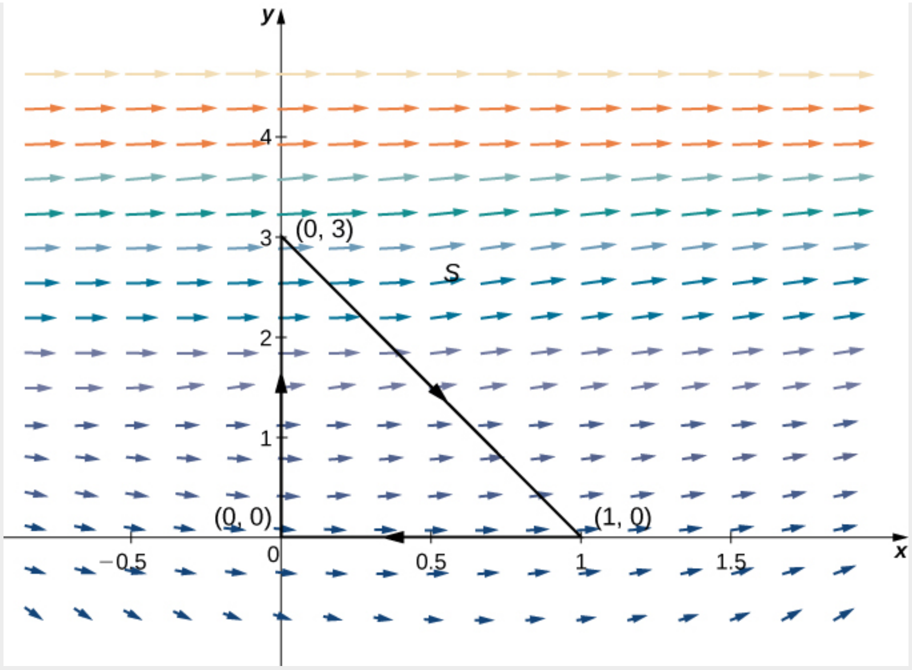
$$\int_C \mathbf{F} \bullet \mathbf{N} ds = \int \int_D 2 dA = 2 \int \int_D dA.$$

Since $\int \int_D dA$ is the area of the circle, $\int \int_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example:
Exercise:

Problem:
Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise ([link]). Calculate the flux of $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x^2 + e^y, x + y \rangle$ across S.



Curve S is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise.

Solution:

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \mathbf{F} \cdot \mathbf{N} ds = -\oint_{-S} \mathbf{F} \cdot \mathbf{N} ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

Equation:

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{N} ds &= \oint_{-S} \mathbf{F} \cdot \mathbf{N} ds \\ &= -\iint_D (P_x + Q_y) dA \\ &= -\iint_D (2x + 1) dA. \end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

Equation:

$$\begin{aligned}-\iint_D (2x+1)dA &= -\int_0^1\int_0^{-3x+3}(2x+1)dydx \\ &= -\int_0^1 (2x+1)(-3x+3)dx = -\int_0^1 \left(-6x^2+3x+3\right)dx \\ &= -\left[-2x^3+\tfrac{3x^2}{2}+3x\right]_0^1 = -\tfrac{5}{2}.\end{aligned}$$

Note:
Exercise:

Problem: Calculate the flux of $\mathbf{F}(x,y) = \langle x^3, y^3\rangle$ across a unit circle oriented counterclockwise.

Solution:

$\frac{3\pi}{2}$

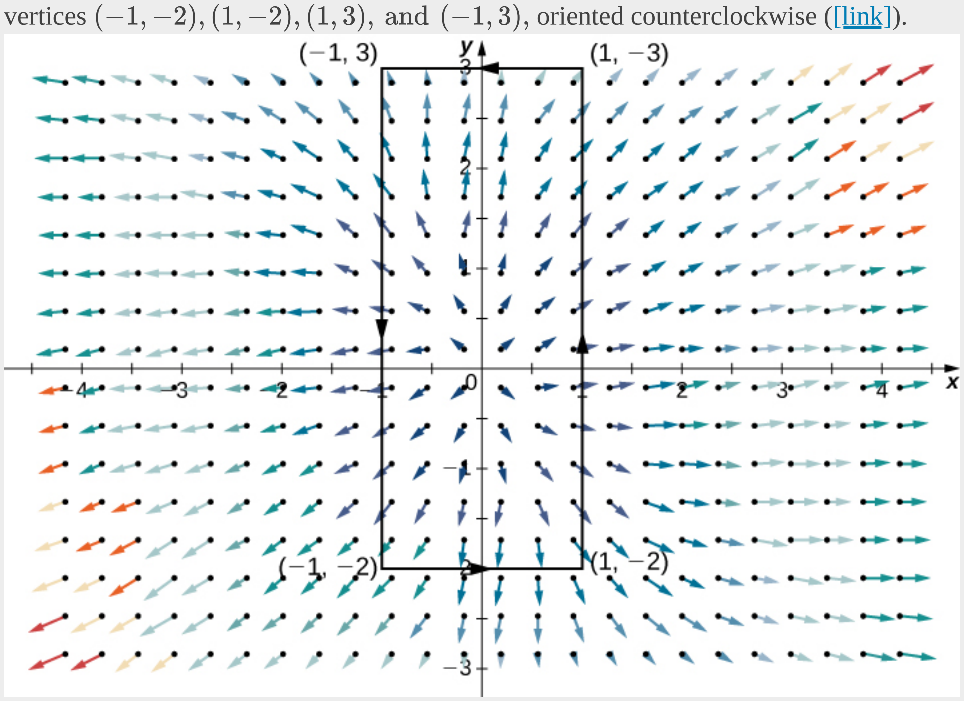
Hint

Apply Green’s theorem and use polar coordinates.

Example:
Exercise:

Problem:
Applying Green’s Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\mathbf{v}(x,y) = \langle 5x+y, x+3y\rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1,-2), (1,-2), (1,3),$ and $(-1,3)$, oriented counterclockwise ([link]).



Water flows across the rectangle with vertices $(-1,-2),(1,-2),(1,3),\text{ and }(-1,3)$, oriented counterclockwise.

Solution:

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \mathbf{v} \bullet d\mathbf{r}$. Let $P(x,y) = 5x + y$ and $Q(x,y) = x + 3y$ so that $\mathbf{v} = (P,Q)$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

Equation:

$$\begin{aligned}\int_C \mathbf{v} \bullet d\mathbf{r} &= \iint_D (P_x + Q_y) dA \\ &= \iint_D 8dA \\ &= 8\,(\text{area of } D) = 80.\end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field **F** is conservative, then **F** does no work around closed curves—that is, the circulation of **F** around a closed curve is zero. In fact, if the domain of **F** is simply connected, then **F** is conservative if and only if the circulation of **F** around any closed curve is zero. If we replace "circulation of **F**" with "flux of **F**," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\mathbf{F} = \langle P, Q\rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \mathbf{F} \cdot \mathbf{N} ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of **F** with the same starting points and endpoints, then $\int_{C_1} \mathbf{F} \cdot \mathbf{N} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{N} ds$. In other words, flux is independent of path.
3. There is a **stream function** $g(x,y)$ for **F**. A stream function for $\mathbf{F} = \langle P, Q\rangle$ is a function g such that $P = g_y$ and $Q = -g_x$. Geometrically, $\mathbf{F} = (a,b)$ is tangential to the level curve of g at (a,b). Since the gradient of g is perpendicular to the level curve of g at (a,b), stream function g has the property $\mathbf{F}(a,b) \bullet \nabla g(a,b) = 0$ for any point (a,b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example:
Exercise:

Problem:
Finding a Stream Function

Verify that rotation vector field $\mathbf{F}(x,y) = \langle y, -x\rangle$ is source free, and find a stream function for **F**.

Solution:

Note that the domain of **F** is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that **F** is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x, y) = y$ and $Q(x, y) = -x$. Then $P_x + 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, **F** is source free.

To find a stream function for **F**, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for **F**. Then $g_y = y$, which implies that

Equation:

$$g(x, y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x$, we have $h\prime(x) = x$. Therefore,

Equation:

$$h(x) = \frac{x^2}{2} + C.$$

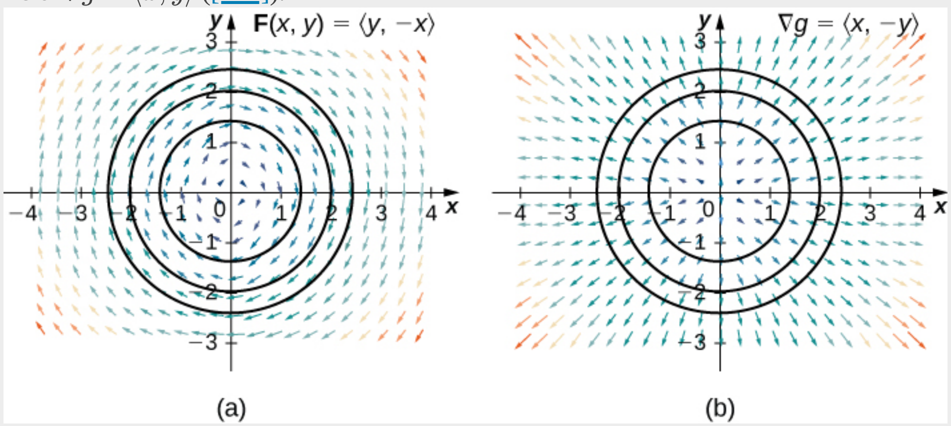
Letting $C = 0$ gives stream function

Equation:

$$g(x, y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for **F**, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ is perpendicular to conservative radial vector field $\nabla g = \langle x, y \rangle$ ([link]).



(a) In this image, we see the three-level curves of g and vector field **F**. Note that the **F** vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field ∇g. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\mathbf{F}(a, b) \bullet \nabla g(a, b) = 0$ for any point in the domain of g.

Note:

Exercise:

Problem: Find a stream function for vector field $\mathbf{F}(x,y) = \langle x \sin y, \cos y \rangle$.

Solution:

$g(x,y) = -x \cos y$

Hint

Follow the outline provided in the previous example.

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\mathbf{F} = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\nabla f = \mathbf{F}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\mathbf{F}$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example:
Exercise:

Problem:
Satisfying Laplace's Equation

For vector field $\mathbf{F}(x,y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for $\mathbf{F}$, and verify that the potential function is harmonic.

Solution:

Let $P(x,y) = e^x \sin y$ and $Q(x,y) = e^x \cos y$. Notice that the domain of $\mathbf{F}$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\mathbf{F}$ to determine whether $\mathbf{F}$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\mathbf{F}$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\mathbf{F}$, let f be a potential function. Then, $\nabla f = \mathbf{F}$, so $f_x = e^x \sin y$. Integrating this equation with respect to *x* gives $f(x,y) = e^x \sin y + h(y)$. Since $f_y = e^x \cos y$, differentiating f with respect to *y* gives $e^x \cos y = e^x \cos y + h\prime(y)$. Therefore, we can take $h(y) = 0$, and $f(x,y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy} = \frac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

Note:
Exercise:

Problem: Is the function $f\left(x, y\right) = e^{x+5y}$ harmonic?

Solution:

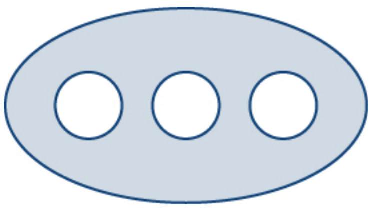
No

Hint

Determine whether the function satisfies Laplace's equation.

Green's Theorem on General Regions

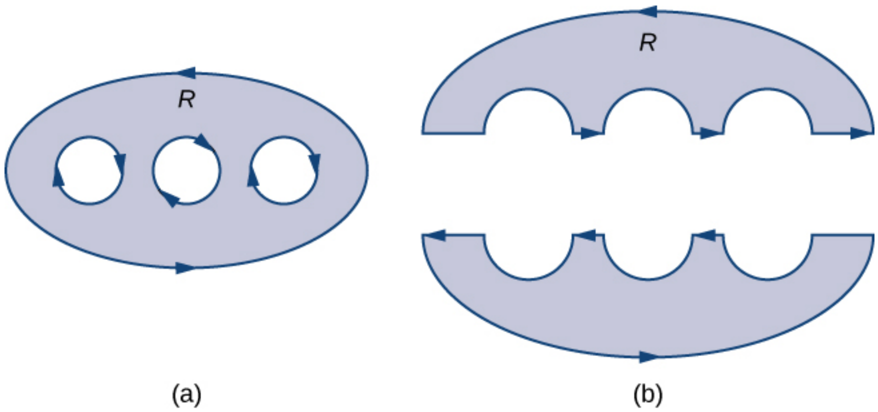
Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes ([link]).



Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D ([link]). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes ([link]).



(a) Region *D* with an oriented boundary has three holes. (b) Region *D* split into two simply connected regions has no holes.

Assume the boundary of *D* is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If **F** is a vector field defined on *D*, then Green's theorem says that

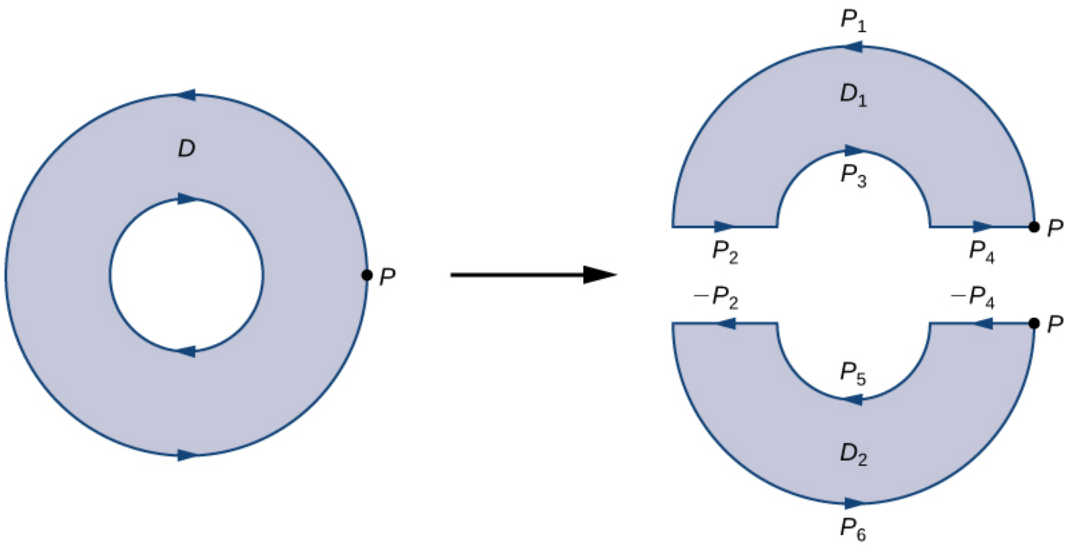
Equation:

$$\begin{aligned}\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_{D_1} Q_x - P_y dA + \iint_{D_2} Q_x - P_y dA \\ &= \iint_D (Q_x - P_y) dA.\end{aligned}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus *D* in [link] and suppose that $\mathbf{F} = \langle P, Q \rangle$ is a vector field defined on this annulus. Region *D* has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise ([link]) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided *D* into two simply connected regions.

We label each piece of these new boundaries as P_i for some *i*, as in [link]. If we begin at *P* and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to *P*. Next, we start at *P* again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.



Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

[link] shows a path that traverses the boundary of *D*. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region *D*. If we had only traversed one portion of the boundary of *D*, then we cannot apply Green's theorem to *D*.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

Equation:

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{-P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} - \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \\
&= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_4} \mathbf{F} \cdot d\mathbf{r} - \int_{P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} - \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \mathbf{F} \\
&= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_{D_1} (Q_x - P_y) dA + \iint_{D_2} (Q_x - P_y) dA \\
&= \iint_{D} (Q_x - P_y) dA.
\end{aligned}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

Equation:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) dA.$$

The same logic implies that the flux form of Green’s theorem can also be extended to a region with finitely many holes:
Equation:

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D (P_x + Q_y) dA.$$

Example:
Exercise:

Problem:
Using Green’s Theorem on a Region with Holes

Calculate integral
Equation:

$$\oint_{\partial D} \left(\sin x - \frac{y^3}{3}\right) dx + \left(\frac{y^3}{3} + \sin y\right) dy,$$

where D is the annulus given by the polar inequalities $1 \leq \mathbf{r} \leq 2$, $0 \leq \theta \leq 2\pi$.

Solution:

Although D is not simply connected, we can use the extended form of Green’s theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:
Equation:

$$\begin{aligned}
\oint_{\partial D} \left(\sin x - \frac{y^3}{3}\right) dx + \left(\frac{x^3}{3} + \sin y\right) dy &= \iint_D (Q_x - P_y) dA \\
&= \iint_D \left(x^2 + y^2\right) dA \\
&= \int_0^{2\pi} \int_1^2 r^3 dr d\theta = \int_0^{2\pi} \frac{15}{4} d\theta \\
&= \tfrac{15\pi}{2}.
\end{aligned}$$

Example:
Exercise:

Problem:
Using the Extended Form of Green’s Theorem

Let $\mathbf{F} = \langle P, Q \rangle = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \mathbf{F} \cdot d\mathbf{r}$?

Solution:

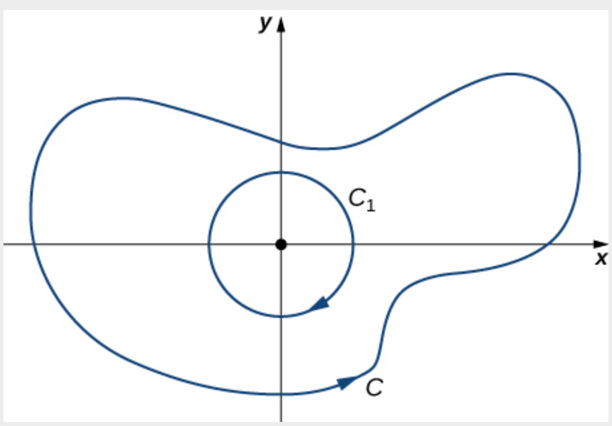
We use the extended form of Green's theorem to show that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of **F** along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: *C* Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of **F** is at the origin. We showed in our discussion of cross-partials that **F** satisfies the cross-partial condition. If we restrict the domain of **F** just to C and the region it encloses, then **F** with this restricted domain is now defined on a simply connected domain. Since **F** satisfies the cross-partial property on its restricted domain, the field **F** is conservative on this simply connected region and hence the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero.

Case 2: *C* Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius *a* centered at the origin so that C_1 is entirely inside the region enclosed by C ([link]). Give C_1 a clockwise orientation.



Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

Equation:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \iint_D Q_x - P_y dA \\ &= \iint_D -\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

Equation:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$. Let

Equation:

$$x = a \cos t, y = a \sin t, 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

Equation:

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}\prime(t) dt \\ &= \int_0^{2\pi} \left\langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a} \right\rangle \cdot \langle -a \sin(t), -a \cos(t) \rangle dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

Therefore, $\int_C \mathbf{F} \cdot ds = -2\pi$.

Note:
Exercise:

Problem:

Calculate integral $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$, where D is the annulus given by the polar inequalities $2 \le r \le 5, 0 \le \theta \le 2\pi$, and $\mathbf{F}\left(x, y\right) = \left\langle x^3, 5x + e^y \sin y \right\rangle$.

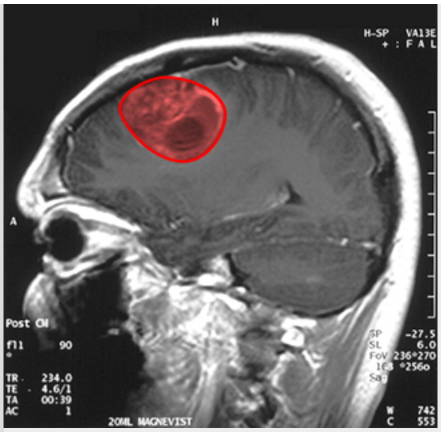
Solution:

105π

Hint

Use the extended version of Green’s theorem.

Note:
Measuring Area from a Boundary: The Planimeter

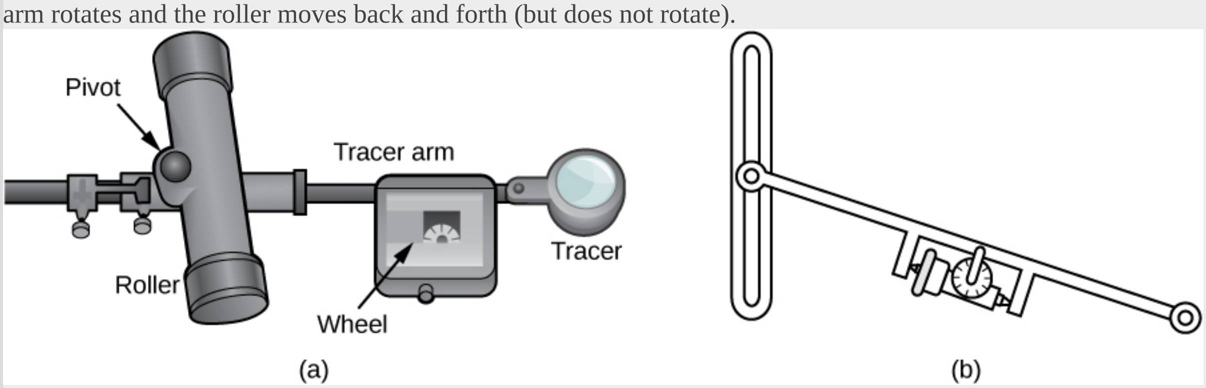


This magnetic resonance image of a patient’s brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Imagine you are a doctor who has just received a magnetic resonance image of your patient’s brain. The brain has a tumor ([link]). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green’s theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region ([link]). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green’s theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

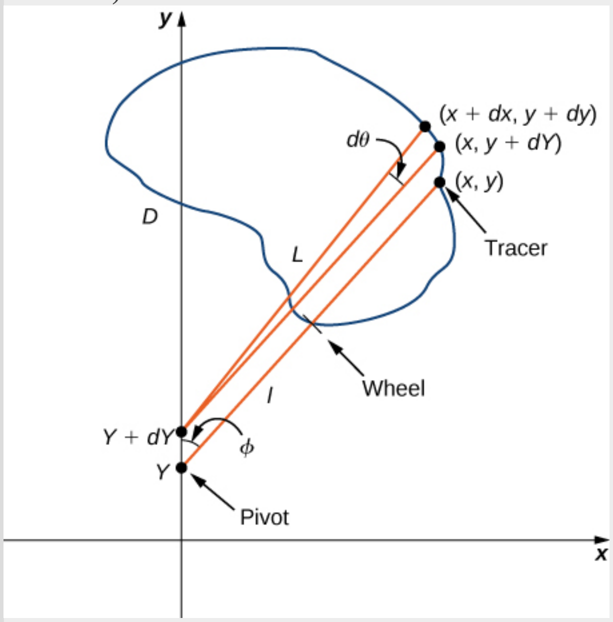


(a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the y-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the y-axis while the tracer arm rotates on the pivot.

Note:
Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) ([link]). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the y-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the x-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let l be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).



Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin \phi dY + l d\theta = \frac{x}{L} dY + l d\theta$.
2. Show that $\oint_C d\theta = 0$.
3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is
   Total wheel roll $= \frac{1}{L} \oint_C x dY$.
   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.

4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in [link]. Explain why $Y \le y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.
6. Use step 5 to show that $dY = dy + \frac{x}{\sqrt{L^2 - x^2}} dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{\sqrt{L^2 - x^2}} dx = 0$.
8. Use step 7 to show that the total wheel roll is
   Total wheel roll $= \frac{1}{L} \oint_C xdy$.
   It took a bit of work, but this equation says that the variable of integration Y in step 3 can be replaced with y.
9. Use Green's theorem to show that the area of D is $\oint_C xdy$. The logic is similar to the logic used to show that the area of $D = \frac{1}{2} \oint_C -ydx + xdy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.
    You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\mathbf{F} \cdot \mathbf{T}$. In the flux form, the integrand is $\mathbf{F} \cdot \mathbf{N}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C Pdx + Qdy = \iint_D Q_x - P_y dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y dA$

For the following exercises, evaluate the line integrals by applying Green's theorem.
Exercise:

Problem:

$\int_C 2xydx + (x + y)dy$, where C is the path from (0, 0) to (1, 1) along the graph of $y = x^3$ and from (1, 1) to (0, 0) along the graph of $y = x$ oriented in the counterclockwise direction

Exercise:

Problem:

$\int_C 2xydx + (x+y)dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

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Solution:

$\int_C 2xydx + (x+y)dy = \frac{32}{3}$

Exercise:

Problem:

$\int_C 2\arctan\left(\frac{y}{x}\right)dx + \ln\left(x^2+y^2\right)dy$, where C is defined by $x = 4 + 2\cos\theta, y = 4\sin\theta$ oriented in the counterclockwise direction

Exercise:

Problem:

$\int_C \sin x \cos ydx + (xy + \cos x \sin y)dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

---

Solution:

$\int_C \sin x \cos ydx + (xy + \cos x \sin y)dy = \frac{1}{12}$

Exercise:

Problem:

$\int_C xydx + (x+y)dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

Exercise:

Problem:

$\oint_C (-ydx + xdy)$, where C consists of line segment C_1 from $(-1, 0)$ to (1, 0), followed by the semicircular arc C_2 from (1, 0) back to (1, 0)

Solution:

$\oint_C (-ydx + xdy) = \pi$

For the following exercises, use Green’s theorem.

Exercise:

Problem:

Let C be the curve consisting of line segments from (0, 0) to (1, 1) to (0, 1) and back to (0, 0). Find the value of $\int_C xydx + \sqrt{y^2+1}dy$.

Exercise:

Problem:

Evaluate line integral $\int_C xe^{-2x}dx + \left(x^4 + 2x^2y^2\right)dy$, where C is the boundary of the region between circles $x^2+y^2=1$ and $x^2+y^2=4$, and is a positively oriented curve.

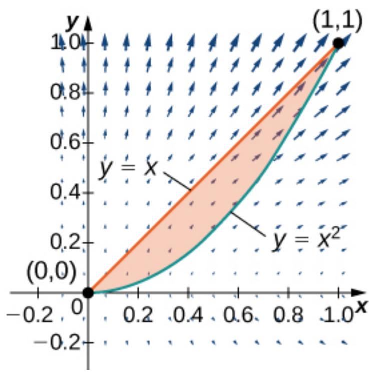
Solution:

$$\int_C xe^{-2x}dx + \left(x^4 + 2x^2y^2\right)dy = 0$$

Exercise:

Problem:

Find the counterclockwise circulation of field $\mathbf{F}(x, y) = xy\mathbf{i} + y^2\mathbf{j}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.



Exercise:

Problem:

Evaluate $\oint_C y^3dx - x^3y^2dy$, where C is the positively oriented circle of radius 2 centered at the origin.

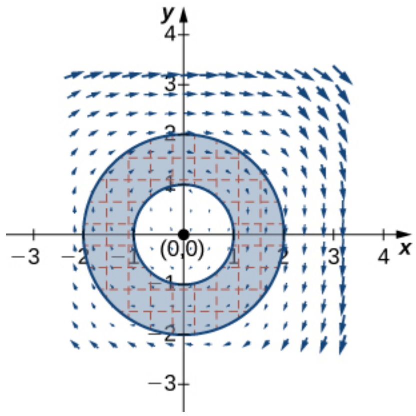
Solution:

$$\oint_C y^3dx - x^3dy = -24\pi$$

Exercise:

Problem:

Evaluate $\oint_C y^3dx - x^3dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



Exercise:

Problem:

Calculate $\oint_C -x^2 y dx + x y^2 dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

Solution:

$\oint_C -x^2 y dx + x y^2 dy = 8\pi$

Exercise:

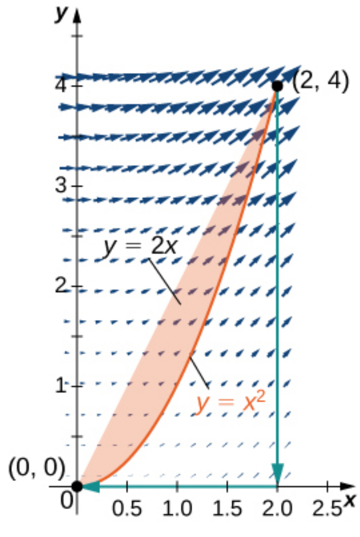
Problem:

Calculate integral $\oint_C 2\left[y + x\sin(y)\right]dx + \left[x^2\cos(y) - 3y^2\right]dy$ along triangle C with vertices (0, 0), (1, 0) and (1, 1), oriented counterclockwise, using Green's theorem.

Exercise:

Problem:

Evaluate integral $\oint_C \left(x^2 + y^2\right)dx + 2xydy$, where C is the curve that follows parabola $y = x^2$ from $(0, 0)$ $(2, 4)$, then the line from (2, 4) to (2, 0), and finally the line from (2, 0) to (0, 0).



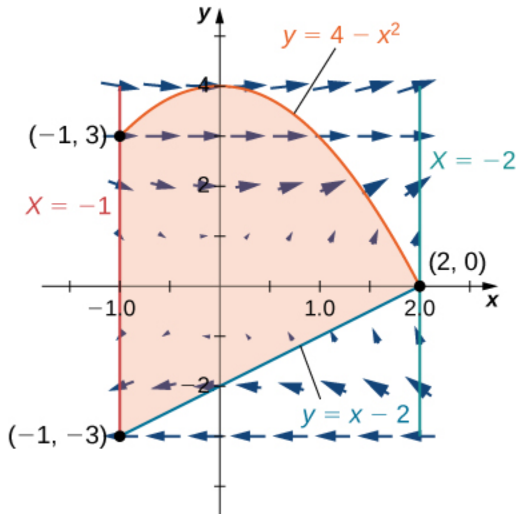
Solution:

$$\oint_C (x^2 + y^2)dx + 2xydy = 0$$

Exercise:

Problem:

Evaluate line integral $\oint_C (y - \sin(y)\cos(y))dx + 2x\sin^2(y)dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1, x = 2, y = 4 - x^2$, and $y = x - 2$.



For the following exercises, use Green's theorem to find the area.

Exercise:

Problem: Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

Solution:

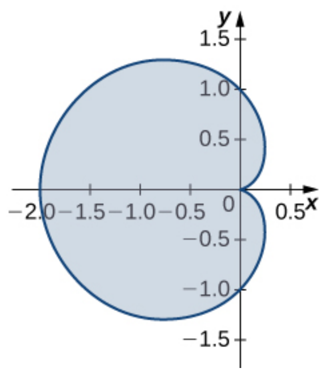
$A = 19\pi$

Exercise:

Problem: Find the area of the region enclosed by parametric equation

Equation:

$$p(\theta) = \left(\cos(\theta) - \cos^2(\theta)\right)\mathbf{i} + (\sin(\theta) - \cos(\theta)\sin(\theta))\mathbf{j} \text{ for } 0 \le \theta \le 2\pi.$$



Exercise:

Problem:

Find the area of the region bounded by hypocycloid $\mathbf{r}(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j}$. The curve is parameterized by $t \in [0, 2\pi]$.

Solution:

$A = \frac{3}{8\pi}$

Exercise:

Problem: Find the area of a pentagon with vertices $(0, 4)$, $(4, 1)$, $(3, 0)$, $(-1, -1)$, and $(-2, 2)$.

Exercise:

Problem:

Use Green's theorem to evaluate $\int_{C+} \left(y^2 + x^3\right)dx + x^4 dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

Solution:

$\int_{C+} \left(y^2 + x^3\right)dx + x^4 dy = 0$

Exercise:

Problem: Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2$.

Exercise:

Problem:

Use Green’s theorem to find the area of one loop of a four-leaf rose $r = 3 \sin 2\theta$. (*Hint*: $xdy - ydx = \mathbf{r}^2 d\theta$).

Solution:

$A = \frac{9\pi}{8}$

Exercise:

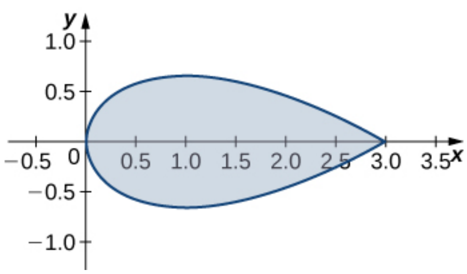
Problem:

Use Green’s theorem to find the area under one arch of the cycloid given by parametric plane $x = t - \sin t, y = 1 - \cos t, t \geq 0$.

Exercise:

Problem: Use Green’s theorem to find the area of the region enclosed by curve
Equation:

$$\mathbf{r}(t) = t^2\mathbf{i} + \left(\frac{t^3}{3} - t\right)\mathbf{j}, \ -\sqrt{3} \leq t\sqrt{3}.$$



Solution:

$A = \frac{8\sqrt{3}}{5}$

Exercise:

Problem:

[T] Evaluate Green’s theorem using a computer algebra system to evaluate the integral $\int_C xe^y dx + e^x dy$, where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

Exercise:

Problem:

Evaluate $\int_C \left(x^2 y - 2xy + y^2\right) ds$, where C is the boundary of the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$, traversed counterclockwise.

Solution:

$\int_C \left(x^2 y - 2xy + y^2\right) ds = 3$

Exercise:

Problem:

Evaluate $\int_C \frac{-(y+2)dx+(x-1)dy}{(x-1)^2+(y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1,-2)$ traversed counterclockwise.

Exercise:

Problem:

Evaluate $\int_C \frac{xdx+ydy}{x^2+y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

Solution:

$$\int_C \frac{xdx+ydy}{x^2+y^2} = 2\pi$$

For the following exercises, use Green's theorem to calculate the work done by force **F** on a particle that is moving counterclockwise around closed path *C*.

Exercise:

Problem: $\mathbf{F}(x,y) = xy\mathbf{i} + (x+y)\mathbf{j}$, $C: x^2+y^2=4$

Exercise:

Problem:

$\mathbf{F}(x,y) = \left(x^{3/2}-3y\right)\mathbf{i} + \left(6x+5\sqrt{y}\right)\mathbf{j}$, C : boundary of a triangle with vertices (0, 0), (5, 0), and (0, 5)

Solution:

$W = \frac{225}{2}$

Exercise:

Problem:

Evaluate $\int_C \left(2x^3-y^3\right)dx + \left(x^3+y^3\right)dy$, where C is a unit circle oriented in the counterclockwise direction.

Exercise:

Problem:

A particle starts at point $(-2,0)$, moves along the *x*-axis to (2, 0), and then travels along semicircle $y=\sqrt{4-x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\mathbf{F}(x,y) = x\mathbf{i} + \left(x^3+3xy^2\right)\mathbf{j}$.

Solution:

$W = 12\pi$

Exercise:

Problem:

David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x, y) (x, y) (x, y) is $\mathbf{F}(x, y) = \left(x^2 y + 10y\right)\mathbf{i} + \left(x^3 + 2xy^2\right)\mathbf{j}$. Use Green's theorem to determine who does more work.

Exercise:

Problem:

Use Green's theorem to find the work done by force field $\mathbf{F}(x, y) = (3y - 4x)\mathbf{i} + (4x - y)\mathbf{j}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4$.

Solution:

$W = 2\pi$

Exercise:

Problem:

Use Green's theorem to evaluate line integral $\oint_C e^{2x} \sin 2y dx + e^{2x} \cos 2y dy$, where C is ellipse $9(x-1)^2 + 4(y-3)^2 = 36$ oriented counterclockwise.

Exercise:

Problem:

Evaluate line integral $\oint_C y^2 dx + x^2 dy$, where C is the boundary of a triangle with vertices $(0, 0), (1, 1), \text{ and } (1, 0)$, with the counterclockwise orientation.

Solution:

$\oint_C y^2 dx + x^2 dy = \frac{1}{3}$

Exercise:

Problem:

Use Green's theorem to evaluate line integral $\int_C \mathbf{h} \cdot d\mathbf{r}$ if $\mathbf{h}(x, y) = e^y \mathbf{i} - \sin \pi x \mathbf{j}$, where C is a triangle with vertices (1, 0), (0, 1), and $(-1, 0)$ $(-1, 0)$ traversed counterclockwise.

Exercise:

Problem:

Use Green's theorem to evaluate line integral $\int_C \sqrt{1 + x^3} dx + 2xy dy$ where C is a triangle with vertices (0, 0), (1, 0), and (1, 3) oriented clockwise.

Solution:

$\int_C \sqrt{1 + x^3} dx + 2xy dy = 3$

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\int_C x^2 y dx - xy^2 dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\int_C (3y - e^{\sin x}) dx + \left(7x + \sqrt{y^4 + 1}\right) dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

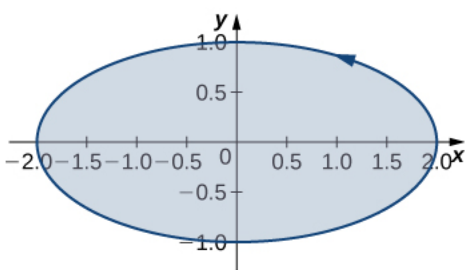
Solution:

$\int_C (3y - e^{\sin x}) dx + \left(7x + \sqrt{y^4 + 1}\right) dy = 36\pi$

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\int_C (3x - 5y) dx + (x - 6y) dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



Exercise:

Problem:

Let C be a triangular closed curve from (0, 0) to (1, 0) to (1, 1) and finally back to (0, 0). Let $\mathbf{F}(x, y) = 4y\mathbf{i} + 6x^2\mathbf{j}$. Use Green’s theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

Solution:

$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2$

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\oint_C y dx - x dy$, where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\oint_C (y+x)dx + (x + \sin y)dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

Solution:

$$\oint_C (y+x)dx + (x+\sin y)dy = 0$$

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\oint_C \left(y - \ln\left(x^2+y^2\right)\right)dx + \left(2\arctan\frac{y}{x}\right)dy$, where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1$.

Exercise:

Problem:

Use Green’s theorem to evaluate $\oint_C xydx + x^3y^3dy$, where C is a triangle with vertices (0, 0), (1, 0), and (1, 2) with positive orientation.

Solution:

$$\oint_C xydx + x^3y^3dy = \frac{22}{21}$$

Exercise:

Problem:

Use Green’s theorem to evaluate line integral $\int_C \sin ydx + x\cos ydy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

Exercise:

Problem:

Let $\mathbf{F}(x,y) = \left(\cos\left(x^5\right)\right) - \frac{1}{3}y^3\mathbf{i} + \frac{1}{3}x^3\mathbf{j}$. Find the counterclockwise circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, half circle $y=\sqrt{1-x^2}$, the line segment joining (1, 0) and (2, 0), and half circle $y=\sqrt{4-x^2}$.

Solution:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \tfrac{15\pi}{4}$$

Exercise:

Problem:

Use Green's theorem to evaluate line integral $\int_C \sin\left(x^3\right) dx + 2ye^{x^2} dy$, where C is a triangular closed curve that connects the points (0, 0), (2, 2), and (0, 2) counterclockwise.

Exercise:

Problem:

Let C be the boundary of square $0 \leq x \leq \pi, 0 \leq y \leq \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x+y)dx + \cos(x+y)dy$.

Solution:

$\int_C \sin(x+y)dx + \cos(x+y)dy = 4$

Exercise:

Problem:

Use Green's theorem to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \left(y^2 - x^2\right)\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j}$, and C is a triangle bounded by $y = 0, x = 3,$ and $y = x$, oriented counterclockwise.

Exercise:

Problem:

Use Green's Theorem to evaluate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \left(xy^2\right)\mathbf{i} + x\mathbf{j}$, and C is a unit circle oriented in the counterclockwise direction.

Solution:

$\int_C \mathbf{F} \cdot d\mathbf{r} = \pi$

Exercise:

Problem:

Use Green's theorem in a plane to evaluate line integral $\oint_C \left(xy + y^2\right)dx + x^2 dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

Exercise:

Problem:

Calculate the outward flux of $\mathbf{F} = -x\mathbf{i} + 2y\mathbf{j}$ over a square with corners $(\pm 1, \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

Solution:

$\oint_C \mathbf{F} \cdot \widehat{\mathbf{n}} ds = 4$

Exercise:

Problem:

[T] Let C be circle $x^2+y^2=4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[\left(3y-e^{\tan^{-1}x}\right)dx+\left(7x+\sqrt{y^4+1}\right)dy\right]$ using a computer algebra system.

Exercise:

Problem:

Find the flux of field $\mathbf{F}=-x\mathbf{i}+y\mathbf{j}$ across $x^2+y^2=16$ oriented in the counterclockwise direction.

Solution:

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds=32\pi$$

Exercise:

Problem:

Let $\mathbf{F}=\left(y^2-x^2\right)\mathbf{i}+\left(x^2+y^2\right)\mathbf{j}$, and let C be a triangle bounded by $y=0$, $x=3$, and $y=x$ oriented in the counterclockwise direction. Find the outward flux of $\mathbf{F}$ through C.

Exercise:

Problem:

[T] Let C be unit circle $x^2+y^2=1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3+\sin(xy)+xy\cos(xy)\right]dx+\left[x^3+x^2\cos(xy)\right]dy$ by using a computer algebra system.

Solution:

$$\int_C \left[-y^3+\sin(xy)+xy\cos(xy)\right]dx+\left[x^3+x^2\cos(xy)\right]dy=4.7124$$

Exercise:

Problem:

[T] Find the outward flux of vector field $\mathbf{F}=xy^2\mathbf{i}+x^2y\mathbf{j}$ across the boundary of annulus $R=\left\{(x,y):1\le x^2+y^2\le 4\right\}=\{(r,\theta):1\le r\le 2, 0\le\theta\le 2\pi\}$ using a computer algebra system.

Exercise:

Problem:

Consider region R bounded by parabolas $y=x^2$ and $x=y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y+e^{\sqrt{x}}\right)dx+\left(2x+\cos\left(y^2\right)\right)dy$.

Solution:

$$\oint_C \left(y+e^{\sqrt{x}}\right)dx+\left(2x+\cos\left(y^2\right)\right)dy=\frac{1}{3}$$

Glossary

Green's theorem

relates the integral over a connected region to an integral over the boundary of the region

stream function
: if $\mathbf{F} = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

Divergence and Curl

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field **F** in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point *P* is a measure of the “outflowing-ness” of the vector field at *P*. If **F** represents the velocity of a fluid, then the divergence of **F** at *P* measures the net rate of change with respect to time of the amount of fluid flowing away from *P* (the tendency of the fluid to flow “out of” *P*). In particular, if the amount of fluid flowing into *P* is the same as the amount flowing out, then the divergence at *P* is zero.

Note:
Definition
If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the **divergence** of **F** is defined by
Equation:

$$\text{div}\,\mathbf{F} = P_x + Q_y + R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$, divergence can be written symbolically as the dot product
Equation:

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}.$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

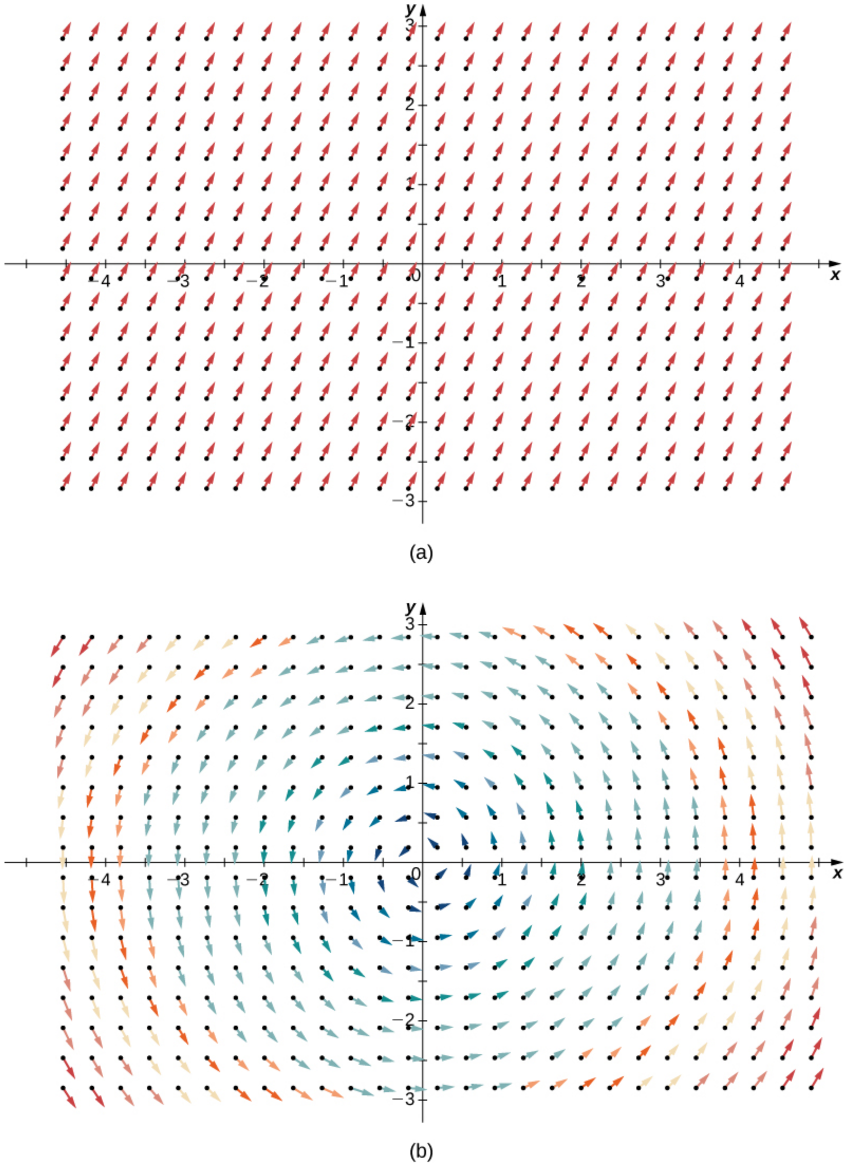
If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of **F** is defined similarly as
Equation:

$$\text{div}\,\mathbf{F} = P_x + Q_y = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \nabla \cdot \mathbf{F}.$$

To illustrate this point, consider the two vector fields in [link]. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the “outflowing-ness” of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

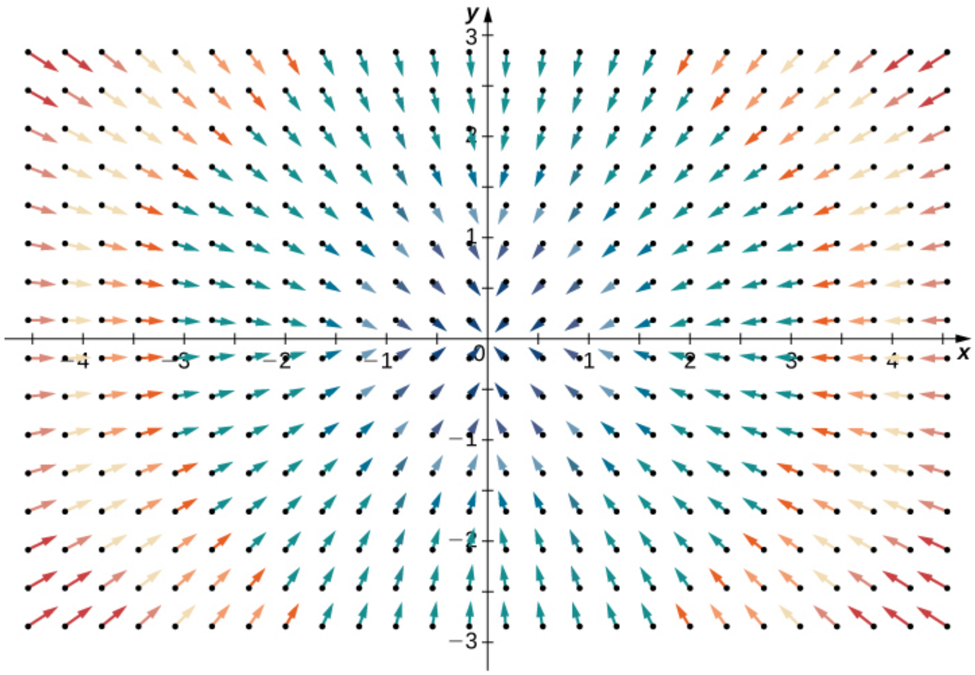
Equation:

$$\operatorname{div}(\langle 1,2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \text{ and } \operatorname{div}(\langle -y,x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$



(a) Vector field $\langle 1,2\rangle$ has zero divergence. (b) Vector field $\langle -y,x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\mathbf{R}(x,y) = \langle -x, -y \rangle$ in [link]. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as $\text{div}(\mathbf{R}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2$.



This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in [link]. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in [link] so that the center of the circle lands at point (3, 3). The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example:
Exercise:

Problem:
Calculating Divergence at a Point

If $\mathbf{F}(x,y,z) = e^x\mathbf{i} + yz\mathbf{j} - y^2\mathbf{k}$, then find the divergence of $\mathbf{F}$ at $(0, 2, -1)$.

Solution:

The divergence of **F** is
Equation:

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}\left(yz^2\right) = e^x + z - 2yz.$$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4$. If $\mathbf{F}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

Note:
Exercise:

Problem: Find div $\mathbf{F}$ for $\mathbf{F}(x, y, z) = \langle xy, 5 - z^2y, x^2 + y^2 \rangle$.

Solution:

$y - z^2$

Hint

Follow [link].

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\mathbf{B}$ is a magnetic field, then $\nabla \cdot \mathbf{B} = 0$; in other words, the divergence of a magnetic field is zero.

Example:
Exercise:

Problem:
Determining Whether a Field Is Magnetic

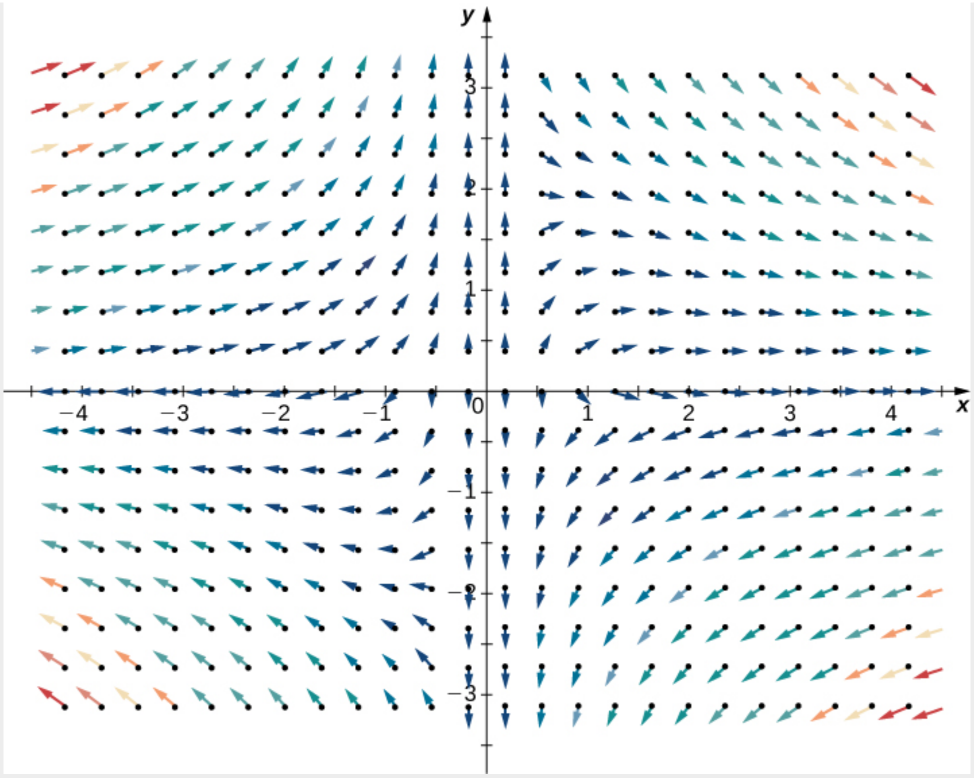
Is it possible for $\mathbf{F}(x, y) = \langle x^2y, y - xy^2 \rangle$ to be a magnetic field?

Solution:

If $\mathbf{F}$ were magnetic, then its divergence would be zero. The divergence of $\mathbf{F}$ is
Equation:

$$\frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}\left(y - xy^2\right) = 2xy + 1 - 2xy = 1$$

and therefore $\mathbf{F}$ cannot model a magnetic field ([link]).



The divergence of vector field $\mathbf{F}(x, y) = \langle x^2 y, y - xy^2 \rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Note:
Divergence of a Source-Free Vector Field
If $\mathbf{F} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then div $\mathbf{F} = 0$.

Proof

Since $\mathbf{F}$ is source free, there is a function $g(x, y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\mathbf{F} = \langle g_y, -g_x \rangle$ and div $\mathbf{F} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of [link] is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Note:
Divergence Test for Source-Free Vector Fields
Let $\mathbf{F} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\text{div } \mathbf{F} = 0$ if and only if $\mathbf{F}$ is source free.

Example:
Exercise:

Problem:
Determining Whether a Field Is Source Free

Is field $\mathbf{F}\left(x, y\right) = \left\langle x^2 y, 5 - xy^2 \right\rangle$ source free?

Solution:

Note the domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is simply connected. Furthermore, $\mathbf{F}$ is continuous with differentiable component functions. Therefore, we can use [link] to analyze $\mathbf{F}$. The divergence of $\mathbf{F}$ is
Equation:

$$\frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(5 - xy^2\right) = 2xy - 2xy = 0.$$

Therefore, $\mathbf{F}$ is source free by [link].

Note:
Exercise:

Problem:

Let $\mathbf{F}\left(x, y\right) = \langle -ay, bx \rangle$ be a rotational field where *a* and *b* are positive constants. Is $\mathbf{F}$ source free?

Solution:

Yes

Hint

Calculate the divergence.

Recall that the flux form of Green's theorem says that
Equation:

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D P_x + Q_y dA,$$

where *C* is a simple closed curve and *D* is the region enclosed by *C*. Since $P_x + Q_y = \text{div } \mathbf{F}$, Green's theorem is sometimes written as
Equation:

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D \text{div}\, \mathbf{F} dA.$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of **F** on a region can be translated into a line integral of **F** along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let **v** be a vector field modeling the velocity of a fluid. Since the divergence of **v** at point *P* measures the "outflowing-ness" of the fluid at *P*, $\text{div}\, \mathbf{v}(P) > 0$ implies that more fluid is flowing out of *P* than flowing in. Similarly, $\text{div}\, \mathbf{v}(P) < 0$ implies the more fluid is flowing in to *P* than is flowing out, and $\text{div}\, \mathbf{v}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

Example:
Exercise:

Problem:
Determining Flow of a Fluid

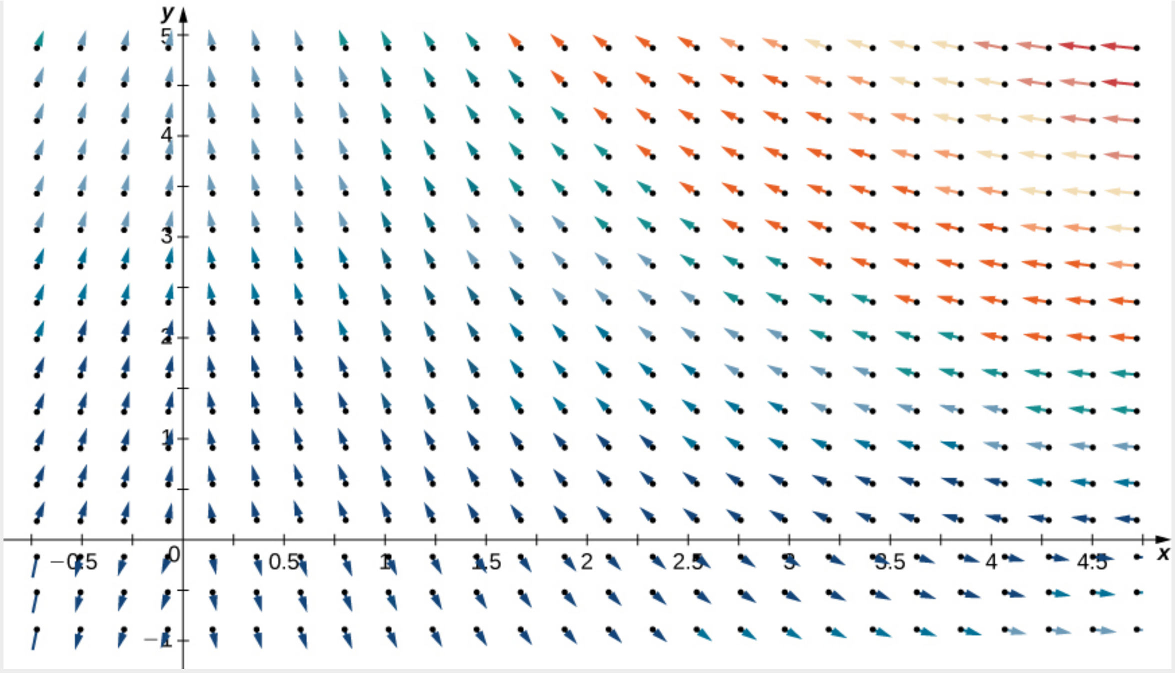
Suppose $\mathbf{v}(x, y) = \langle -xy, y \rangle, y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution:

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of **v** at $(1, 4)$:
Equation:

$$\text{div}\,(\mathbf{v}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$, substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of **v** at $(1, 4)$ is negative, more fluid is flowing in than flowing out ([link]).



Vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle$ has negative divergence at $(1, 4)$.

Note:
Exercise:

Problem:

For vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle, y > 0$, find all points *P* such that the amount of fluid flowing in to *P* equals the amount of fluid flowing out of *P*.

Solution:

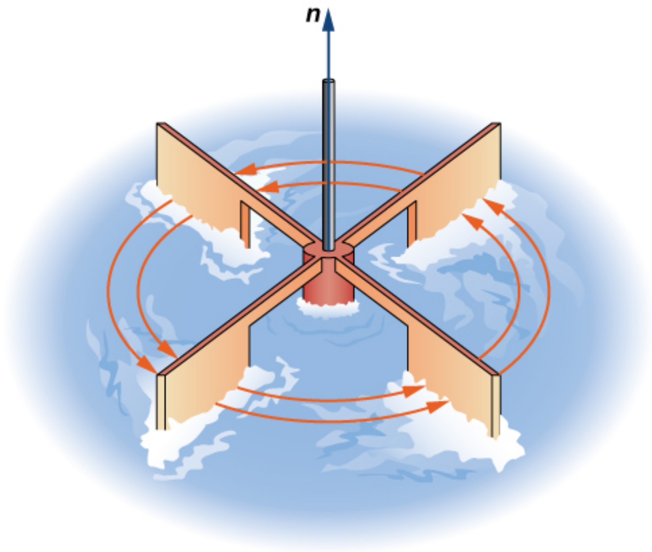
All points on line $y = 1$.

Hint

Find where the divergence is zero.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that **F** represents the velocity field of a fluid. Then, the curl of **F** at point *P* is a vector that measures the tendency of particles near *P* to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at *P* measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at *P*, with the axis of the paddlewheel aligned with the curl vector ([link]). The curl measures the tendency of the paddlewheel to rotate.



To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in [link]. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\mathbf{k}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Note:
Definition
If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the **curl** of **F** is defined by
Equation:

$$\begin{aligned} \text{curl}\,\mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \end{aligned}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\nabla \times \mathbf{F}$ to stand for a "determinant" that gives the curl formula:
Equation:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The determinant of this matrix is
Equation:

$$(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} = \operatorname{curl} \mathbf{F}.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\mathbf{F}$, by definition, is
Equation:

$$\operatorname{curl} \mathbf{F} = (Q_x - P_y)\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}.$$

Example:
Exercise:

Problem:
Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\mathbf{F}\,(P, Q, R) = \langle x^2 z, e^y + xz, xyz \rangle$.

Solution:

The curl is
Equation:

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= (xz - x)\mathbf{i} + \left(x^2 - yz\right)\mathbf{j} + z\mathbf{k}. \end{aligned}$$

Note:
Exercise:

Problem: Find the curl of $\mathbf{F} = \langle \sin x \cos z, \sin y \sin z, \cos x \cos y \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution:

$-\mathbf{i}$

Hint

Find the determinant of matrix $\nabla \times \mathbf{F}$.

Example:
Exercise:

Problem:
Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\mathbf{F} = \langle P, Q \rangle = \langle y, 0 \rangle$.

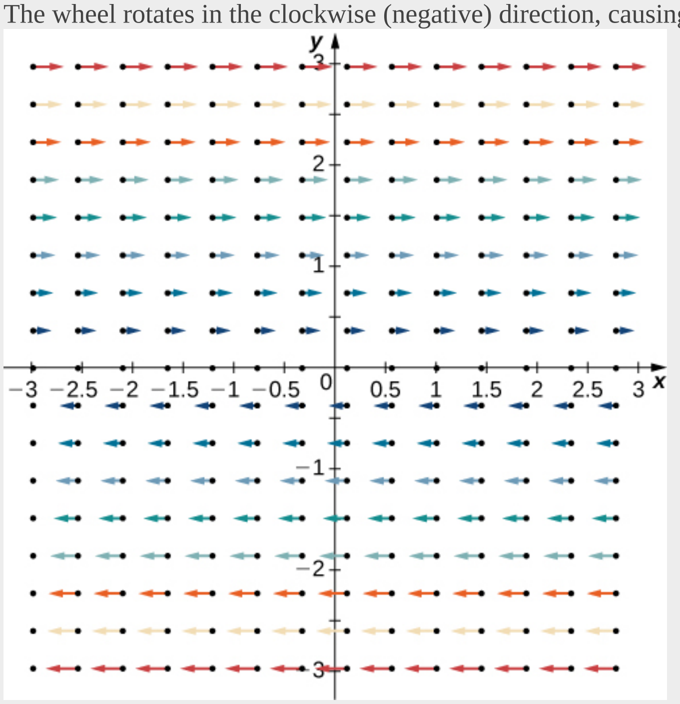
Solution:

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the *x*-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

Equation:

$$\text{curl}\,\mathbf{F} = (Q_x - P_y)\mathbf{k} = -\mathbf{k} \neq 0.$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant ([link]). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.



Vector field $\mathbf{F}(x, y) = \langle y, 0 \rangle$ consists of vectors that are all parallel.

Note that if $\mathbf{F} = \langle P, Q \rangle$ is a vector field in a plane, then $\text{curl}\,\mathbf{F} \cdot \mathbf{k} = (Q_x - P_y)\mathbf{k} \cdot \mathbf{k} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

Equation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\,\mathbf{F} \cdot \mathbf{k} dA,$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of **F** on a region can be translated into a line integral of **F** along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

Equation:

$$\mathbf{F}(x, y, z) = -Gm_2m_2 \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

Example:

Exercise:

Problem:

Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution:

To show that **F** has no spin, we calculate its curl. Let $P(x, y, z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$, $Q(x, y, z) = \frac{y}{(x^2+y^2+z^2)^{3/2}}$, and $R(x, y, z) = \frac{z}{(x^2+y^2+z^2)^{3/2}}$. Then,

Equation:

$$\begin{aligned} \text{curl}\,\mathbf{F} &= -Gm_1m_2\left[(R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}\right] \\ &= -Gm_1m_2 \begin{bmatrix} \left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}} - \left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{i} \\ + \left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}} - \left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{j} \\ + \left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}} - \left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{k} \end{bmatrix} \\ &= 0. \end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Note:

Exercise:

Problem:

Field $\mathbf{v}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Solution:

$\text{curl}\, \mathbf{v} = \mathbf{0}$

Hint

Calculate the curl.

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, then the curl of $\mathbf{F}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\mathbf{v}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\text{div curl }(\mathbf{v}) = 0$, the net rate of flow in vector field curl($\mathbf{v}$) at any point is zero. Taking the curl of vector field $\mathbf{F}$ eliminates whatever divergence was present in $\mathbf{F}$.

Note:
Divergence of the Curl
Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then, $\text{div curl }(\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,
Equation:

$$\begin{aligned} \text{div curl}\, \mathbf{F} &= \text{div}\, [(R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}] \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} \\ &= 0. \end{aligned}$$

□

Example:
Exercise:

Problem:
Showing That a Vector Field Is Not the Curl of Another

Show that $\mathbf{F}(x, y, z) = e^x\mathbf{i} + yz\mathbf{j} + xz^2\mathbf{k}$ is not the curl of another vector field. That is, show that there is no other vector $\mathbf{G}$ with curl $\mathbf{G} = \mathbf{F}$.

Solution:

Notice that the domain of **F** is all of $\mathbb{R}^3$ and the second-order partials of **F** are all continuous. Therefore, we can apply the previous theorem to **F**.

The divergence of **F** is $e^x + z + 2xz$. If **F** were the curl of vector field **G**, then div $\mathbf{F} =$ div curl $\mathbf{G} = 0$. But, the divergence of **F** is not zero, and therefore **F** is not the curl of any other vector field.

Note:
Exercise:

Problem: Is it possible for $\mathbf{G}(x, y, z) = \langle \sin x, \cos y, \sin(xyz) \rangle$ to be the curl of a vector field?

Solution:

No

Hint

Find the divergence of **G**.

With the next two theorems, we show that if **F** is a conservative vector field then its curl is zero, and if the domain of **F** is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Note:
Curl of a Conservative Vector Field
If $\mathbf{F} = \langle P, Q, R \rangle$ is conservative, then curl $\mathbf{F} = 0$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of **F** are equal. Therefore,
Equation:

$$\begin{aligned} \text{curl}\,\mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= 0. \end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that curl $(\nabla f) = 0$ for any scalar function f. In terms of our curl notation, $\nabla \times \nabla(f) = 0$. This equation makes

sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\nabla \times \nabla(f) = 0$ is simplified as $\nabla \times \nabla = 0$.

Note:
Curl Test for a Conservative Field
Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If curl $\mathbf{F} = 0$, then $\mathbf{F}$ is conservative.

Proof

Since curl $\mathbf{F} = 0$, we have that $R_y = Q_z, P_z = R_x$, and $Q_x = P_y$. Therefore, $\mathbf{F}$ satisfies the cross-partials property on a simply connected domain, and [link] implies that $\mathbf{F}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\mathbf{F}$ is a vector field in a plane or in space and the domain is simply connected, then $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = 0$.

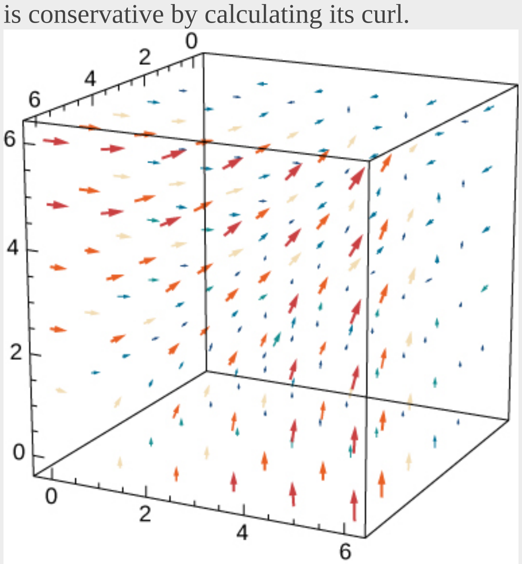
Example:
Exercise:

Problem:
Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution:

Note that the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, which is simply connected ([link]). Therefore, we can test whether $\mathbf{F}$ is conservative by calculating its curl.



The curl of vector field $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of $\mathbf{F}$ is
Equation:

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\mathbf{i} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\mathbf{j} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\mathbf{k} = (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = 0.$$

Thus, $\mathbf{F}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy}$. We abbreviate this “double dot product” as ∇^2. This operator is called the *Laplace operator*, and in this notation Laplace’s equation becomes $\nabla^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then
Equation:

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}.$$

Using this notation we get Laplace’s equation for harmonic functions of three variables:
Equation:

$$\nabla^2 f = 0.$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example:
Exercise:

Problem:
Analyzing a Function

Is it possible for $f(x, y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution:

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Note:
Exercise:

Problem:

Is it possible for function $f(x, y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Solution:

Yes

Hint

Determine whether the function is harmonic.

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If **v** is the velocity field of a fluid, then the divergence of **v** at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point *P* measures the tendency of particles at *P* to rotate about the axis that points in the direction of the curl at *P*.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**
  $\nabla \times \mathbf{F} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$
- **Divergence**
  $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$
- **Divergence of curl is zero**
  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
- **Curl of a gradient is the zero vector**
  $\nabla \times (\nabla f) = 0$

For the following exercises, determine whether the statement is *true or false*.

Exercise:

Problem:

If the coordinate functions of $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl}\,(\text{div}(\mathbf{F}))$ equals zero.

Exercise:

Problem: $\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$

Solution:

False

Exercise:

Problem: All vector fields of the form $\mathbf{F}\,(x, y, z) = f\,(x)\mathbf{i} + g\,(y)\mathbf{j} + h\,(z)\mathbf{k}$ are conservative.

Exercise:

Problem: If curl $\mathbf{F} = 0$, then $\mathbf{F}$ is conservative.

Solution:

True

Exercise:

Problem: If $\mathbf{F}$ is a constant vector field then div $\mathbf{F} = 0$.

Exercise:

Problem: If $\mathbf{F}$ is a constant vector field then curl $\mathbf{F} = 0$.

Solution:

True

For the following exercises, find the curl of $\mathbf{F}$.

Exercise:

Problem: $\mathbf{F}\left(x, y, z\right) = xy^2z^4\mathbf{i} + \left(2x^2y + z\right)\mathbf{j} + y^3z^2\mathbf{k}$

Exercise:

Problem: $\mathbf{F}\left(x, y, z\right) = x^2z\mathbf{i} + y^2x\mathbf{j} + \left(y + 2z\right)\mathbf{k}$

Solution:

$\text{curl}\,\mathbf{F} = \mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$

Exercise:

Problem: $\mathbf{F}\left(x, y, z\right) = 3xyz^2\mathbf{i} + y^2\sin z\mathbf{j} + xe^{2z}\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$

Solution:

$\text{curl}\,\mathbf{F} = \left(xz^2 - xy^2\right)\mathbf{i} + \left(x^2y - yz^2\right)\mathbf{j} + \left(y^2z - x^2z\right)\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = (x\cos y)\mathbf{i} + xy^2\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

Solution:

$\text{curl}\,\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

Solution:

$\text{curl}\,\mathbf{F} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants *a*, *b*, *c*

Solution:

$\text{curl}\,\mathbf{F} = 0$

For the following exercises, find the divergence of **F**.

Exercise:

Problem: $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + y^2x\mathbf{j} + (y + 2z)\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = 3xyz^2\mathbf{i} + y^2\sin z\mathbf{j} + xe^{2z}\mathbf{k}$

Solution:

$\text{div}\,\mathbf{F} = 3yz^2 + 2y\sin z + 2xe^{2z}$

Exercise:

Problem: $\mathbf{F}(x, y) = (\sin x)\mathbf{i} + (\cos y)\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

Solution:

$\text{div}\,\mathbf{F} = 2(x + y + z)$

Exercise:

Problem: $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$

Solution:

div $\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}}$

Exercise:

Problem: $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants a, b, c

Solution:

div $\mathbf{F} = a + b$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

Solution:

div $\mathbf{F} = x + y + z$

For the following exercises, determine whether each of the given scalar functions is harmonic.

Exercise:

Problem: $u(x, y, z) = e^{-x}(\cos y - \sin y)$

Exercise:

Problem: $w(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$

Solution:

Harmonic

Exercise:

Problem: If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find curl $(\mathbf{F} \times \mathbf{G})$.

Exercise:

Problem: If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find div $(\mathbf{F} \times \mathbf{G})$.

Solution:

div $(\mathbf{F} \times \mathbf{G}) = 2z + 3x$

Exercise:

Problem: Find div $\mathbf{F}$, given that $\mathbf{F} = \nabla f$, where $f(x, y, z) = xy^3z^2$.

Exercise:

Problem:

Find the divergence of $\mathbf{F}$ for vector field
$\mathbf{F}(x, y, z) = \left(y^2 + z^2\right)(x + y)\mathbf{i} + \left(z^2 + x^2\right)(y + z)\mathbf{j} + \left(x^2 + y^2\right)(z + x)\mathbf{k}$.

Solution:

$\text{div}\,\mathbf{F} = 2r^2$

Exercise:

Problem: Find the divergence of $\mathbf{F}$ for vector field $\mathbf{F}(x, y, z) = f_1(y, z)\mathbf{i} + f_2(x, z)\mathbf{j} + f_3(x, y)\mathbf{k}$.

For the following exercises, use $r = |\mathbf{r}|$ and $\mathbf{r} = (x, y, z)$.

Exercise:

Problem: Find the curl $\mathbf{r}$.

Solution:

$\text{curl}\,\mathbf{r} = 0$

Exercise:

Problem: Find the curl $\frac{\mathbf{r}}{r}$.

Exercise:

Problem: Find the curl $\frac{\mathbf{r}}{r^3}$.

Solution:

$\text{curl}\,\frac{\mathbf{r}}{r^3} = 0$

Exercise:

Problem: Let $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$, where $\mathbf{F}$ is defined on $\{(x, y) \in \mathbb{R} | (x, y) \neq (0, 0)\}$. Find curl $\mathbf{F}$.

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

Exercise:

Problem: **[T]** $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\mathbf{j} + \mathbf{k}$

Solution:

$\text{curl}\,\mathbf{F} = \frac{2x}{x^2 + y^2}\mathbf{k}$

Exercise:

Problem: **[T]** $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

For the following exercises, find the divergence of **F** at the given point.

Exercise:

Problem: $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$

Solution:

$\text{div}\,\mathbf{F} = 0$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at $(1, 2, 3)$

Exercise:

Problem: $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at $(3, 2, 0)$

Solution:

$\text{div}\,\mathbf{F} = 2 - 2e^{-6}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at (1, 2, 1)

Exercise:

Problem: $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$ at (0, 0, 3)

Solution:

$\text{div}\,\mathbf{F} = 0$

For the following exercises, find the curl of **F** at the given point.

Exercise:

Problem: $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ at $(1, 2, 3)$

Solution:

$\text{curl}\,\mathbf{F} = \mathbf{j} - 3\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at (3, 2, 0)

Exercise:

Problem: $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at (1, 2, 1)

Solution:

$\text{curl}\,\mathbf{F} = 2\mathbf{j} - \mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$ at (0, 0, 3)

Exercise:

Problem: Let $\mathbf{F}(x, y, z) = \left(3x^2y + az\right)\mathbf{i} + x^3\mathbf{j} + \left(3x + 3z^2\right)\mathbf{k}$. For what value of *a* is $\mathbf{F}$ conservative?

Solution:

$a = 3$

Exercise:

Problem:

Given vector field $\mathbf{F}(x, y) = \frac{1}{x^2+y^2}(-y, x)$ on domain $D = \frac{\mathbb{R}^2}{\{(0,0)\}} = \left\{(x, y) \in \mathbb{R}^2 \middle| (x, y) \neq (0, 0)\right\}$, is $\mathbf{F}$ conservative?

Exercise:

Problem: Given vector field $\mathbf{F}(x, y) = \frac{1}{x^2+y^2}(x, y)$ on domain $D = \frac{\mathbb{R}^2}{\{(0,0)\}}$, is $\mathbf{F}$ conservative?

Solution:

$\mathbf{F}$ is conservative.

Exercise:

Problem:

Find the work done by force field $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving an object from *P*(0, 1) to *Q*(2, 0). Is the force field conservative?

Exercise:

Problem: Compute divergence $\mathbf{F} = (\sinh x)\mathbf{i} + (\cosh y)\mathbf{j} - xyz\mathbf{k}$.

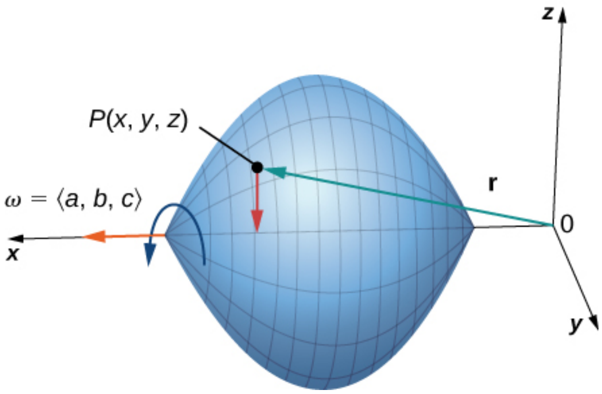
Solution:

$\text{div}\,\mathbf{F} = \cosh x + \sinh y - xy$

Exercise:

Problem: Compute curl $\mathbf{F} = (\sinh x)\mathbf{i} + (\cosh y)\mathbf{j} - xyz\mathbf{k}$.

For the following exercises, consider a rigid body that is rotating about the *x*-axis counterclockwise with constant angular velocity $\omega = \langle a, b, c\rangle$. If *P* is a point in the body located at $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the velocity at *P* is given by vector field $\mathbf{F} = \omega \times \mathbf{r}$.



Exercise:

Problem: Express $\mathbf{F}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ vectors.

Solution:

$(bz - cy)\mathbf{i}(cx - az)\mathbf{j} + (ay - bx)\mathbf{k}$

Exercise:

Problem: Find div $\mathbf{F}$.

Exercise:

Problem: Find curl $\mathbf{F}$

Solution:

curl $\mathbf{F} = 2\omega$

In the following exercises, suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$.

Exercise:

Problem: Does $\mathbf{F} + \mathbf{G}$ necessarily have zero divergence?

Exercise:

Problem: Does $\mathbf{F} \times \mathbf{G}$ necessarily have zero divergence?

Solution:

$\mathbf{F} \times \mathbf{G}$ does not have zero divergence.

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\mathbf{F} = -k\nabla T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$.

Exercise:

Problem: Compute the heat flow vector field.

Exercise:

Problem: Compute the divergence.

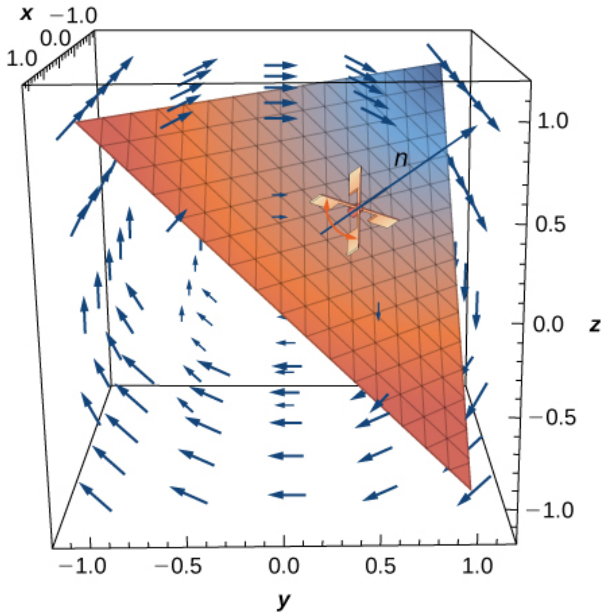
Solution:

$\nabla \cdot \mathbf{F} = -200k \left[1 + 2\left(x^2 + y^2 + z^2\right)\right] e^{-x^2+y^2+z^2}$

Exercise:

Problem:

[T] Consider rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



Glossary

curl

the curl of vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\nabla \times \mathbf{F}$, is the “determinant” of the matrix $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ and is given by the expression $(R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence

the divergence of a vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\nabla \times \mathbf{F}$, is $P_x + Q_y + R_z$; it measures the “outflowing-ness” of a vector field

Stokes’ Theorem

- Explain the meaning of Stokes’ theorem.
- Use Stokes’ theorem to evaluate a line integral.
- Use Stokes’ theorem to calculate a surface integral.
- Use Stokes’ theorem to calculate a curl.

In this section, we study Stokes’ theorem, a higher-dimensional generalization of Green’s theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green’s theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes’ theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes’ theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S.

In addition to allowing us to translate between line integrals and surface integrals, Stokes’ theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes’ theorem to derive Faraday’s law, an important result involving electric fields.

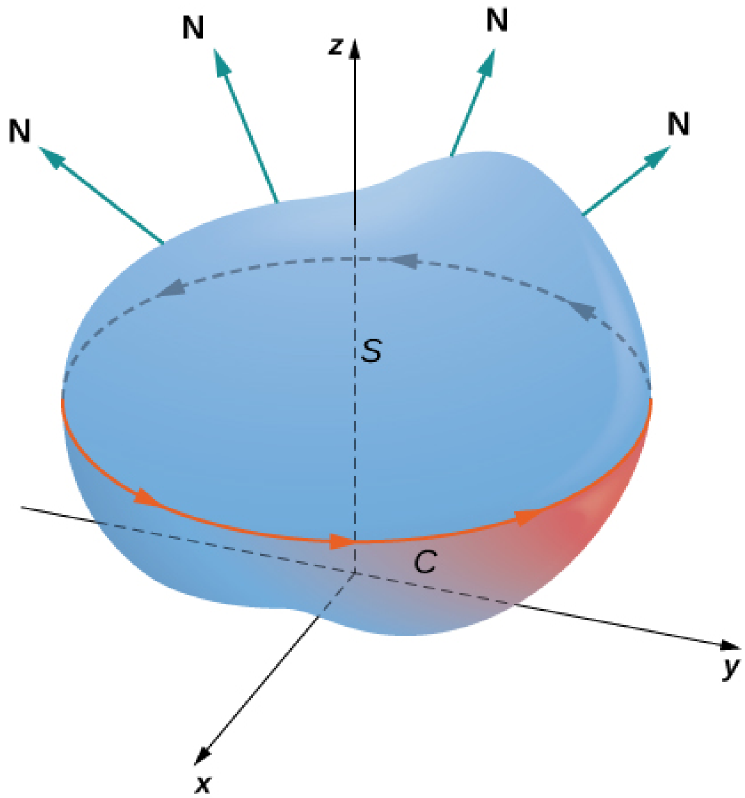
Stokes’ Theorem

Stokes’ theorem says we can calculate the flux of curl $\mathbf{F}$ across surface S by knowing information only about the values of $\mathbf{F}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\mathbf{F}$ along the boundary of surface S by translating to a double integral of the curl of $\mathbf{F}$ over S.

Let S be an oriented smooth surface with unit normal vector $\mathbf{N}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\mathbf{N}$, the surface is always on your left. With this definition in place, we can state Stokes’ theorem.

Note:
Stokes’ Theorem
Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation ([link]). If $\mathbf{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then
Equation:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}.$$



Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\mathbf{k}$ and surface integral $\iint\limits_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$ is actually the double integral $\iint\limits_S \text{curl}\, \mathbf{F} \cdot \mathbf{k} dA$. In this special case, Stokes' theorem gives $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{k} dA$. However, this is the flux form of Green's theorem,

which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface *S* is a portion of a graph of a function, and *S*, the boundary of *S*, and **F** are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let *S* be a surface and let *D* be a small piece of the surface so that *D* does not share any points with the boundary of *S*. We choose *D* to be small enough so that it can be approximated by an oriented square *E*. Let *D* inherit its orientation from *S*, and give *E* the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

Equation:

$$\int_{E_l+E_d+E_r+E_u} \mathbf{F}\cdot d\mathbf{r} = \iint_E \operatorname{curl}\mathbf{F}\cdot\mathbf{N}dS = \iint_E \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.$$

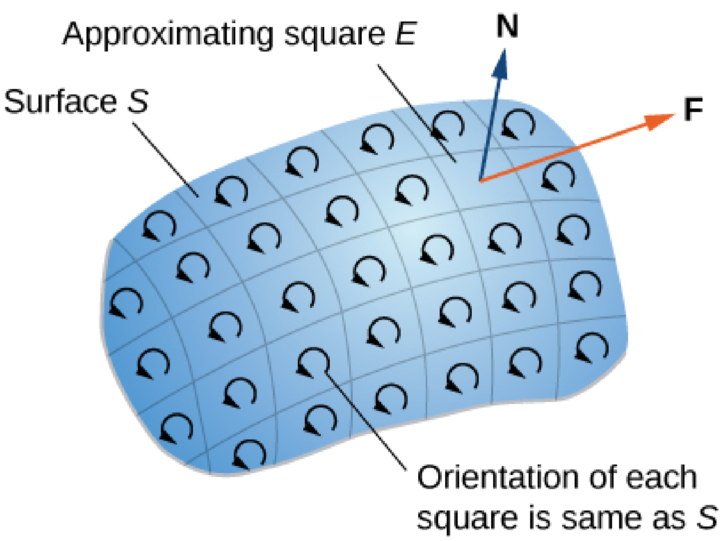
To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface ([link]). By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let *F* be an approximating square with an orientation inherited from *S* and with a right side E_l (so *F* is to the left of *E*). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of *E*, just oriented in the opposite direction. Therefore,

Equation:

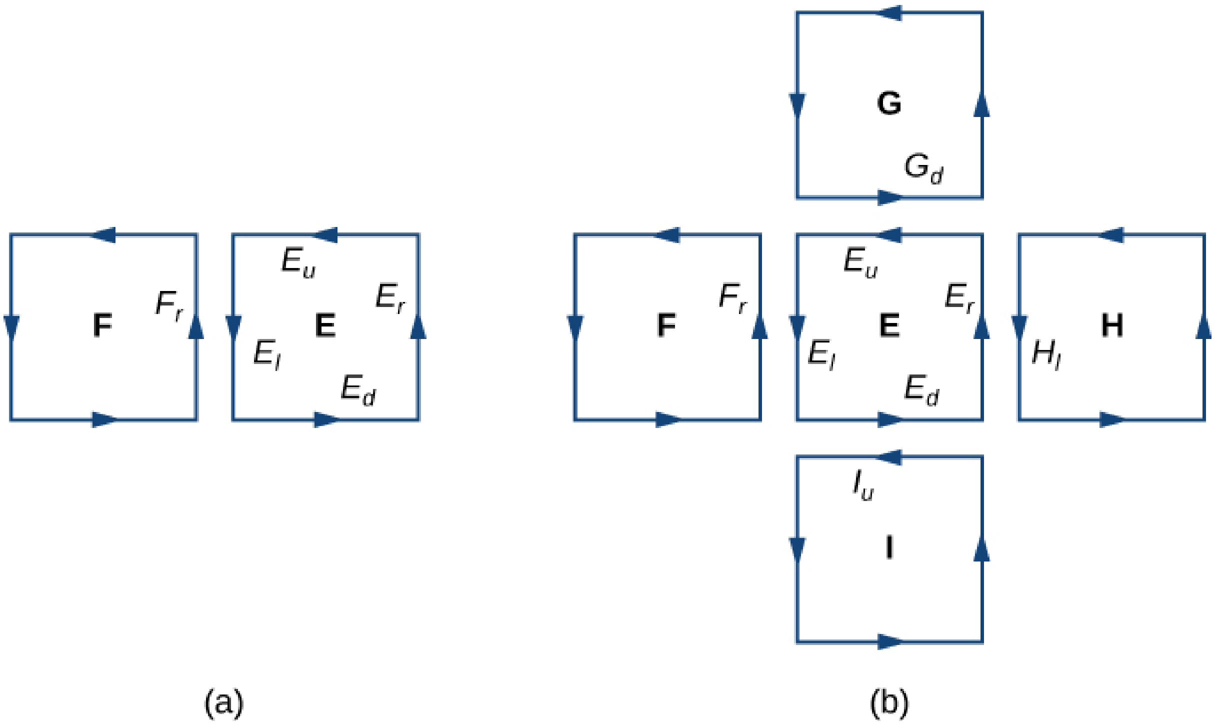
$$\int_{E_l} \mathbf{F}\cdot d\mathbf{r} = -\int_{F_r} \mathbf{F}\cdot d\mathbf{r}.$$

As we add up all the fluxes over all the squares approximating surface *S*, line integrals $\int_{E_l} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{F_r} \mathbf{F}\cdot d\mathbf{r}$ cancel each other out. The same goes for the line

integrals over the other three sides of *E*. These three line integrals cancel out with the line integral of the lower side of the square above *E*, the line integral over the left side of the square to the right of *E*, and the line integral over the upper side of the square below *E* ([link]). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of *S*. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of *S*. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

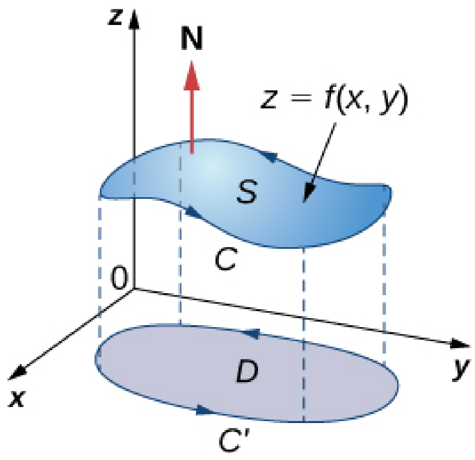


Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.



(a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area ([link]). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ be a vector field with component functions that have continuous partial derivatives.



D is the “shadow,” or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S : x = x, y = y, z = g\left(x, y\right)$. The tangent vectors are $\mathbf{t}_x = \langle 1, 0, g_x \rangle$ and $\mathbf{t}_y = \langle 0, 1, g_y \rangle$, and therefore, $\mathbf{t}_x \cdot \mathbf{t}_y = \langle -g_x, -g_y, 1 \rangle$. By [link],

Equation:

$$\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-\left(R_y - Q_z\right)z_x - \left(P_z - R_x\right)z_y + \left(Q_x - P_y\right)\right]dA,$$

where the partial derivatives are all evaluated at $\left(x, y, g\left(x, y\right)\right)$, making the integrand depend on x and y only. Suppose $\langle x\left(t\right), y\left(t\right)\rangle, a \leq t \leq b$ is a parameterization of C'. Then, a parameterization of C is $\langle x\left(t\right), y\left(t\right), g\left(x\left(t\right), y\left(t\right)\right)\rangle, a \leq t \leq b$. Armed with these parameterizations, the Chain rule, and Green’s theorem, and keeping in mind that P, Q, and R are all functions of x and y, we can evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$:

Equation:

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \left(Px'(t)+Qy'(t)+Rz'(t)\right)dt \\
&= \int_a^b \left[Px'(t)+Qy'(t)+R\left(\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt \\
&= \int_a^b \left[\left(P+R\frac{\partial z}{\partial x}\right)x'(t)+\left(Q+R\frac{\partial z}{\partial y}\right)y'(t)\right]dt \\
&= \int\limits_{C'} \left(P+R\frac{\partial z}{\partial x}\right)dx+\left(Q+R\frac{\partial z}{\partial y}\right)dy \\
&= \iint\limits_D \left[\frac{\partial}{\partial x}\left(Q+R\frac{\partial z}{\partial y}\right)-\frac{\partial}{\partial y}\left(P+R\frac{\partial z}{\partial x}\right)\right]dA \\
&= \iint\limits_D \begin{pmatrix} \left(\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial R}{\partial x}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+R\frac{\partial^2 z}{\partial x\partial y}\right) \\ -\left(\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+R\frac{\partial^2 z}{\partial y\partial x}\right) \end{pmatrix} dA.
\end{aligned}$$

By Clairaut's theorem, $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$. Therefore, four of the terms disappear from this double integral, and we are left with

Equation:

$$\iint\limits_D \left[-\left(R_y-Q_z\right)z_x-\left(P_z-R_x\right)z_y+\left(Q_x-P_y\right)\right]dA,$$

which equals $\iint\limits_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

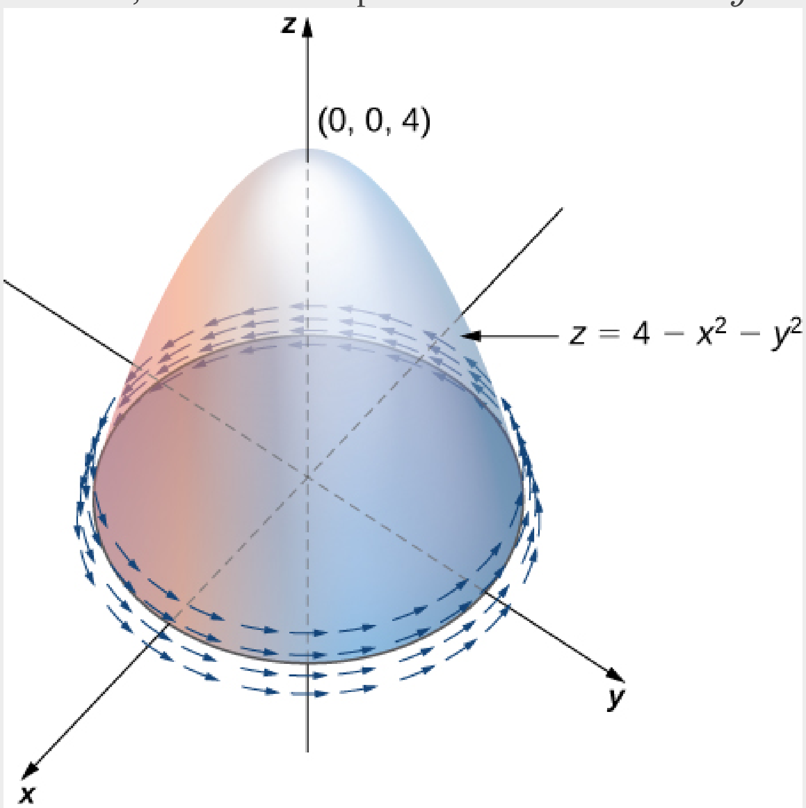
□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field **F** is conservative. If **F** is conservative, the curl of **F** is zero, so $\iint\limits_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$. Since the boundary of S is a closed curve, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is also zero.

Example:
Exercise:

Problem:
Verifying Stokes’ Theorem for a Specific Case

Verify that Stokes’ theorem is true for vector field $\mathbf{F}\left(x, y, z\right) = \left\langle y, 2z, x^2 \right\rangle$ and surface *S*, where *S* is the paraboloid $z = 4 - x^2 - y^2$.



Verifying Stokes’ theorem for a hemisphere in a vector field.

Solution:

As a surface integral, you have$g\left(x, y\right) = 4 - x^2 - y^2, g_x = -2y$ and
Equation:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix} = \langle -2, -2x, -1 \rangle.$$

By [link],

Equation:

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_D \text{curl } \mathbf{F}\left(\mathbf{r}\left(\phi,\theta\right)\right) \cdot (\mathbf{t}_\phi \times \mathbf{t}_\theta) dA \\ &= \iint_D \langle -2, -2x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle dA \\ &= \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-4x - 4xy - 1) dy dx \\ &= \int_{-2}^{2} \left(-8x\sqrt{4-x^2} - 2\sqrt{4-x^2}\right) dx \\ &= -4\pi \end{aligned}$$

As a line integral, you can parameterize C by $\mathbf{r}\left(t\right) = \langle 2\cos t, 2\sin t, 0 \rangle$ $0 \le t \le 2\pi$. By [link],

Equation:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle 2\sin t, 0, 4\cos^2 t \right\rangle \cdot \left\langle -2\sin t, 2\cos t, 0 \right\rangle dt \\ &= \int_0^{2\pi} -4\sin^2 t dt = -4\pi \end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

Note:
Exercise:

Problem:

Verify that Stokes' theorem is true for vector field $\mathbf{F}(x, y, z) = \langle y, x, -z \rangle$ and surface *S*, where *S* is the upwardly oriented portion of the graph of $f(x, y) = x^2 y$ over a triangle in the *xy*-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

Solution:

Both integrals give $-\frac{136}{45}$.

Hint

Calculate the double integral and line integral separately.

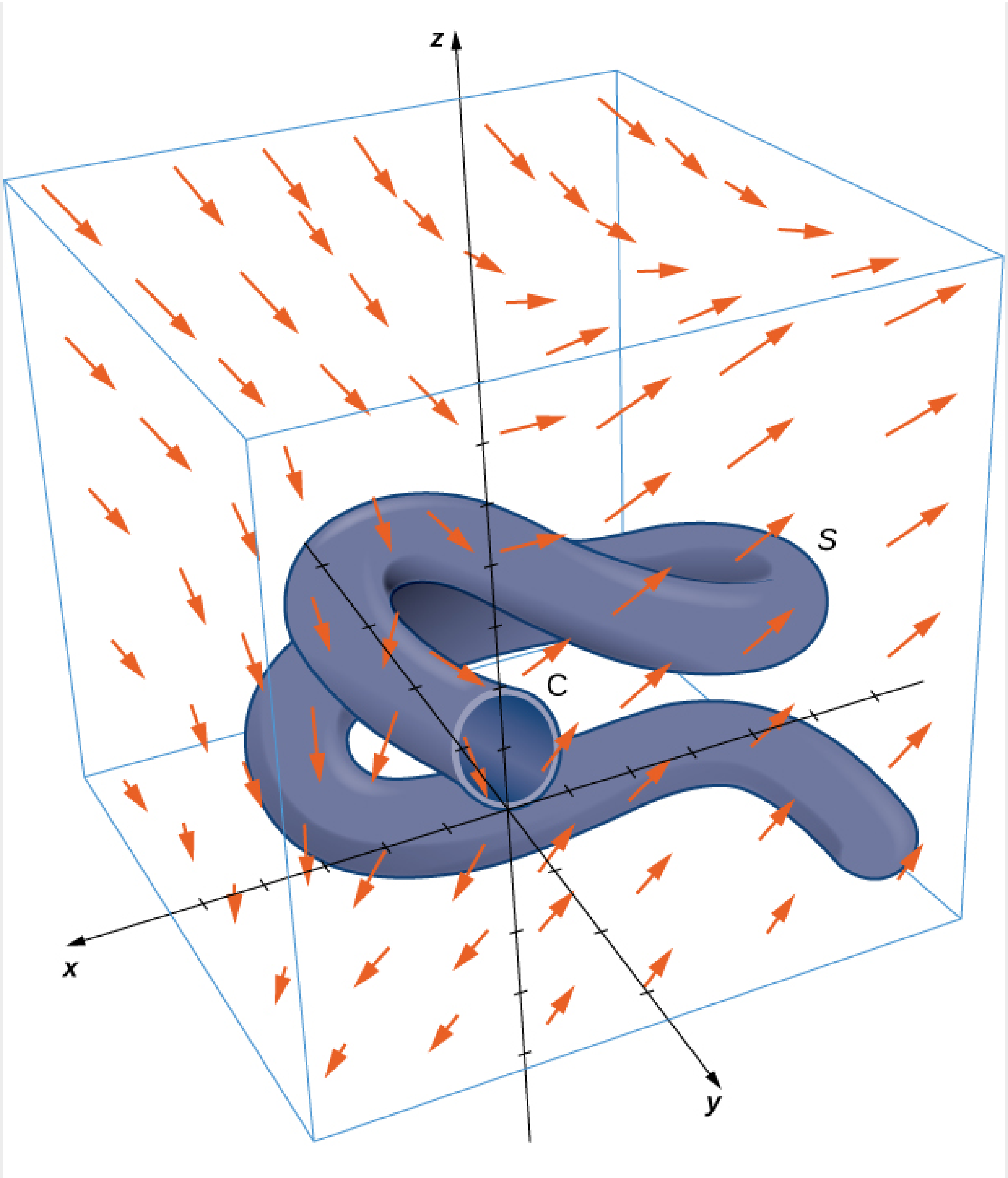
Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface *S* to a line integral around the boundary of *S*. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example:
Exercise:

Problem:
Calculating a Surface Integral

Calculate surface integral $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where *S* is the surface, oriented outward, in [link] and $\mathbf{F} = \langle z, 2xy, x + y \rangle$.



A complicated surface in a vector field.

Solution:

Note that to calculate $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ without using Stokes' theorem, we would need to use [link]. Use of this equation requires a parameterization of S.

Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\mathbf{F}$ is $\langle 1, 1, 2y \rangle$. By Stokes' theorem,

Equation:

$$\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C has parameterization $\mathbf{r}\,(t) = \langle -\sin t, 0, 1 - \cos t \rangle, 0 \leq t < 2\pi$. By [link],

Equation:

$$\begin{aligned}\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \langle 1 - \cos t\,, 0, -\sin t \rangle \cdot \langle -\cos t, 0, \sin t \rangle dt \\ &= \int_0^{2\pi} \left(-\cos t + \cos^2 t - \sin^2 t \right) dt \\ &= \left[-\sin t + \tfrac{1}{2} \sin\,(2t) \right]_0^{2\pi} \\ &= (-\sin(2\pi) + \tfrac{1}{2}\sin(4\pi)) - (-\sin 0 + \tfrac{1}{2}\sin 0) \\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In [link], we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

Equation:

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Therefore, if $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is difficult to calculate but $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In [link], we could have calculated $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ by calculating $\iint_{S'} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where S' is the disk enclosed by boundary curve C (a much more simple surface with which to work).

[link] shows that flux integrals of curl vector fields are **surface independent** in the same way that line integrals of gradient fields are path independent. Recall that if **F** is a two-dimensional conservative vector field defined on a simply connected domain, f is a potential function for **F**, and C is a curve in the domain of **F**, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$. In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that **G** is a three-dimensional vector field that can be written as the curl of another vector field **F** (so that **F** is like a “potential field” of **G**). By [link],

Equation:

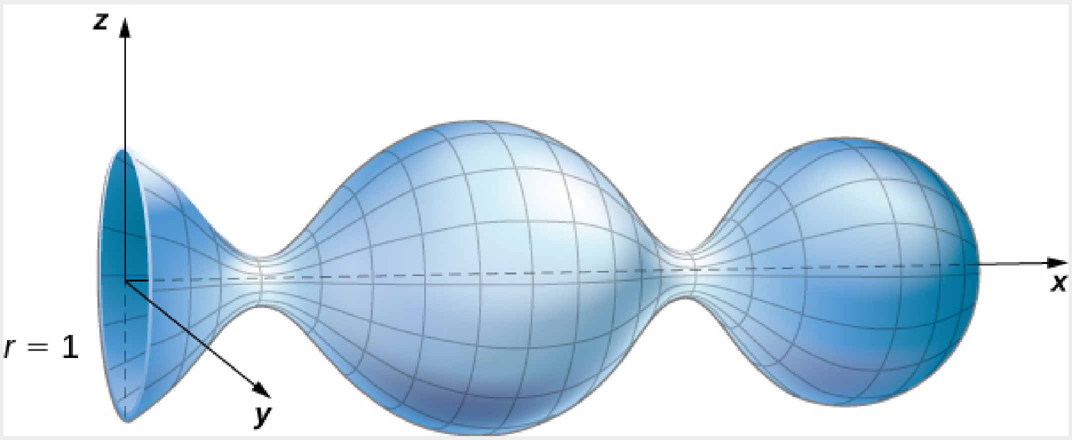
$$\iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S'} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \mathbf{G} \cdot d\mathbf{S}.$$

Therefore, the flux integral of **G** does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

Note:
Exercise:

Problem:

Use Stokes' theorem to calculate surface integral $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle z, x, y \rangle$ and S is the surface as shown in the following figure. The boundary curve, C, is oriented clockwise.



Solution:

$-\pi$

Hint

Parameterize the boundary of S and translate to a line integral.

Example:
Exercise:

Problem:
Calculating a Line Integral

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, x^2 + y^2 + z^2, yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, 0, -1)$, and $(2, 1, -2)$.

Solution:

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let *S* denote the surface of the parallelogram. Note that *S* is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$, and $(2, 1)$ in the *xy*-plane. Therefore, a parameterization of *S* is $\langle x, y, 1 - x - y \rangle, 0 \le x \le 2, 0 \le y \le 1$. The curl of **F** is $-\langle z, 0, x \rangle$, and Stokes' theorem and [link] give

Equation:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} \\ &= \int_0^2 \int_0^1 \text{curl}\, \mathbf{F}\,(x, y) \cdot (\mathbf{t}_x \times \mathbf{t}_y) dy dx \\ &= \int_0^2 \int_0^1 \langle -(1 - x - y), 0, x \rangle \cdot (\langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle) dy dx \\ &= \int_0^2 \int_0^1 \langle x + y - 1, 0, x \rangle \cdot \langle 1, 1, 1 \rangle dy dx \\ & \int_0^2 \int_0^1 2x + y - 1 dy dx \\ &= 3. \end{aligned}$$

Note:
Exercise:

Problem:

Use Stokes' theorem to calculate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z, x, y \rangle$ and *C* is oriented clockwise and is the boundary of a triangle with vertices $(0, 0, 1)$, $(3, 0, -2)$, and $(0, 1, 2)$.

Solution:

$\frac{3}{2}$

Hint

This triangle lies in plane $z = 1 - x + y$.

Interpretation of Curl

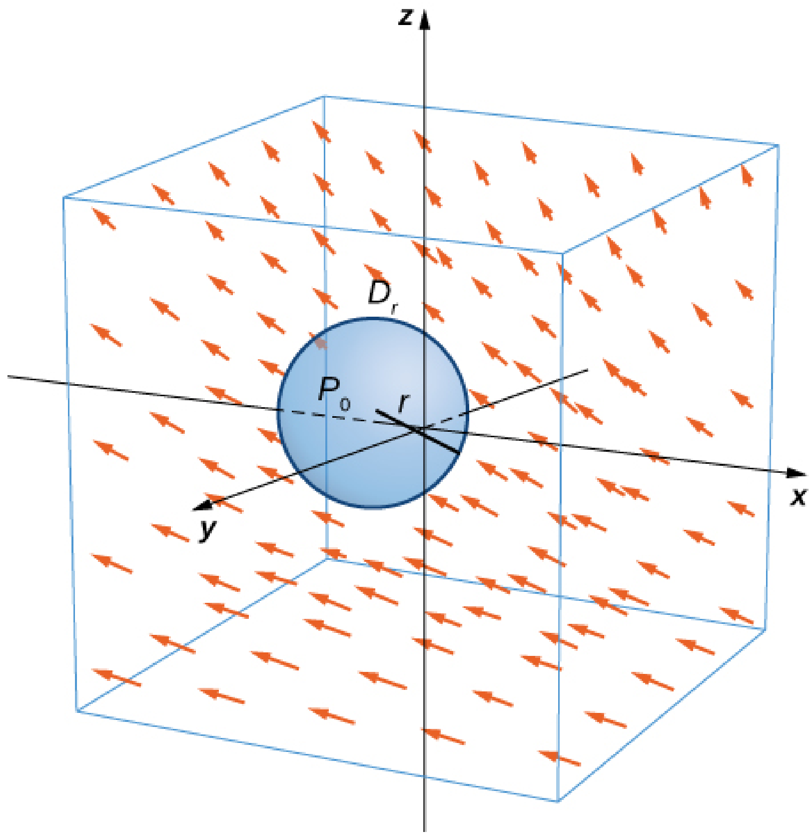
In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and **F** is a vector field defined on C, then the circulation of **F** around C is line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. If **F** represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let **F** be a continuous vector field and let D_r be a small disk of radius r with center P_0 ([link]). If D_r is small enough, then $(\text{curl}\,\mathbf{F})(P) \approx (\text{curl}\,\mathbf{F})(P_0)$ for all points P in D_r because the curl is continuous. Let C_r be the boundary circle of D_r. By Stokes' theorem,

Equation:

$$\int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_r} \text{curl}\,\mathbf{F} \cdot \mathbf{N} dS \approx \iint_{D_r} (\text{curl}\,\mathbf{F})\,(P_0) \cdot \mathbf{N}\,(P_0) dS.$$



Disk D_r is a small disk in a continuous vector field.

The quantity $(\text{curl}\,\mathbf{F})\,(P_0)\cdot\mathbf{N}\,(P_0)$ is constant, and therefore
Equation:

$$\iint_{D_r}(\text{curl}\,\mathbf{F})\,(P_0)\cdot\mathbf{N}\,(P_0)dS=\pi r^2\left[(\text{curl}\,\mathbf{F})\,(P_0)\cdot\mathbf{N}\,(P_0)\right].$$

Thus
Equation:

$$\int_{C_r}\mathbf{F}\cdot d\mathbf{r}\approx\pi r^2\left[(\text{curl}\,\mathbf{F})\,(P_0)\cdot\mathbf{N}\,(P_0)\right],$$

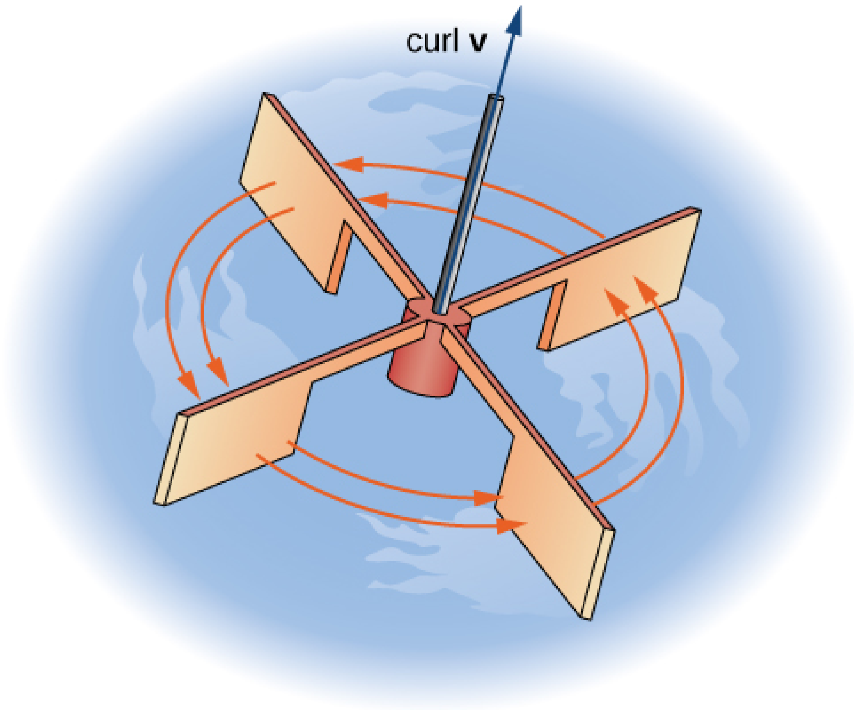
and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes’ theorem implies that

Equation:

$$(\text{curl}\,\mathbf{F})\,(P_0)\cdot\mathbf{N}\,(P_0)=\lim_{r\to 0^+}\frac{1}{\pi r^2}\int_{C_r}\mathbf{F}\cdot d\mathbf{r}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if **F** is the velocity field of a fluid, then circulation $\oint_{C_r}\mathbf{F}\cdot d\mathbf{r}=\oint_{C_r}\mathbf{F}\cdot\mathbf{T}ds$ is a measure of the tendency of the fluid to move around C_r. The reason for this is that $\mathbf{F}\cdot\mathbf{T}$ is a component of **F** in the direction of **T**, and the closer the direction of **F** is to **T**, the larger the value of $\mathbf{F}\cdot\mathbf{T}$ (remember that if **a** and **b** are vectors and **b** is fixed, then the dot product $\mathbf{a}\cdot\mathbf{b}$ is maximal when **a** points in the same direction as **b**). Therefore, if **F** is the velocity field of a fluid, then $\text{curl}\,\mathbf{F}\cdot\mathbf{N}$ is a measure of how the fluid rotates about axis **N**. The effect of the curl is largest about the axis that points in the direction of **N**, because in this case $\text{curl}\,\mathbf{F}\cdot\mathbf{N}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 ([link]). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl**F**. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector **N**, and Stokes’ theorem justifies this interpretation.



To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time *t*, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times *t*. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$ is in a magnetic field $\mathbf{B}(t)$ that can also change over time. In other words, **B** has the form

Equation:

$$\mathbf{B}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle,$$

where *P*, *Q*, and *R* can all vary continuously over time. We can produce current along the wire by changing field $\mathbf{B}(t)$ (this is a consequence of Ampere's law). Flux

$\phi(t) = \iint_{D(t)} \mathbf{B}(t) \cdot d\mathbf{S}$ creates electric field $\mathbf{E}(t)$ that does work. The integral form of Faraday's law states that

Equation:

$$\text{Work} = \int_{C(t)} \mathbf{E}(t) \cdot d\mathbf{r} = -\frac{\partial \phi}{\partial t}.$$

In other words, the work done by $\mathbf{E}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

Equation:

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

Equation:

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \mathbf{E}(t) \cdot d\mathbf{r} = \iint_{D(t)} \operatorname{curl} \mathbf{E}(t) \cdot d\mathbf{S}.$$

Since $\phi(t) = \iint_{D(t)} \mathbf{B}(t) \cdot d\mathbf{S}$, then as long as the integration of the surface does not vary with time we also have

Equation:

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

Therefore,

Equation:

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \operatorname{curl} \mathbf{E} \cdot d\mathbf{S}.$$

To derive the differential form of Faraday's law, we would like to conclude that $\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. In general, the equation

Equation:

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \operatorname{curl} \mathbf{E} \cdot d\mathbf{S}$$

is not enough to conclude that $\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x dx$ and $\int_0^1 f(x) dx$, where

Equation:

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & 1/2 \leq x \leq 1. \end{cases}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x dx = \int_0^1 f(x) dx$. However, $x \neq f(x)$. Analogously, with our equation $\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \operatorname{curl} \mathbf{E} \cdot d\mathbf{S}$, we cannot simply conclude that $\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ just because their integrals are equal. However, in our context, equation $\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \operatorname{curl} \mathbf{E} \cdot d\mathbf{S}$ is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If **F** and **G** are three-dimensional vector fields such that $\iint_s \mathbf{F} \cdot d\mathbf{S} = \iint_s \mathbf{G} \cdot d\mathbf{S}$ for any surface *S*, then it is possible to show that $\mathbf{F} = \mathbf{G}$ by shrinking the area of *S* to zero by taking a limit (the smaller the area of *S*, the closer the value of $\iint_s \mathbf{F} \cdot d\mathbf{S}$ to the value of **F** at a point inside *S*). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

Equation:

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example:
Exercise:

Problem:
Using Faraday's Law

Calculate the curl of electric field $\mathbf{E}$ if the corresponding magnetic field is constant field $\mathbf{B}\left(t\right)=\langle 1,-4,2\rangle$.

Solution:

Since the magnetic field does not change with respect to time, $-\frac{\partial \mathbf{B}}{\partial t}=\mathbf{0}$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

Note:
Exercise:

Problem:

Calculate the curl of electric field $\mathbf{E}$ if the corresponding magnetic field is $\mathbf{B}(t)=\langle tx,ty,-2tz\rangle,0\le t<\infty$.

Solution:

curl $\mathbf{E}=\langle x,y,-2z\rangle$

Hint

Use the differential form of Faraday's law.
Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary *C*.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**
  $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

For the following exercises, without using Stokes' theorem, calculate directly both the flux of $\operatorname{curl} \mathbf{F} \cdot \mathbf{N}$ over the given surface and the circulation integral around its boundary, assuming all boundaries are oriented clockwise as viewed from above.

Exercise:

Problem:

$\mathbf{F}(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$; *S* is the first-octant portion of plane $x + y + z = 1$.

Exercise:

Problem: $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$; *S* is hemisphere $z = \left(a^2 - x^2 - y^2\right)^{1/2}$.

Solution:

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})dS = \pi a^2$$

Exercise:

Problem:

$\mathbf{F}(x, y, z) = y^2\mathbf{i} + 2x\mathbf{j} + 5\mathbf{k}$; S is hemisphere $z = \left(4 - x^2 - y^2\right)^{1/2}$.

Exercise:

Problem:

$\mathbf{F}(x, y, z) = z\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$; S is upper hemisphere $z = \sqrt{9 - x^2 - y^2}$.

Solution:

$$\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})dS = 18\pi$$

Exercise:

Problem:

$\mathbf{F}(x, y, z) = (x + 2z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k}$; S is a triangular region with vertices (3, 0, 0), (0, 3/2, 0), and (0, 0, 3).

Exercise:

Problem:

$\mathbf{F}(x, y, z) = 2y\mathbf{i} - 6z\mathbf{j} + 3x\mathbf{k}$; S is a portion of paraboloid $z = 4 - x^2 - y^2$ and is above the *xy*-plane.

Solution:

$$\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})dS = -8\pi$$

For the following exercises, use Stokes' theorem to evaluate $\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})dS$ for the vector fields and surface.

Exercise:

Problem:

$\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j}$ and S is the surface of the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$, except for the face where $z = 0$, and using the outward unit normal vector.

Exercise:

Problem:

$\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$; and C is the intersection of paraboloid $z = x^2 + y^2$ and plane $z = y$, and using the outward normal vector.

Solution:

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N}) dS = 0$$

Exercise:

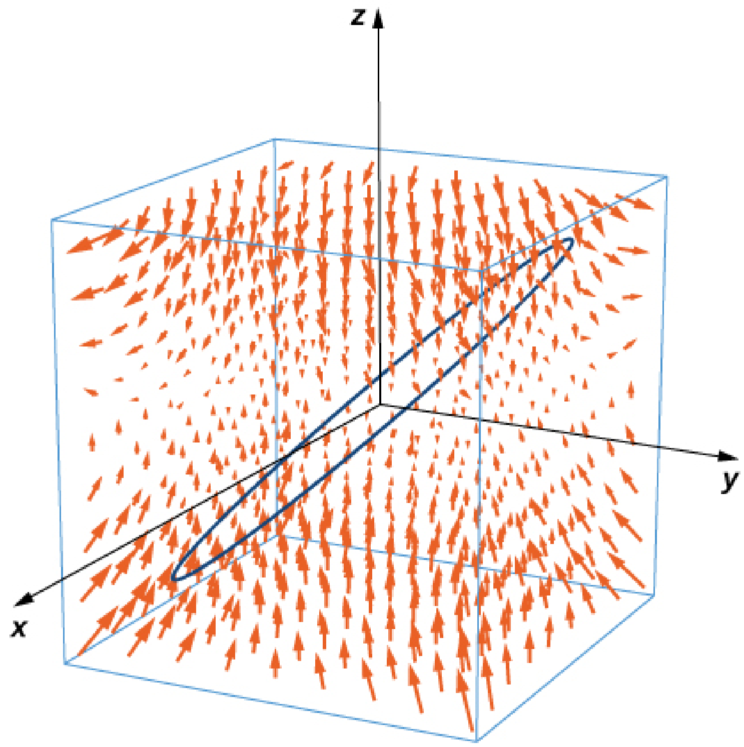
Problem:

$\mathbf{F}(x, y, z) = 4y\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ and C is the intersection of sphere $x^2 + y^2 + z^2 = 4$ with plane $z = 0$, and using the outward normal vector

Exercise:

Problem:

Use Stokes' theorem to evaluate $\int\limits_C \left[2xy^2zdx + 2x^2yzdy + \left(x^2y^2 - 2z\right)dz\right]$, where C is the curve given by $x = \cos t, y = \sin t, z = \sin t, 0 \leq t \leq 2\pi$, traversed in the direction of increasing t.



Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{S} = 0$$

Exercise:

Problem:

[T] Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int\limits_C (ydx + zdy + xdz)$, where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2\,(x + y)$, traversed counterclockwise viewed from the origin.

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to approximate line integral $\int_C (3ydx + 2zdy - 5xdz)$, where C is the intersection of the *xy*-plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive *z*-axis toward the *xy*-plane.

Solution:

$$\int_C \mathbf{F} \cdot d\mathbf{S} = -9.4248$$

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to approximate line integral $\int_C [(1 + y)zdx + (1 + z)xdy + (1 + x)ydz]$, where C is a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise.

Exercise:

Problem:

Use Stokes' theorem to evaluate $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = e^{xy}\cos z\mathbf{i} + x^2 z\mathbf{j} + xy\mathbf{k}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive *x*-axis.

Solution:

$$\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = 0$$

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to evaluate $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS$, where $\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + xy^2\mathbf{j} + z^3\mathbf{k}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to evaluate $\iint\limits_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where

$\mathbf{F}(x, y, z) = \left(\sin(y + z) - yx^2 - \frac{y^3}{3}\right)\mathbf{i} + x\cos(y + z)\mathbf{j} + \cos(2y)\mathbf{k}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

Solution:

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 2.6667$$

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to evaluate $\iint\limits_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where

$\mathbf{F}(x, y, z) = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$, and S is oriented upward.

Exercise:

Problem:

Use Stokes' theorem to evaluate $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS$, where $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and S is a triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) with counterclockwise orientation.

Solution:

$$\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})dS = -\frac{1}{6}$$

Exercise:

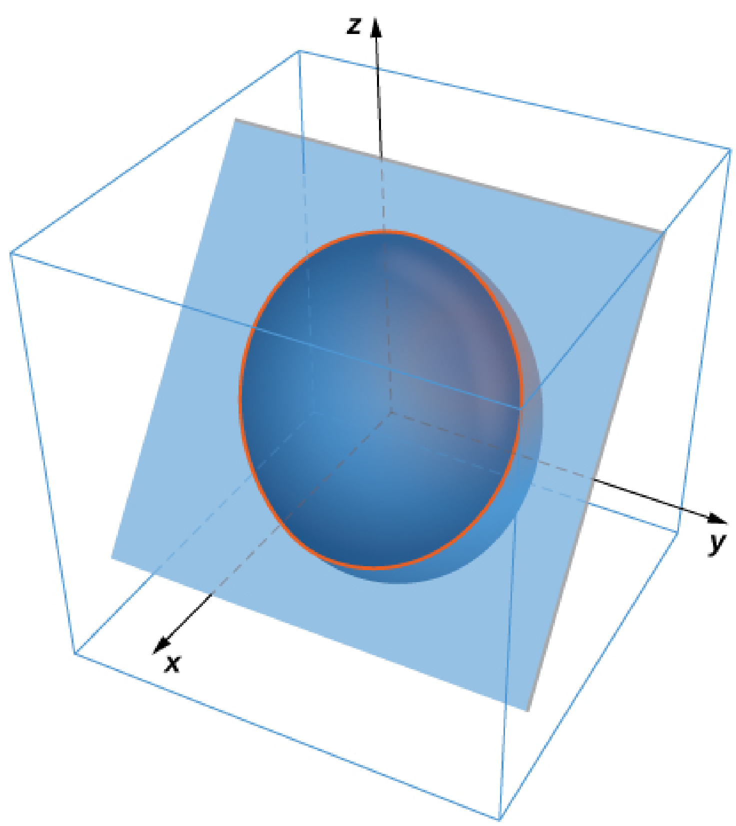
Problem:

Use Stokes' theorem to evaluate line integral $\int_C (zdx + xdy + ydz)$, where C is a triangle with vertices (3, 0, 0), (0, 0, 2), and (0, 6, 0) traversed in the given order.

Exercise:

Problem:

Use Stokes' theorem to evaluate $\int_C \left(\frac{1}{2}y^2dx + zdy + xdz\right)$, where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.



Solution:

$$\int_C \left(\frac{1}{2}y^2dx + zdy + xdz\right) = -\frac{\pi}{4}$$

Exercise:

Problem:

Use Stokes' theorem to evaluate $\iint_S (\text{curl}\,\mathbf{F}\cdot\mathbf{N})dS$, where $\mathbf{F}(x,y,z) = x\mathbf{i} + y^2\mathbf{j} + ze^{xy}\mathbf{k}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0,$ oriented counterclockwise.

Exercise:

Problem:

Use Stokes' theorem for vector field $\mathbf{F}(x,y,z) = z\mathbf{i} + 3x\mathbf{j} + 2z\mathbf{k}$ where S is surface $z = 1 - x^2 - 2y^2, z \geq 0,$ C is boundary circle $x^2 + y^2 = 1,$ and S is oriented in the positive *z*-direction.

Solution:

$$\iint_S (\text{curl}\,\mathbf{F}\cdot\mathbf{N})dS = -3\pi$$

Exercise:

Problem:

Use Stokes' theorem for vector field $\mathbf{F}(x,y,z) = -\frac{3}{2}y^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), traversed counterclockwise as viewed from above.

Exercise:

Problem:

A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the *xy*-plane. Let c be a constant and let $\mathbf{R}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Use Stokes' theorem to evaluate $\int_C (c\mathbf{k} \times \mathbf{R})\cdot d\mathbf{S}$.

Solution:

$$\int_C (c\mathbf{k}\ \times\ \mathbf{R}) \cdot d\mathbf{S} = 2\pi c$$

Exercise:

Problem:

Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \leq x \leq 2$ and $0 \leq y \leq 1$, oriented with upward facing normal. Define

Equation:

$\mathbf{F}(x, y, z) = \left[\sin\left(x^3\right) + xz\right]\mathbf{i} + (x - yz)\mathbf{j} + \cos\left(z^4\right)\mathbf{k}$ and evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{S}$.

Exercise:

Problem:

Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \geq 0$, oriented upward. Let $\mathbf{F}(x, y, z) = x^2 e^{yz}\mathbf{i} + y^2 e^{xz}\mathbf{j} + z^2 e^{xy}\mathbf{k}$ be a vector field. Use Stokes' theorem to evaluate $\displaystyle\iint_S \text{curl}\ \mathbf{F} \cdot d\mathbf{S}$.

Solution:

$$\iint_S \text{curl}\ \mathbf{F} \cdot d\mathbf{S} = 0$$

Exercise:

Problem:

Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + \left(e^{z^2} + y\right)\mathbf{j} + (x + y)\mathbf{k}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \leq 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral $\displaystyle\iint_S \text{curl}\ \mathbf{F} \cdot d\mathbf{S}$.

Exercise:

Problem:

Use Stokes' theorem to evaluate $\oint \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and C is a triangle with vertices (0, 0, 0), (2, 0, 0) and $(0, -2, 2)$ oriented counterclockwise when viewed from above.

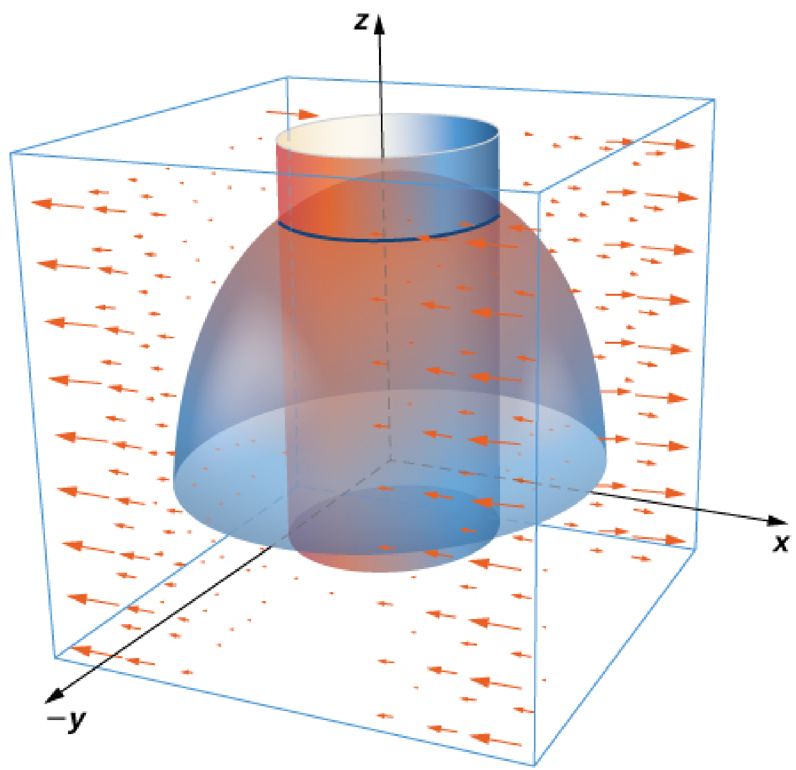
Solution:

$$\oint \mathbf{F} \cdot d\mathbf{S} = -4$$

Exercise:

Problem:

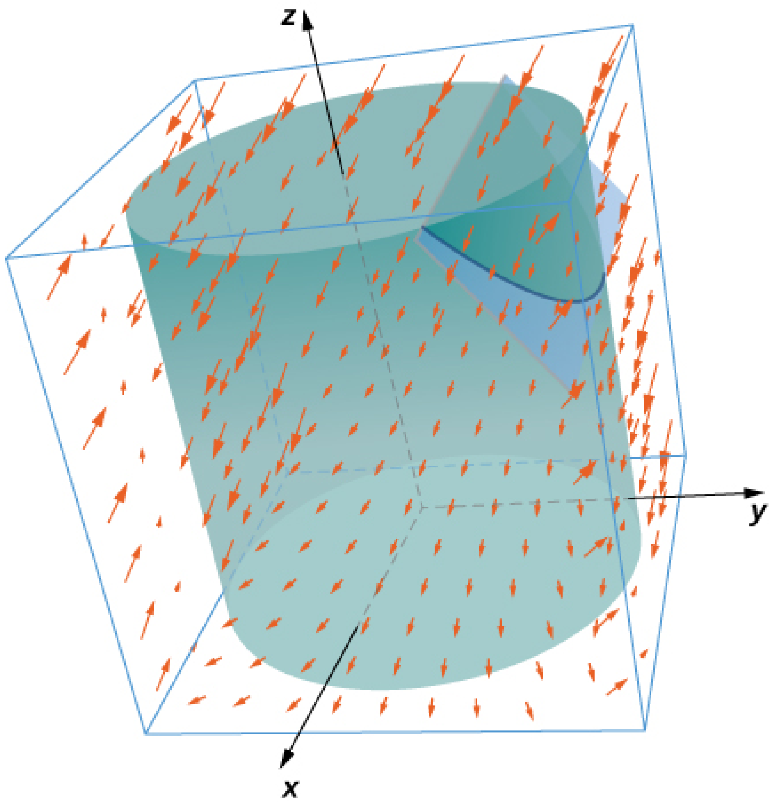
Use the surface integral in Stokes' theorem to calculate the circulation of field $\mathbf{F}$, $\mathbf{F}(x, y, z) = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, oriented counterclockwise when viewed from above.



Exercise:

Problem:

Use Stokes' theorem to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \mathbf{i} + xy^2\mathbf{j} + xy^2\mathbf{k}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.



Solution:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

Exercise:

Problem:

Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \geq 0, y \geq 0, z \geq 0$.

Exercise:

Problem:

Let $\mathbf{F}(x,y,z) = xy\mathbf{i} + 2z\mathbf{j} - 2y\mathbf{k}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\mathbf{F}$ over C using Stokes' theorem.

Solution:

$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = -36\pi$$

Exercise:

Problem:

[T] Use a CAS and let $\mathbf{F}(x,y,z) = xy^2\mathbf{i} + (yz - x)\mathbf{j} + e^{yxz}\mathbf{k}$. Use Stokes' theorem to compute the surface integral of curl $\mathbf{F}$ over surface S with inward orientation consisting of cube $[0,1] \times [0,1] \times [0,1]$ with the right side missing.

Exercise:

Problem:

Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\mathbf{F}$ be a vector field with component functions that have continuous partial derivatives.

Solution:

$$\iint_S \text{curl}\,\mathbf{F} \cdot \mathbf{N} = 0$$

Exercise:

Problem:

Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\mathbf{F}(x,y,z) = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$.

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F}(x, y, z) = (3z - \sin x)\mathbf{i} + \left(x^2 + e^y\right)\mathbf{j} + \left(y^3 - \cos z\right)\mathbf{k}$, where C is the curve given by $x = \cos t, y = \sin t, z = 1; 0 \leq t \leq 2\pi$.

Solution:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Exercise:

Problem:

[T] Use a CAS and Stokes' theorem to evaluate $\mathbf{F}(x, y, z) = 2y\mathbf{i} + e^z\mathbf{j} - \arctan x\mathbf{k}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the *xy*-plane oriented counterclockwise.

Exercise:

Problem:

[T] Use a CAS to evaluate $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$ and S is the surface parametrically by $\mathbf{r}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + \left(4 - r^2\right)\mathbf{k}$ $(0 \leq \theta \leq 2\pi, 0 \leq r \leq 3)$.

Solution:

$$\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = 84.8230$$

Exercise:

Problem:

Let S be paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\mathbf{F} = \langle x - y, y + z, z - x\rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS$ have its maximum value?

For the following application exercises, the goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, where $\mathbf{F} = \langle xz, -xz, xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1,$ where $z \geq 0$.

Exercise:

Problem:

Evaluate a surface integral over a more convenient surface to find the value of A.

Solution:

$$A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$$

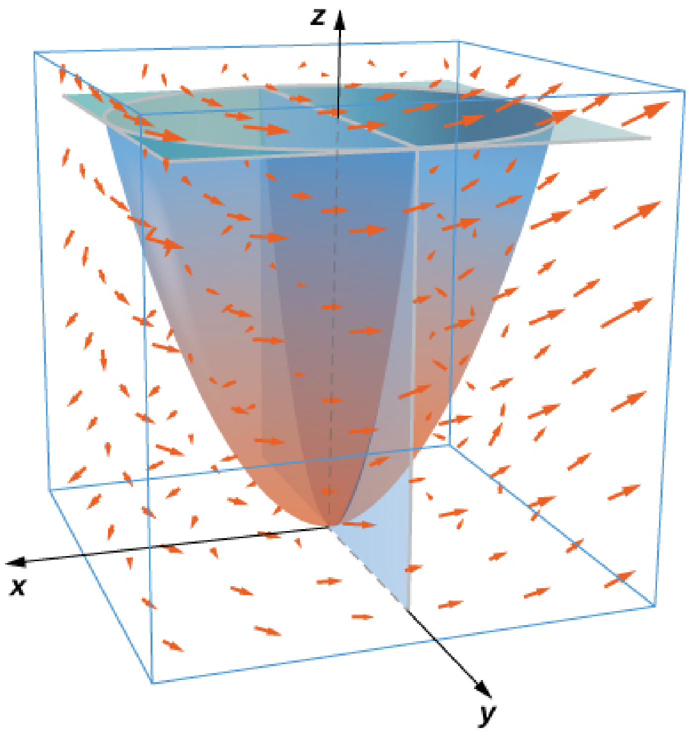
Exercise:

Problem: Evaluate A using a line integral.

Exercise:

Problem:

Take paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \geq 0$, including the planar surface in the *xz*-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y, 2x + z, 2y + x \rangle$. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.



Solution:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 2\pi$$

For the following exercises, let S be the disk enclosed by curve

$C : \mathbf{r}(t) = \langle \cos\varphi \cos t, \sin t, \sin\varphi \cos t \rangle$, for $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

Exercise:

Problem: What is the length of C in terms of φ?

Exercise:

Problem:

What is the circulation of C of vector field $\mathbf{F} = \langle -y, -z, x \rangle$ as a function of φ?

Solution:

$$C = \pi (\cos\varphi - \sin\varphi)$$

Exercise:

Problem: For what value of φ is the circulation a maximum?

Exercise:

Problem:

Circle C in plane $x + y + z = 8$ has radius 4 and center (2, 3, 3). Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $F = \langle 0, -z, 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Solution:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 48\pi$$

Exercise:

Problem:

Velocity field $\mathbf{v} = \left\langle 0, 1 - x^2, 0 \right\rangle$, for $|x| \leq 1$ and $|z| \leq 1$, represents a horizontal flow in the *y*-direction. Compute the curl of **v** in a clockwise rotation.

Exercise:

Problem:

Evaluate integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, where $\mathbf{F} = -xz\mathbf{i} + yz\mathbf{j} + xye^z\mathbf{k}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and **n** points in the positive *z*-direction on *S*.

Solution:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} = 0$$

For the following exercises, use Stokes’ theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

Exercise:

Problem: $\mathbf{F} = \nabla \left(x \sin ye^z \right)$

Exercise:

Problem: $\mathbf{F} = \langle y^2 z^3, z2xyz^3, 3xy^2 z^2 \rangle$

Solution:

0

Glossary

Stokes’ theorem
: relates the flux integral over a surface *S* to a line integral around the boundary *C* of the surface *S*

surface independent
: flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

The Divergence Theorem

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a “derivative” of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. **The Fundamental Theorem of Calculus**:
   Equation:

   $$\int_a^b f'(x)dx = f(b) - f(a).$$

   This theorem relates the integral of derivative f' over line segment $[a, b]$ along the *x*-axis to a difference of f evaluated on the boundary.
2. **The Fundamental Theorem for Line Integrals**:
   Equation:

   $$\int_C \nabla f \cdot d\mathbf{r} = f(P_1) - f(P_0),$$

   where P_0 is the initial point of *C* and P_1 is the terminal point of *C*. The Fundamental Theorem for Line Integrals allows path *C* to be a path in a plane or in space, not just a line segment on the *x*-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path *C* to a difference of f evaluated on the boundary of *C*.
3. **Green’s theorem, circulation form**:
   Equation:

   $$\iint_D (Q_x - P_y)dA = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Since $Q_x - P_y = \text{curl}\,\mathbf{F} \cdot \mathbf{k}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl**F** over planar region *D* to an integral of **F** over the boundary of *D*.

4. **Green's theorem, flux form**:
   Equation:

$$\iint_D (P_x + Q_y)dA = \int_C \mathbf{F} \cdot \mathbf{N}ds.$$

   Since $P_x + Q_y = \text{div}\,\mathbf{F}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div**F** over planar region *D* to an integral of **F** over the boundary of *D*.
5. **Stokes' theorem**:
   Equation:

$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

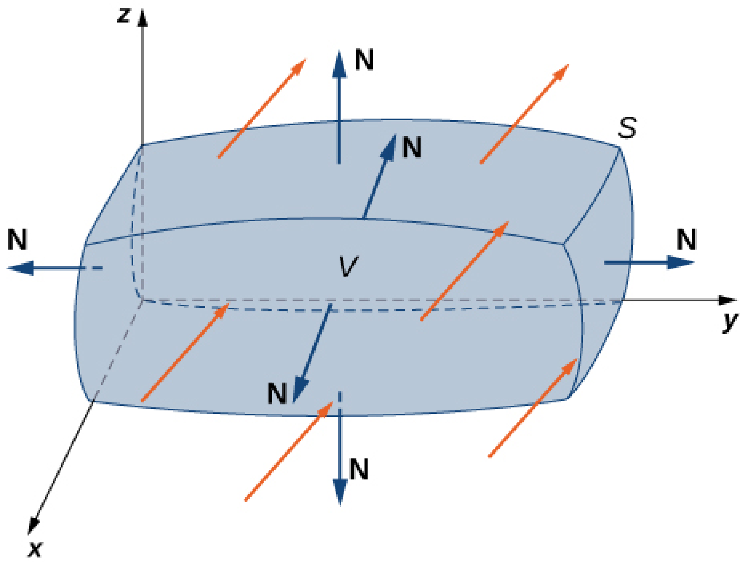
   If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl**F** over surface *S* (not necessarily planar) to an integral of **F** over the boundary of *S*.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the **divergence theorem** relates a triple integral of derivative div**F** over a solid to a flux integral of **F** over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field **F** over a closed surface *S* to a triple integral of the divergence of **F** over the solid enclosed by *S*.

Note:
The Divergence Theorem
Let *S* be a piecewise, smooth closed surface that encloses solid *E* in space. Assume that *S* is oriented outward, and let **F** be a vector field with continuous partial derivatives on an open region containing *E* ([link]). Then
Equation:

$$\iiint_E \text{div}\,\mathbf{F}dV = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$



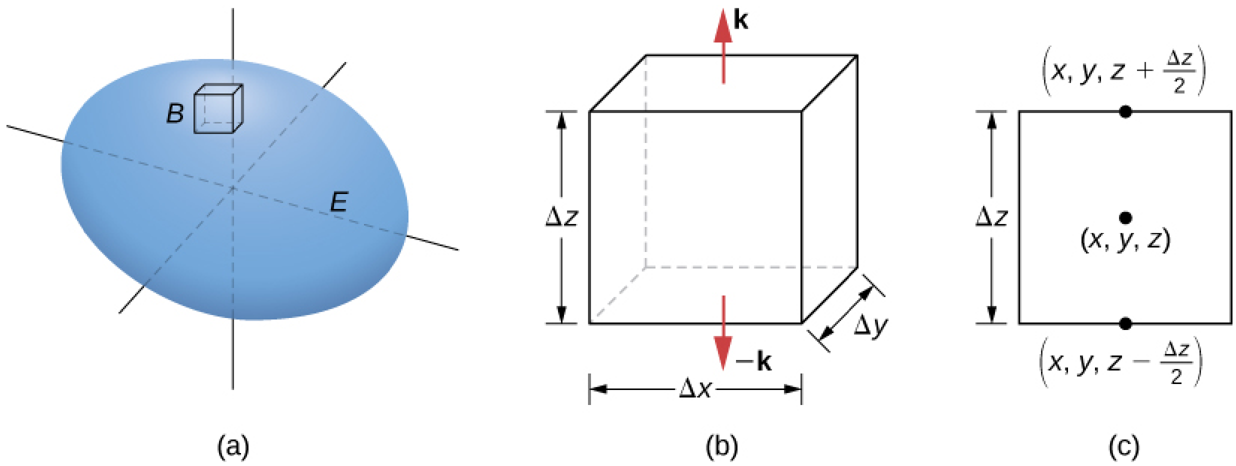
The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that $\iint_D \text{div}\,\mathbf{F}dA = \int_C \mathbf{F}\cdot\mathbf{N}ds.$ Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E ([link]). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz ([link](b)). The normal vector out of the top of the box is $\mathbf{k}$ and the normal vector out of the bottom of the box is $-\mathbf{k}$. The dot product of $\mathbf{F} = \langle P, Q, R\rangle$ with $\mathbf{k}$ is R and the dot product with $-\mathbf{k}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x\Delta y$.



(a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths Δx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\Delta z/2$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\Delta z/2$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x, y, z + \frac{\Delta z}{2}\right)\Delta x\Delta y$ ([link](c)) and the flux out of the bottom of the box is $-R\left(x, y, z - \frac{\Delta z}{2}\right)\Delta x\Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\Delta x\Delta y$. However,

Equation:

$$\Delta R\Delta x\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\Delta x\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the *x*-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\Delta V$ and the net flux in the *y*-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

Equation:

$$\text{Total flux} \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\Delta V = \text{div}\,\mathbf{F}\Delta V.$$

This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

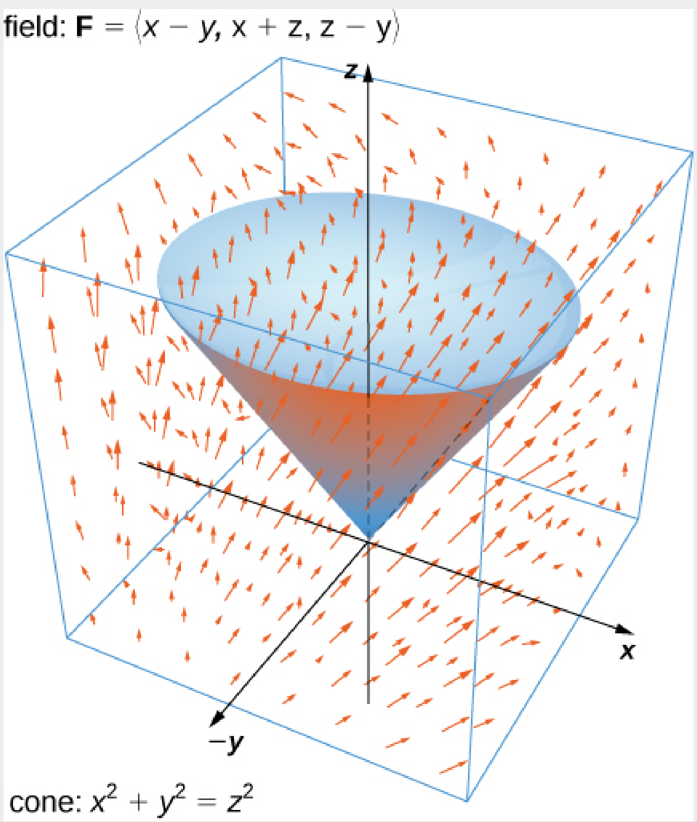
The sum of div $\mathbf{F}\Delta V$ over all the small boxes approximating E is approximately $\iiint_E \text{div } \mathbf{F} dV$. On the other hand, the sum of div $\mathbf{F}\Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example:
Exercise:

Problem:
Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\mathbf{F} = \langle x - y, x + z, z - y \rangle$ and surface S that consists of cone $x^2 + y^2 = z^2, 0 \leq z \leq 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution:

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that $\iiint_E \operatorname{div} \mathbf{F} dV = \iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ by calculating each integral separately.

To compute the triple integral, note that $\operatorname{div} \mathbf{F} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

Equation:

$$\begin{aligned} \iiint_E \operatorname{div} \mathbf{F} dV & = 2 \iiint_E dV \\ & = 2\,(\text{volume of } E). \end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

Equation:

$$\iiint_E \operatorname{div} \mathbf{F} dV = 2\,(\text{volume of } E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

Equation:

$$\mathbf{r}\,(u, v) = \langle u \cos v, u \sin v, 1 \rangle, 0 \leq u \leq 1, 0 \leq v \leq 2\pi.$$

Then, the tangent vectors are $\mathbf{t}_u = \langle \cos v, \sin v, 0 \rangle$ and $\mathbf{t}_v = \langle -u \cos v, u \sin v, 0 \rangle$. Therefore, the flux across S_1 is

Equation:

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} & = \int_0^1 \int_0^{2\pi} \mathbf{F}\,(\mathbf{r}\,(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v) dA \\ & = \int_0^1 \int_0^{2\pi} \langle u \cos v - u \sin v, u \cos v + 1, 1 - u \sin v \rangle \cdot \langle 0, 0, u \rangle dv du \\ & = \int_0^1 \int_0^{2\pi} u - u^2 \sin v \, dv du = \pi. \end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is
Equation:

$$\mathbf{r}\left(u,v\right)=\langle u\cos v,u\sin v,u\rangle ,0\le u\le 1,0\le v\le 2\pi .$$

The tangent vectors are $\mathbf{t}_u=\langle \cos v,\sin v,1\rangle$ and $\mathbf{t}_v=\langle -u\sin v,u\cos v,0\rangle$, so the cross product is
Equation:

$$\mathbf{t}_u\,\times\,\mathbf{t}_v=\langle -u\cos v,-u\sin v,u\rangle .$$

Notice that the negative signs on the *x* and *y* components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\mathbf{t}_v\,\times\,\mathbf{t}_u=\langle u\cos v,u\sin v,-u\rangle$ in the flux integral. The flux across S_2 is then
Equation:

$$\begin{aligned}\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} &=\int_0^1\int_0^{2\pi}\mathbf{F}\left(\mathbf{r}\left(u,v\right)\right)\cdot(\mathbf{t}_v\,\times\,\mathbf{t}_u)dA\\ &=\int_0^1\int_0^{2\pi}\langle u\cos v-u\sin v,u\cos v+u,u-\sin v\rangle\cdot\langle u\cos v,u\sin v,-u\rangle\\ &=\int_0^1\int_0^{2\pi}u^2\cos^2v+2u^2\sin v-u^2dvdu=-\tfrac{\pi}{3}.\end{aligned}$$

The total flux across *S* is
Equation:

$$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}=\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}+\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}=\frac{2\pi}{3}=\iiint_E\text{div }\mathbf{F}dV,$$

and we have verified the divergence theorem for this example.

Note:
Exercise:

Problem:

Verify the divergence theorem for vector field $\mathbf{F}(x,y,z)=\langle x+y+z,y,2x-y\rangle$ and surface *S* given by the cylinder $x^2+y^2=1,0\le z\le 3$ plus the circular top and bottom of the cylinder. Assume that *S* is positively oriented.

Solution:

Both integrals equal 6π.

Hint

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Recall that the divergence of continuous field $\mathbf{F}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\mathbf{F}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_r be a ball of small radius r centered at P ([link]). Let S_r be the boundary sphere of B_r. Since the radius is small and $\mathbf{F}$ is continuous, $\text{div}\,\mathbf{F}\,(Q) \approx \text{div}\,\mathbf{F}\,(P)$ for all other points Q in the ball. Therefore, the flux across S_r can be approximated using the divergence theorem:

Equation:

$$\iint_{S_r} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_r} \text{div}\,\mathbf{F} dV \approx \iiint_{B_r} \text{div}\,\mathbf{F}\,(P) dV.$$

Since $\text{div}\,\mathbf{F}\,(P)$ is a constant,

Equation:

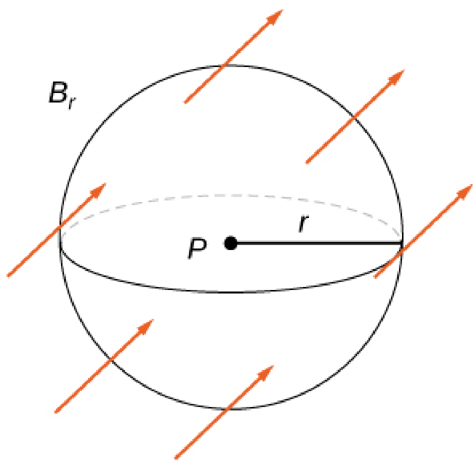
$$\iiint_{B_r} \text{div}\,\mathbf{F}\,(P) dV = \text{div}\,\mathbf{F}\,(P) V\,(B_r).$$

Therefore, flux $\iint_{S_r} \mathbf{F} \cdot d\mathbf{S}$ can be approximated by $\text{div}\,\mathbf{F}\,(P) V\,(B_r)$. This approximation gets better as the radius shrinks to zero, and therefore

Equation:

$$\text{div}\,\mathbf{F}\,(P) = \lim_{r \to 0} \frac{1}{V\,(B_r)} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S}.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.



Ball B_r of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example:
Exercise:

Problem:
Applying the Divergence Theorem

Calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is cylinder $x^2 + y^2 = 1, 0 \leq z \leq 2$, including the circular top and bottom, and $\mathbf{F} = \left\langle \frac{x^3}{3} + yz, \frac{y^3}{3} - \sin(xz), z - x - y \right\rangle$.

Solution:

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product, and using [link].

By contrast, the divergence theorem allows us to calculate the single triple integral $\iiint_E \text{div } \mathbf{F} dV$, where E is the solid enclosed by the cylinder. Using the divergence theorem and converting to cylindrical coordinates, we have

Equation:

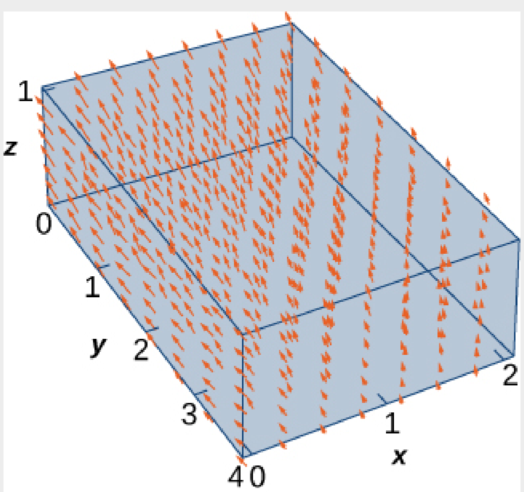
$$\begin{aligned}\iint_s \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \text{div } \mathbf{F}\, dV \\ &= \iiint_E \left(x^2+y^2+1\right) dV \\ &= \int_0^{2\pi}\int_0^1\int_0^2 \left(r^2+1\right) r\, dz\, dr\, d\theta \\ &= \tfrac{3}{2}\int_0^{2\pi} d\theta = 3\pi.\end{aligned}$$

Note:

Exercise:

Problem:

Use the divergence theorem to calculate flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the boundary of the box given by $0 \le x \le 2, 1 \le y \le 4, 0 \le z \le 1$, and $\mathbf{F} = \left\langle x^2+yz, y-z, 2x+2y+2z\right\rangle$ (see the following figure).



Solution:

30

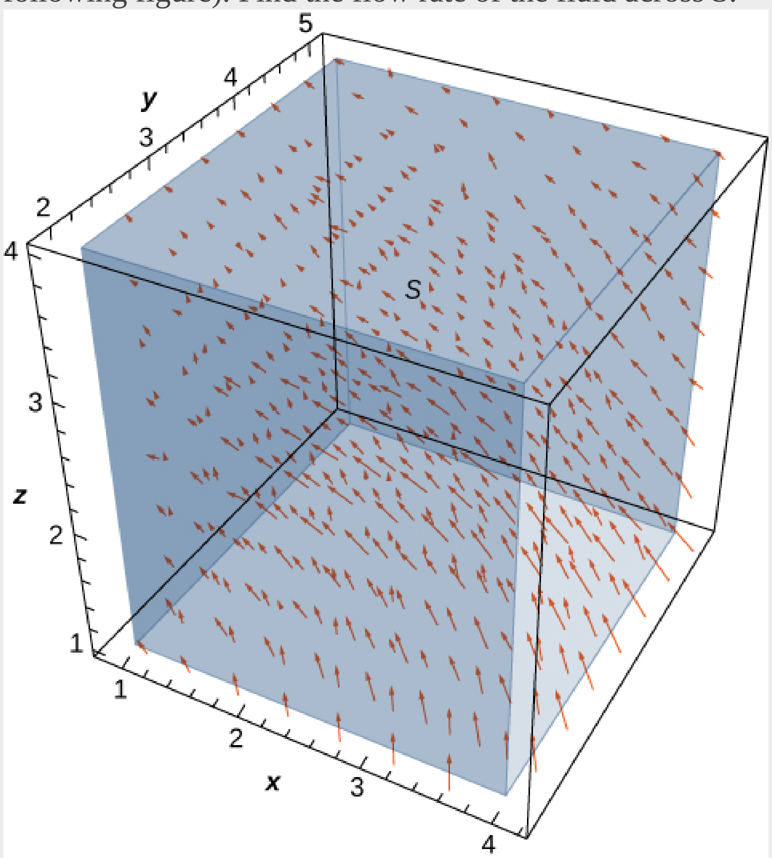
Hint

Calculate the corresponding triple integral.

Example:
Exercise:

Problem:
Applying the Divergence Theorem

Let $\mathbf{v} = \left\langle -\frac{y}{z}, \frac{x}{z}, 0 \right\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \leq x \leq 4, 2 \leq y \leq 5, 1 \leq z \leq 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Vector field $\mathbf{v} = \left\langle -\frac{y}{z}, \frac{x}{z}, 0 \right\rangle$.

Solution:

The flow rate of the fluid across S is $\iint_S \mathbf{v} \cdot d\mathbf{S}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on [link], we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the *xy*-plane that has a center on the *z*-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product, and use [link]. However, using the divergence theorem makes this calculation go much more quickly:

Equation:

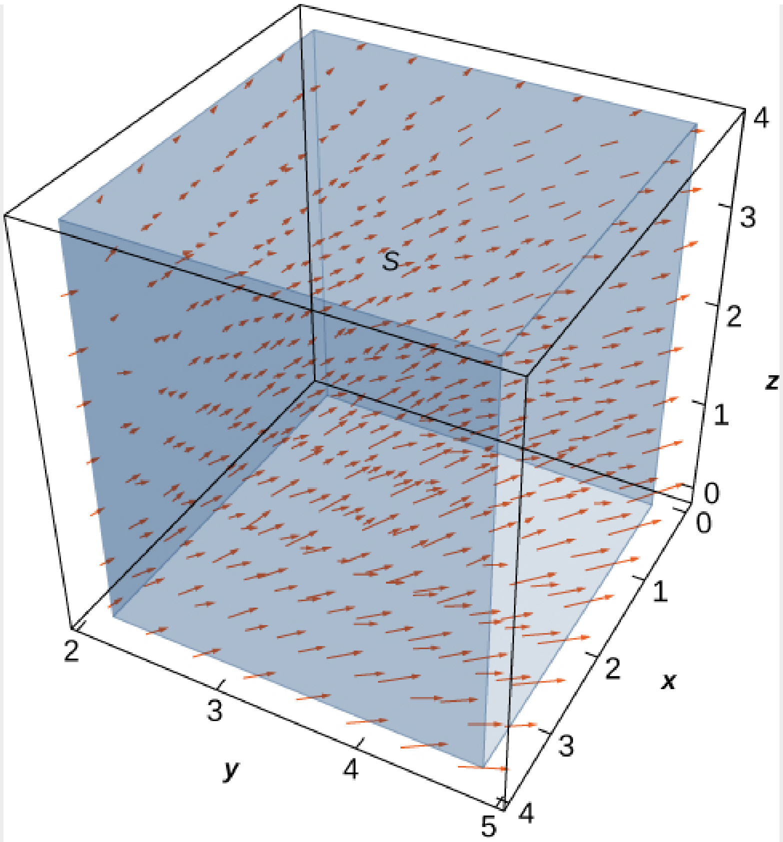
$$\begin{aligned}\iint_S \mathbf{v} \cdot d\mathbf{S} &= \iiint_C \operatorname{div}(\mathbf{v}) dV \\ &= \iiint_C 0\, dV = 0.\end{aligned}$$

Therefore the flux is zero, as expected.

Note:
Exercise:

Problem:

Let $\mathbf{v} = \left\langle \frac{x}{z}, \frac{y}{z}, 0 \right\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \leq x \leq 4, 2 \leq y \leq 5, 1 \leq z \leq 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Solution:

$9 \ln (16)$

Hint

Use the divergence theorem and calculate a triple integral.

[link] illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\mathbf{F}$ be a vector field defined on an open region containing the surface enclosed by S. If $\mathbf{F}$ has the form $\mathbf{F} = \langle f(y, z), g(x, z), h(x, y) \rangle$, then the divergence of $\mathbf{F}$ is zero. By the divergence theorem, the flux of $\mathbf{F}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is a cube and

Equation:

$$\mathbf{F} = \left\langle \sin(y) e^{yz}, x^2 z^2, \cos(xy) e^{\sin x} \right\rangle.$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if **F** is a continuous three-dimensional vector field and *P* is a point in the domain of **F**, then the divergence of **F** at *P* is a measure of the "outflowing-ness" of **F** at *P*. If **F** represents the velocity field of a fluid, then the divergence of **F** at *P* is a measure of the net flow rate out of point *P* (the flow of fluid out of *P* less the flow of fluid in to *P*). To see how the divergence theorem justifies this interpretation, let B_r be a ball of very small radius *r* with center *P*, and assume that B_r is in the domain of **F**. Furthermore, assume that B_r has a positive, outward orientation. Since the radius of B_r is small and **F** is continuous, the divergence of **F** is approximately constant on B_r. That is, if P' is any point in B_r, then $\text{div}\,\mathbf{F}(P) \approx \text{div}\,\mathbf{F}(P')$. Let S_r denote the boundary sphere of B_r. We can approximate the flux across S_r using the divergence theorem as follows:
Equation:

$$\begin{aligned}\iint_{S_r} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{B_r} \text{div}\,\mathbf{F}\,dV \\ &\approx \iiint_{B_r} \text{div}\,\mathbf{F}(P) dV \\ &= \text{div}\,\mathbf{F}(P) V(B_r).\end{aligned}$$

As we shrink the radius *r* to zero via a limit, the quantity $\text{div}\,\mathbf{F}(P)V(B_r)$ gets arbitrarily close to the flux. Therefore,
Equation:

$$\text{div}\,\mathbf{F}(P) = \lim_{r \to 0} \frac{1}{V(B_r)} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S}$$

and we can consider the divergence at *P* as measuring the net rate of outward flux per unit volume at *P*. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to electrostatic fields. An important result in this subject is **Gauss' law**. This law states that if *S* is a closed surface in electrostatic field **E**, then the flux of **E** across *S* is the total charge enclosed by *S* (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\mathbf{F}_r$ denote radial vector field $\mathbf{F}_r = \frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$. The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ and is scaled by the quantity $1/r^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field **E** given by

Equation:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r,$$

where the approximation $\varepsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ε_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that **E** is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

Equation:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle\right),$$

we say that electrostatic fields obey an **inverse-square law**. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface *S* is zero if the charge is outside of *S*, and that the flux is q/ε_0 if the charge is inside of *S*. In other words, the flux across *S* is the charge inside the surface divided by constant ε_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across *S* is the charge inside the surface divided by constant ε_0, we need two intermediate steps. First we show that the divergence of $\mathbf{F}_r$ is zero and then we show that the flux of $\mathbf{F}_r$ across any smooth surface *S* is either zero or 4π. We can then justify this special case of Gauss' law.

Example:
Exercise:

Problem:
The Divergence of $\mathbf{F}_r$ Is Zero

Verify that the divergence of $\mathbf{F}_r$ is zero where $\mathbf{F}_r$ is defined (away from the origin).

Solution:

Since $r = \sqrt{x^2+y^2+z^2}$, the quotient rule gives us
Equation:

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{\left(x^2+y^2+z^2\right)^{3/2} - x\left[\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2x\right]}{\left(x^2+y^2+z^2\right)^3}\\ &= \frac{r^3-3x^2r}{r^6} = \frac{r^2-3x^2}{r^5}.\end{aligned}$$

Similarly,
Equation:

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{r^2-3y^2}{r^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{r^2-3z^2}{r^5}.$$

Therefore,
Equation:

$$\begin{aligned}\operatorname{div}\mathbf{F}_r &= \frac{r^2-3x^2}{r^5} + \frac{r^2-3y^2}{r^5} + \frac{r^2-3z^2}{r^5}\\ &= \frac{3r^2-3\left(x^2+y^2+z^2\right)}{r^5}\\ &= \frac{3r^2-3r^2}{r^5} = 0.\end{aligned}$$

Notice that since the divergence of $\mathbf{F}_r$ is zero and $\mathbf{E}$ is $\mathbf{F}_r$ scaled by a constant, the divergence of electrostatic field $\mathbf{E}$ is also zero (except at the origin).

Note:
Flux across a Smooth Surface
Let S be a connected, piecewise smooth closed surface and let $\mathbf{F}_r = \frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\rangle$. Then,
Equation:

$$\iint_S \mathbf{F}_r \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not encompass the origin} \\ 4\pi & \text{if } S \text{ encompasses the origin.} \end{cases}$$

In other words, this theorem says that the flux of $\mathbf{F}_r$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\mathbf{F}_r$, and since the divergence of $\mathbf{F}_r$ is zero, we can immediately apply the divergence theorem and find that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

Equation:

$$\mathbf{t}_\phi \times \mathbf{t}_\theta = \langle a^2 \cos\theta \sin^2\phi, a^2 \sin\theta \sin^2\phi, a^2 \sin\phi \cos\phi \rangle$$

(see [link]). Therefore, on the surface of the sphere, the dot product $\mathbf{F}_r \cdot \mathbf{N}$ (in spherical coordinates) is

Equation:

$$\begin{aligned} \mathbf{F}_r \cdot \mathbf{N} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \cdot \langle a^2 \cos\theta \sin^2\phi, a^2 \sin\theta \sin^2\phi, a^2 \sin\phi \cos\phi \rangle \\ &= \sin\phi \left(\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle \right) \\ &= \sin\phi. \end{aligned}$$

The flux of $\mathbf{F}_r$ across S_a is

Equation:

$$\iint_{S_a} \mathbf{F}_r \cdot \mathbf{N} dS = \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi d\theta = 4\pi.$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

Equation:

$$\begin{aligned} \iint_{S-S_a} \mathbf{F}_r \cdot d\mathbf{S} &= \iint_S \mathbf{F}_r \cdot d\mathbf{S} - \iint_{S_a} \mathbf{F}_r \cdot d\mathbf{S} \\ &= \iiint_E \operatorname{div}\mathbf{F}_r \, dV \\ &= \iiint_E 0 dV = 0. \end{aligned}$$

Therefore,

Equation:

$$\iint_S \mathbf{F}_r \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F}_r \cdot d\mathbf{S} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

Equation:

$$\begin{aligned} \iint_S \mathbf{E} \cdot d\mathbf{S} &= \iint_S \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r \cdot d\mathbf{S} \\ &= \frac{q}{4\pi\varepsilon_0} \iint_S \mathbf{F}_r \cdot d\mathbf{S} \\ &= \frac{q}{\varepsilon_0}. \end{aligned}$$

If S does not encompass the origin, then

Equation:

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0} \iint_S \mathbf{F}_r \cdot d\mathbf{S} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ε_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\mathbf{E}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\mathbf{E}$ across S is Q/ε_0.

Example:
Exercise:

Problem:
Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0, 1, 1), (1, 1, 4), (-1, 0, 0), \text{ and } (-2, -2, 2)$. Let $\mathbf{E}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find $\iint_S \mathbf{E} \cdot d\mathbf{S}$.

Solution:

According to Gauss' law, the flux of $\mathbf{E}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0, 1, 1)$ and the charge at $(-1, 0, 0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,
Equation:

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.518 \times 10^9 \text{ V-m.}$$

Note:
Exercise:

Problem:

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Solution:

$\approx 6.777 \times 10^9$

Hint

Use Gauss' law.

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**
$$\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

For the following exercises, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \mathbf{F} \cdot \mathbf{n} ds$ for the given choice of **F** and the boundary surface *S*. For each closed surface, assume **N** is the outward unit normal vector.

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of cube $0 \le x \le 1, 0 \le y \le 1, 0 < z \le 1.$

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = (\cos yz)\mathbf{i} + e^{xz}\mathbf{j} + 3z^2\mathbf{k}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 4$ in the *xy*-plane.

Solution:

$$\int_S \mathbf{F} \cdot \mathbf{n} ds = 75.3982$$

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = \left(x^2 + y^2 - x^2\right)\mathbf{i} + x^2y\mathbf{j} + 3z\mathbf{k}$; S is the surface of the five faces of unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 < z \leq 1$.

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 9$.

Solution:

$$\int_S \mathbf{F} \cdot \mathbf{n} ds = 127.2345$$

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \left(z^2 - 1\right)\mathbf{k}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Solution:

$$\int_S \mathbf{F} \cdot \mathbf{n} ds = 37.6991$$

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yz^2\mathbf{j} + x^2z\mathbf{k}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \frac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

Exercise:

Problem:

[T] $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + 3a^2z\mathbf{k}$ (constant $a > 0$); S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Solution:

$$\int_S \mathbf{F} \cdot \mathbf{n}ds = \frac{9\pi a^4}{2}$$

Exercise:

Problem:

[T] Surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\mathbf{F}(x, y, z) = \left(x + y^2z^2\right)\mathbf{i} + \left(y + z^2x^2\right)\mathbf{j} + \left(z + x^2y^2\right)\mathbf{k}$

Exercise:

Problem:

Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(e^{y^2}\right)\mathbf{i} + \left(y + \sin\left(z^2\right)\right)\mathbf{j} + (z - 1)\mathbf{k}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented upward.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{3}$$

Exercise:

Problem:

Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Exercise:

Problem:

Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = x^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + xz^4 \mathbf{k}$ and S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$$

Exercise:

Problem:

Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = z \tan^{-1}\left(y^2\right)\mathbf{i} + z^3 \ln\left(x^2 + 1\right)\mathbf{j} + z\mathbf{k}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

Exercise:

Problem:

[T] Use a CAS and the divergence theorem to calculate flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(x^3 + y^3\right)\mathbf{i} + \left(y^3 + z^3\right)\mathbf{j} + \left(z^3 + x^3\right)\mathbf{k}$ and S is a sphere with center (0, 0) and radius 2.

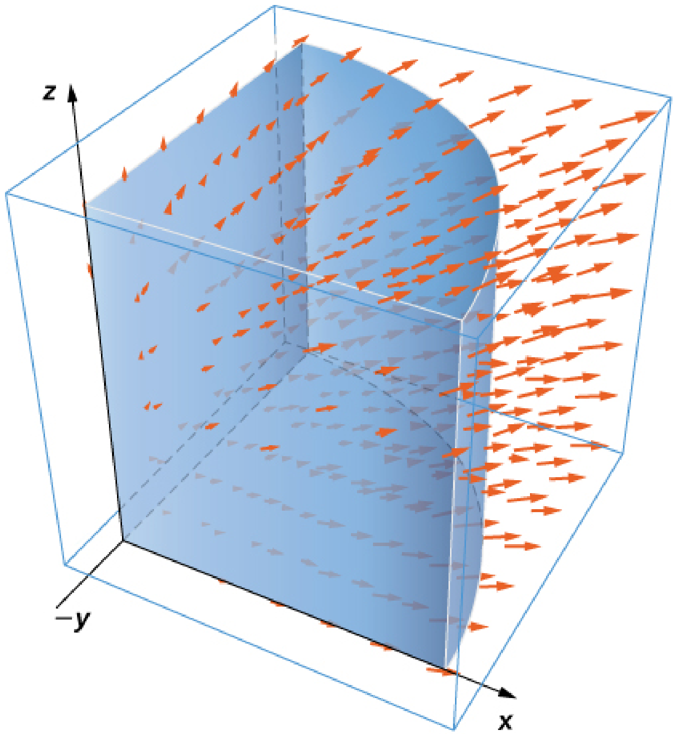
Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 241.2743$$

Exercise:

Problem:

Use the divergence theorem to compute the value of flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(y^3 + 3x\right)\mathbf{i} + (xz + y)\mathbf{j} + \left[z + x^4 \cos\left(x^2 y\right)\right]\mathbf{k}$ and S is the area of the region bounded by $x^2 + y^2 = 1, x \geq 0, y \geq 0,$ and $0 \leq z \leq 1$.



Exercise:

Problem:

Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{j} - z\mathbf{k}$ and S consists of the union of paraboloid $y = x^2 + z^2, 0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1, y = 1$, oriented outward. What is the flux through just the paraboloid?

Solution:

$$\iint_D \mathbf{F} \cdot d\mathbf{S} = -\pi$$

Exercise:

Problem:

Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x + y\mathbf{j} + z^4\mathbf{k}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$, oriented downward.

Exercise:

Problem:

Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}$, where *S* is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3}$$

Exercise:

Problem:

Consider $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + (z + 1)\mathbf{k}$. Let *E* be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside *E*. Compute flux ***F*** across the boundary of *E* using the divergence theorem.

For the following exercises, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces *S*.

Exercise:

Problem: **[T]** $\mathbf{F} = \langle x, -2y, 3z \rangle$; *S* is sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 6\right\}$.

Solution:

$16\sqrt{6}\pi$

Exercise:

Problem:

[T] $\mathbf{F} = \langle x, 2y, z \rangle$; *S* is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

Exercise:

Problem:

[T] $\mathbf{F} = \left\langle y - 2x, x^3 - y, y^2 - z \right\rangle$; *S* is sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 4\right\}$.

Solution:

$-\frac{128}{3}\pi$

Exercise:

Problem:

[T] $\mathbf{F} = \langle x, y, z \rangle$; *S* is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the *xy*-plane.

For the following exercises, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions *D*.

Exercise:

Problem:

[T] $\mathbf{F} = \langle z - x, x - y, 2y - z \rangle$; *D* is the region between spheres of radius 2 and 4 centered at the origin.

Solution:

-703.7168

Exercise:

Problem:

[T] $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x,y,z \rangle}{\sqrt{x^2+y^2+z^2}}$; *D* is the region between spheres of radius 1 and 2 centered at the origin.

Exercise:

Problem:

[T] $\mathbf{F} = \left\langle x^2, -y^2, z^2 \right\rangle$; *D* is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

Solution:

20

Exercise:

Problem:

Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3xy\mathbf{j} + xz^2\mathbf{k}$. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where *S* is the surface of the cube with corners at $(0, 0, 0), (1, 0, 0), (0, 1, 0),$ $(1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1),$ and $(1, 1, 1)$, oriented outward.

Exercise:

Problem:

Use the divergence theorem to find the outward flux of field $\mathbf{F}(x, y, z) = \left(x^3 - 3y\right)\mathbf{i} + (2yz + 1)\mathbf{j} + xyz\mathbf{k}$ through the cube bounded by planes $x = \pm 1, y = \pm 1,$ and $z = \pm 1.$

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 8$$

Exercise:

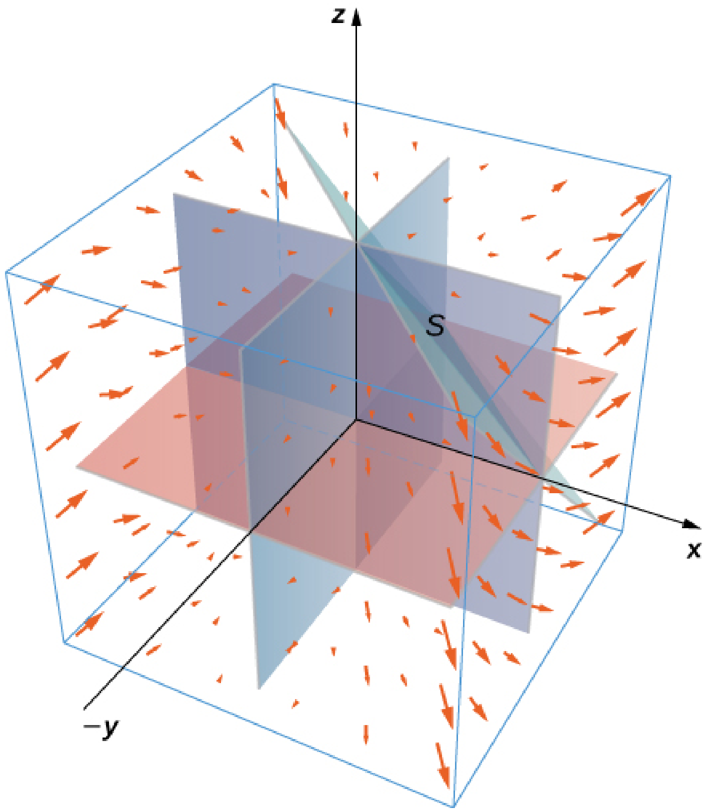
Problem:

Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + 5z\mathbf{k}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the *xy*-plane. Use the divergence theorem.

Exercise:

Problem:

Evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector **N**.



Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \frac{1}{8}$$

Exercise:

Problem:

Find the net outward flux of field $\mathbf{F} = \langle bz - cy, cx - az, ay - bx \rangle$ across any smooth closed surface in $\mathbf{R}^3$, where a, b, and c are constants.

Exercise:

Problem:

Use the divergence theorem to evaluate $\iint_S ||\mathbf{R}||\mathbf{R} \cdot nds$, where $\mathbf{R}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Solution:

$$\iint_S ||\mathbf{R}||\mathbf{R} \cdot n ds = 4\pi a^4$$

Exercise:

Problem:

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + y^3\mathbf{j} + xz\mathbf{k}$ and S is the boundary of the cube defined by $-1 \leq x \leq 1, -1 \leq y \leq 1, \text{ and } 0 \leq z \leq 2.$

Exercise:

Problem:

Let R be the region defined by $x^2 + y^2 + z^2 \leq 1$. Use the divergence theorem to find $\iiint_R z^2 dV$.

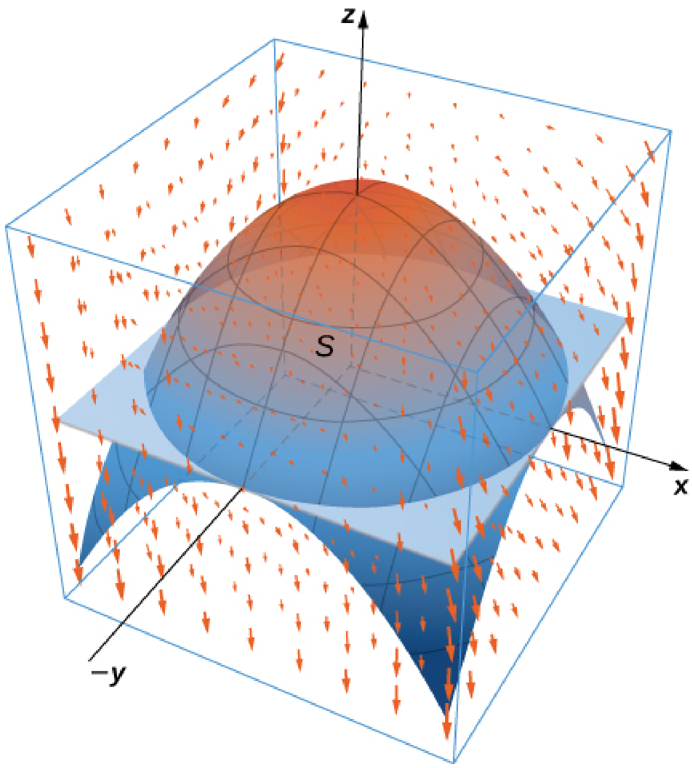
Solution:

$\iiint_R z^2 d\text{V} = \frac{4\pi}{15}$

Exercise:

Problem:

Let E be the solid bounded by the *xy*-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the *xy*-plane that forms its bottom. If $\mathbf{F}(x, y, z) = \left(xz \sin(yz) + x^3\right)\mathbf{i} + \cos(yz)\mathbf{j} + \left(3zy^2 - e^{x^2+y^2}\right)\mathbf{k}$, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.



Exercise:

Problem:

Let E be the solid unit cube with diagonally opposite corners at the origin and (1, 1, 1), and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem if $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + 3ye^z\mathbf{j} + x\sin z\mathbf{k}$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 6.5759$$

Exercise:

Problem:

Use the divergence theorem to calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through sphere $x^2 + y^2 + z^2 = 1$.

Exercise:

Problem:

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the outwardly oriented surface obtained by removing cube $[1, 2] \times [1, 2] \times [1, 2]$ from cube $[0, 2] \times [0, 2] \times [0, 2]$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 21$$

Exercise:

Problem:

Consider radial vector field $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius a centered at the origin.

Exercise:

Problem:

Compute the flux of water through parabolic cylinder $S : y = x^2$, from $0 \leq x \leq 2, 0 \leq z \leq 3$, if the velocity vector is $\mathbf{F}(x, y, z) = 3z^2\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 72$$

Exercise:

Problem:

[T] Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + z\mathbf{j} + \sqrt{x^2 + y^2}\mathbf{k}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \frac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

Exercise:

Problem:

[T] Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = (e^y + x)\mathbf{i} + (3\cos(xz) - y)\mathbf{j} + z\mathbf{k}$ through surface S, where S is given by $z^2 = 4x^2 + 4y^2$ from $0 \leq z \leq 4$, oriented so the unit normal vector points downward.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -33.5103$$

Exercise:

Problem:

[T] Use a CAS to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ and S is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \le z \le 1$.

Exercise:

Problem:

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = bxy^2\mathbf{i} + bx^2y\mathbf{j} + \left(x^2 + y^2\right)z^2\mathbf{k}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \pi a^4 b^2$$

Exercise:

Problem:

[T] Use a CAS to calculate the flux of $\mathbf{F}(x, y, z) = \left(x^3 + y \sin z\right)\mathbf{i} + \left(y^3 + z \sin x\right)\mathbf{j} + 3z\mathbf{k}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$, and plane $z = 0$.

Exercise:

Problem:

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j} + z\mathbf{k}$ and S is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2, 1 \le z \le 4$ on the top; $x^2 + y^2 = 1, 0 \le z \le 1$ on the sides; and $z = 0$ on the bottom.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{5}{2}\pi$$

Exercise:

Problem:

[T] Use a CAS and the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (2x + y\cos z)\mathbf{i} + \left(x^2 - y\right)\mathbf{j} + y^2 z\mathbf{k}$ and S is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

Exercise:

Problem:

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2$, cylinder $x^2 + z^2 = 1$, and plane $x + y = 2$, and S is oriented outward.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{21\pi}{2}$$

For the following exercises, Fourier's law of heat transfer states that the heat flow vector $\mathbf{F}$ at a point is proportional to the negative gradient of the temperature; that is, $\mathbf{F} = -k\nabla T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D is given. Use the divergence theorem to find net outward heat flux $\iint_S \mathbf{F} \cdot \mathbf{N} dS = -k \iint_S \nabla T \cdot \mathbf{N} dS$ across the boundary S of D, where $k = 1$.

Exercise:

Problem:

$T(x, y, z) = 100 + x + 2y + z$; $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

Exercise:

Problem: $T(x, y, z) = 100 + e^{-z}$; $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

Solution:

$-\left(1 - e^{-1}\right)$

Exercise:

Problem: $T(x, y, z) = 100e^{-x^2 - y^2 - z^2}$; D is the sphere of radius a centered at the origin.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.
Exercise:

Problem: Vector field $\mathbf{F}(x, y) = x^2 y\mathbf{i} + y^2 x\mathbf{j}$ is conservative.

Solution:

False

Exercise:

Problem:

For vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, if $P_y(x, y) = Q_x(x, y)$ in open region D, then $\int_{\partial D} P dx + Q dy = 0$.

Exercise:

Problem: The divergence of a vector field is a vector field.

Solution:

False

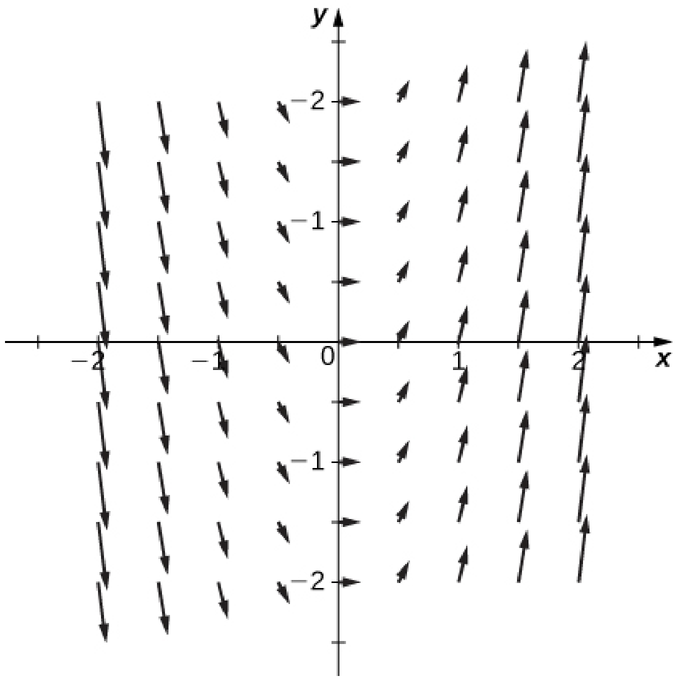
Exercise:

Problem: If curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Draw the following vector fields.
Exercise:

Problem: $\mathbf{F}\left(x, y\right) = \frac{1}{2}\mathbf{i} + 2x\mathbf{j}$

Solution:



Exercise:

Problem: $\mathbf{F}\left(x,y\right)=\sqrt{\frac{y\mathbf{i}+3x\mathbf{j}}{x^2+y^2}}$

Are the following the vector fields conservative? If so, find the potential function f such that $\mathbf{F}=\nabla f$.

Exercise:

Problem: $\mathbf{F}\left(x,y\right)=y\mathbf{i}+\left(x-2e^y\right)\mathbf{j}$

Solution:

Conservative, $f(x,y)=xy-2e^y$

Exercise:

Problem: $\mathbf{F}\left(x,y\right)=\left(6xy\right)\mathbf{i}+\left(3x^2-ye^y\right)\mathbf{j}$

Exercise:

Problem: $\mathbf{F}\left(x,y,z\right)=\left(2xy+z^2\right)\mathbf{i}+\left(x^2+2yz\right)\mathbf{j}+\left(2xz+y^2\right)\mathbf{k}$

Solution:

Conservative, $f(x,y,z)=x^2y+y^2z+z^2x$

Exercise:

Problem: $\mathbf{F}(x,y,z)=(e^x y)\mathbf{i}+(e^x+z)\mathbf{j}+\left(e^x+y^2\right)\mathbf{k}$

Evaluate the following integrals.

Exercise:

Problem: $\int_C x^2dy+(2x-3xy)dx$, along $C: y=\frac{1}{2}x$ from (0, 0) to (4, 2)

Solution:

$-\frac{16}{3}$

Exercise:

Problem: $\int_C ydx+xy^2dy$, where $C: x=\sqrt{t}, y=t-1, 0\le t\le 1$

Exercise:

Problem: $\iint_S xy^2dS$, where S is surface $z=x^2-y, 0\le x\le 1, 0\le y\le 4$

Solution:

$\frac{32\sqrt{2}}{9}\left(3\sqrt{3}-1\right)$

Find the divergence and curl for the following vector fields.

Exercise:

Problem: $\mathbf{F}(x,y,z)=3xyz\mathbf{i}+xye^z\mathbf{j}-3xy\mathbf{k}$

Exercise:

Problem: $\mathbf{F}(x,y,z)=e^x\mathbf{i}+e^{xy}\mathbf{j}+e^{xyz}\mathbf{k}$

Solution:

Divergence: $e^x+xe^{xy}+xye^{xyz}$, curl: $xze^{xyz}\mathbf{i}-yze^{xyz}\mathbf{j}+ye^{xy}\mathbf{k}$

Use Green’s theorem to evaluate the following integrals.

Exercise:

Problem:

$\int_C 3xydx + 2xy^2dy$, where C is a square with vertices (0, 0), (0, 2), (2, 2) and (2, 0)

Exercise:

Problem: $\oint_C 3ydx + (x + e^y)dy$, where C is a circle centered at the origin with radius 3

Solution:

-2π

Use Stokes' theorem to evaluate $\int\int_S \text{curl}\,\mathbf{F} \cdot dS$.

Exercise:

Problem: $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, where S is the upper half of the unit sphere

Exercise:

Problem:

$\mathbf{F}(x, y, z) = y\mathbf{i} + xyz\mathbf{j} - 2zx\mathbf{k}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Solution:

$-\pi$

Use the divergence theorem to evaluate $\int\int_S \mathbf{F} \cdot dS$.

Exercise:

Problem:

$\mathbf{F}(x, y, z) = \left(x^3y\right)\mathbf{i} + (3y - e^x)\mathbf{j} + (z + x)\mathbf{k}$, over cube S defined by $-1 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 2$

Exercise:

Problem:

$\mathbf{F}(x, y, z) = (2xy)\mathbf{i} + \left(-y^2\right)\mathbf{j} + \left(2z^3\right)\mathbf{k}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Solution:

$31\pi/2$

Exercise:

Problem:

Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

Exercise:

Problem:

Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2},$ and a density function of $\rho\left(x, y\right) = y + x^2$.

Solution:

$\sqrt{2}\left(2 + \pi\right)$

Exercise:

Problem:

Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \geq 0$ with a density function $\rho\left(x, y, z\right) = x + y + z$.

Exercise:

Problem:

Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 2y\mathbf{j} + 2x\mathbf{k}$.

Solution:

$2\pi/3$

Glossary

divergence theorem
: a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law
: if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field $\mathbf{E}$ across S is Q/ε_0

inverse-square law

the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

Double Integrals over Rectangular Regions

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

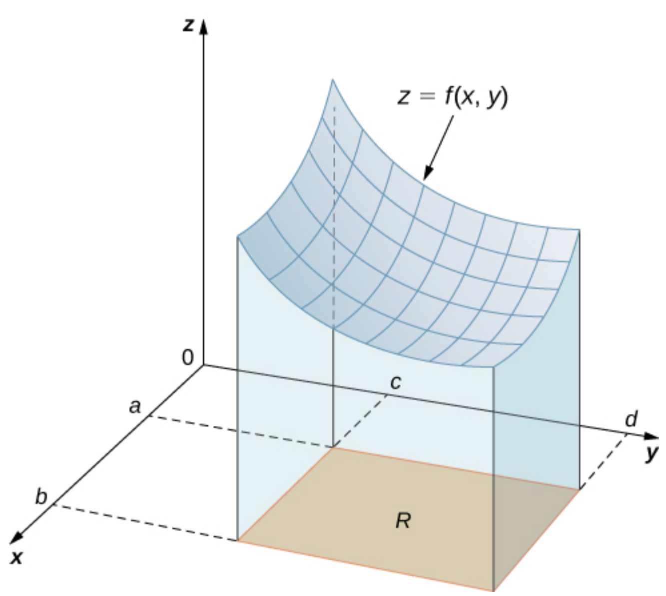
Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

Equation:

$$R = [a,b] \times [c,d] = \left\{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\right\}$$

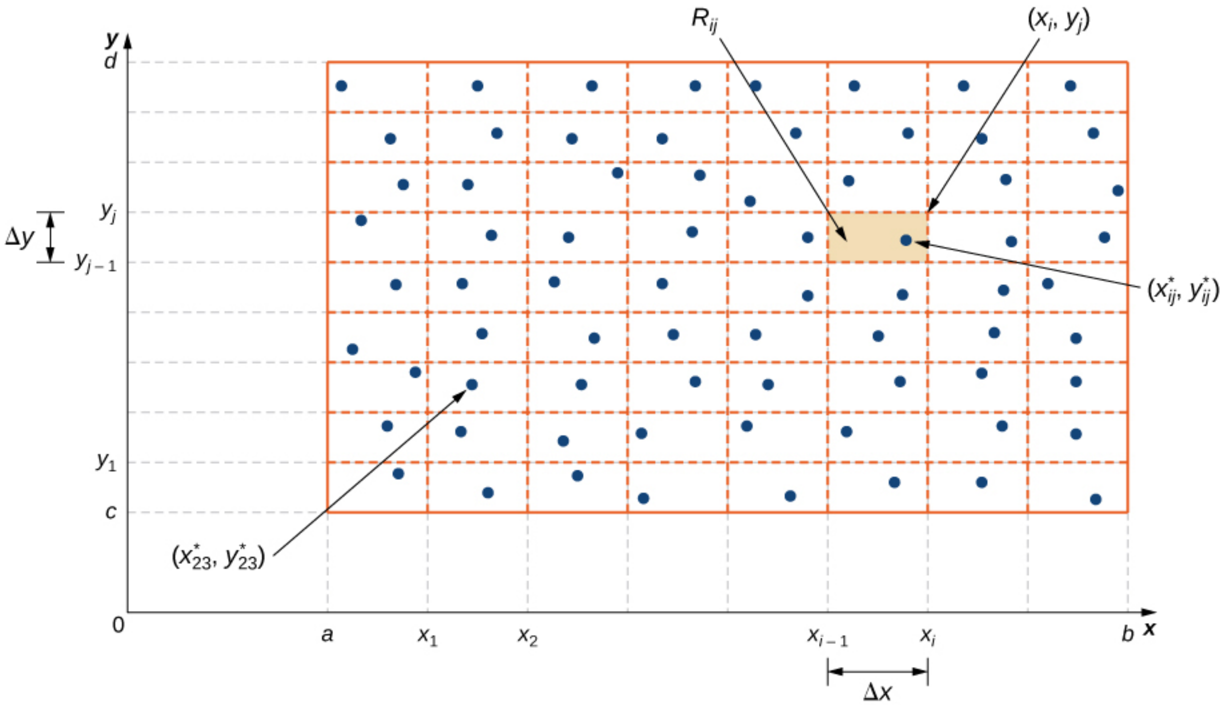
Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f ([link]). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.



The graph of $f(x,y)$ over the rectangle R in the xy-plane is a curved surface.

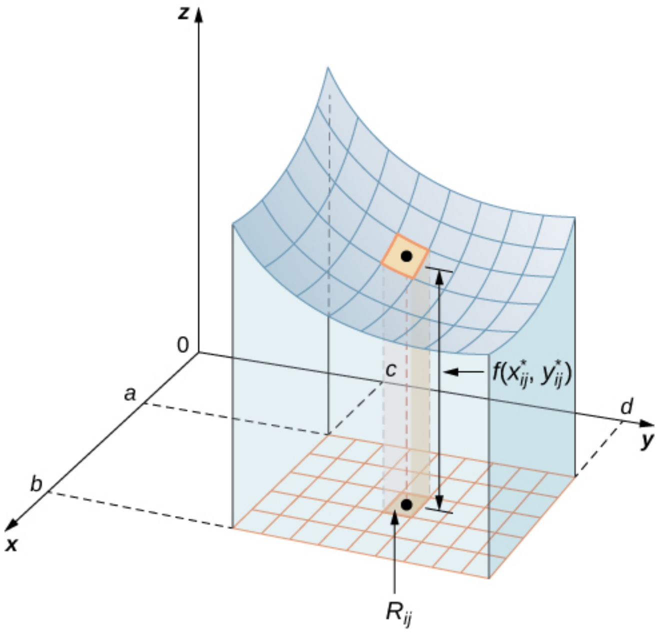
We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy ([link]). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence

$\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.



Rectangle R is divided into small rectangles R_{ij}, each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure.



A thin rectangular box above R_{ij} with height $f\left(x^*_{ij}, y^*_{ij}\right)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A$. This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if *m* and *n* become larger.

Equation:

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A \text{ or } V = \lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base *R*. Now we are ready to define the double integral.

Note:
Definition
The **double integral** of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as
Equation:

$$\iint_R f(x, y) dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A.$$

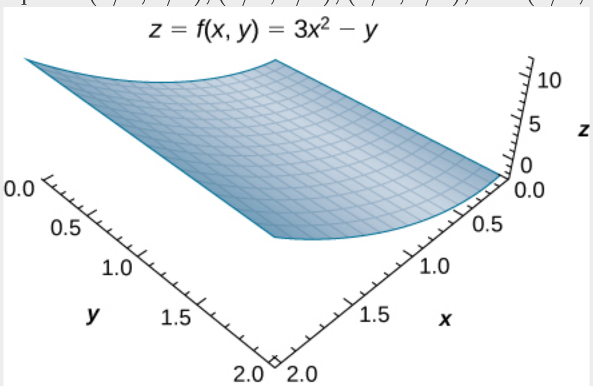
If $f(x, y) \geq 0$, then the volume *V* of the solid *S*, which lies above R in the xy-plane and under the graph of *f*, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example:
Exercise:

Problem:
Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ ([link]).

a. Set up a double integral for finding the value of the signed volume of the solid *S* that lies above R and "under" the graph of f.
b. Divide *R* into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square $(1, 1), (2, 1), (1, 2)$, and $(2, 2)$ ([link]) to approximate the signed volume of the solid *S* that lies above R and "under" the graph of f.
c. Divide *R* into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: $(1/2, 1/2), (3/2, 1/2), (1/2, 3/2)$, and $(3/2, 3/2)$ to approximate the signed volume.



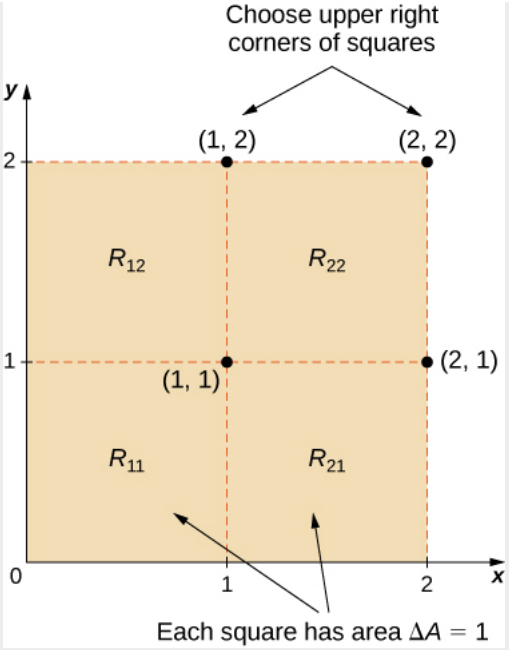
The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution:

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of *S*, we need to divide the region *R* into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as
Equation:

$$V = \iint_R (3x^2 - y)\,dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[3\left(x_{ij}^*\right)^2 - y_{ij}^*\right] \Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.



Subrectangles for the rectangular region $R = [0, 2] \times [0, 2]$.

Hence,

Equation:

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^{*}, y_{i1}^{*})+f(x_{i2}^{*}, y_{i2}^{*}))\Delta A \\ &= f(x_{11}^{*}, y_{11}^{*})\Delta A + f(x_{21}^{*}, y_{21}^{*})\Delta A + f(x_{12}^{*}, y_{12}^{*})\Delta A + f(x_{22}^{*}, y_{22}^{*})\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$, we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).

Hence

Equation:

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A \\ &= f(x_{11}^{*}, y_{11}^{*})\Delta A + f(x_{21}^{*}, y_{21}^{*})\Delta A + f(x_{12}^{*}, y_{12}^{*})\Delta A + f(x_{22}^{*}, y_{22}^{*})\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= (\tfrac{3}{4} - \tfrac{1}{4})(1) + (\tfrac{27}{4} - \tfrac{1}{2})(1) + (\tfrac{3}{4} - \tfrac{3}{2})(1) + (\tfrac{27}{4} - \tfrac{3}{2})(1) \\ &= \tfrac{2}{4} + \tfrac{25}{4} + \left(-\frac{3}{4}\right) + \tfrac{21}{4} = \tfrac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Note:
Exercise:

Problem: Use the same function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide *R* into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square $(0, 1)$, $(1, 1)$, $(0, 2)$, and $(1, 2)$ ([link]) to approximate the signed volume of the solid *S* that lies above R and "under" the graph of f.

Solution:

$$V = \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A = 0$$

Hint

Follow the steps of the previous example.

Note that we developed the concept of double integral using a rectangular region *R*. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into *R*, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region *R*. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid *S* approach 0 as *m* and *n* approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over *R* except on a finite number of smooth curves, then the double integral exists and we say that f is integrable over *R*.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\, dy$ or $dy\, dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint\limits_R f(x, y) dA$ can be written as $\iint\limits_R f(x, y) dx\, dy$ or $\iint\limits_R f(x, y) dy\, dx$.

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Note:
Properties of Double Integrals
Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region *R*; *S* and *T* are subregions of *R*; and assume that *m* and *M* are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and
Equation:

$$\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA.$$

ii. If *c* is a constant, then $cf(x, y)$ is integrable and
Equation:

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then
Equation:

$$\iint_R f(x,y)dA = \iint_S f(x,y)dA + \iint_T f(x,y)dA.$$

iv. If $f(x, y) \geq g(x, y)$ for (x, y) in R, then
Equation:

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA.$$

v. If $m \leq f(x, y) \leq M$, then
Equation:

$$m \times A(R) \leq \iint_R f(x,y)dA \leq M \times A(R).$$

vi. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, the double integral can be written as
Equation:

$$\iint_R f(x,y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n. Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Note:
Definition
Assume $a, b, c,$ and d are real numbers. We define an **iterated integral** for a function $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ as

a.
Equation:

$$\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

b.
Equation:

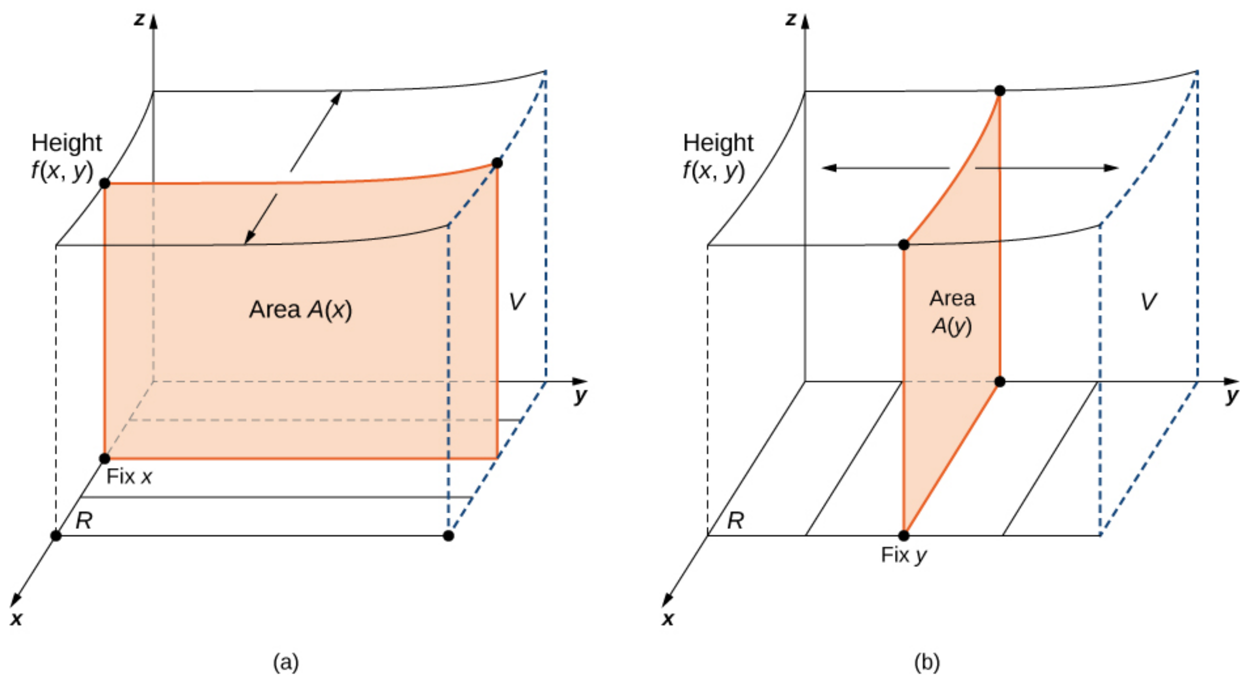
$$\int_c^d \int_a^b f(x,y)dx\,dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)dy\right] dx$ means that we integrate $f(x, y)$ with respect to *y* while holding *x* constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)dx\right] dy$ means that we integrate $f(x, y)$ with respect to *x* while holding *y* constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Note:
Fubini's Theorem
Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, \mathrm{c} \leq \mathrm{y} \leq \mathrm{d}\}$. Then we see from [link] that the double integral of f over the region equals an iterated integral,
Equation:

$$\iint_R f(x,y)dA = \iint_R f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy.$$

More generally, **Fubini's theorem** is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.



(a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example:
Exercise:

Problem:
Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)dA$ where $f(x, y) = x$ and $R = [0, 2] \times [0, 1]$.

Solution:

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dA &= \iint\limits_R f(x,y)dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=1} 2dy = 2y|_{y=0}^{y=1} = 2.\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see [link]).

Example:
Exercise:

Problem:
Illustrating Properties i and ii

Evaluate the double integral $\iint\limits_R (xy - 3xy^2)dA$ where $R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$.

Solution:

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.
Equation:

$$\begin{aligned}
&\iint_R (xy - 3xy^2)\,dA \\
&= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the int} \\
&= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2 dx\,dy && \text{Convert double integrals to iterated integrals.} \\
&= \int_{y=1}^{y=2} \left(\tfrac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2} \left(\tfrac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x, \text{ holding } y \text{ constant.} \\
&= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integ} \\
&= \int_1^2 y\,dy - 6\int_1^2 y^2 dy && \text{Integrate with respect to } y. \\
&= 2\tfrac{y^2}{2}\Big|_1^2 - 6\tfrac{y^3}{3}\Big|_1^2 \\
&= y^2\Big|_1^2 - 2y^3\Big|_1^2 \\
&= (4-1) - 2(8-1) \\
&= 3 - 2(7) = 3 - 14 = -11.
\end{aligned}$$

Example:
Exercise:

Problem:
Illustrating Property v.

Over the region $R = \{(x,y) | 1 \le x \le 3, 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\,dA$.

Solution:

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.
Equation:

$$\begin{aligned}
\int_1^2\int_1^3 2dx\,dy &= \int_1^2 \left[2x|_1^3\right] dy = \int_1^2 2(2)dy = 4y|_1^2 = 4(2-1) = 4 \\
\int_1^2\int_1^3 13dx\,dy &= \int_1^2 \left[13x|_1^3\right] dy = \int_1^2 13(2)dy = 26y|_1^2 = 26(2-1) = 26.
\end{aligned}$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\,dA \le 26.$

Example:
Exercise:

Problem:
Illustrating Property vi

Evaluate the integral $\iint\limits_R e^y \cos x \, dA$ over the region $R = \left\{(x,y) \middle| 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1\right\}$.

Solution:

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.
Equation:

$$\begin{aligned} \iint\limits_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right) \left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= \left(e^y\big|_0^1\right) \left(\sin x\big|_0^{\pi/2}\right) \\ &= e - 1. \end{aligned}$$

Note:
Exercise:

Problem:

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral
Equation:

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) dy \, dx.$$

b. Show that $0 \le \iint\limits_R \sin \pi x \cos \pi y \, dA \le \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

Solution:

a. 26 b. Answers may vary.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint\limits_R f(x,y)dA$ can be written as $\iint\limits_R f(x,y)dx\,dy$ or $\iint\limits_R f(x,y)dy\,dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example:
Exercise:

Problem:
Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from [link], this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint\limits_R f(x,y)dA$ in two different ways:

a. First integrate with respect to *y* and then with respect to *x*;
b. First integrate with respect to *x* and then with respect to *y*.

Solution:

[link] shows how the calculation works in two different ways.

a. First integrate with respect to *y* and then integrate with respect to *x*:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=3}(3x^2-y)dy\,dx \\ &= \int_{x=0}^{x=2}\left(\int\limits_{y=0}^{y=3}(3x^2-y)dy\right)dx = \int_{x=0}^{x=2}\left[3x^2y-\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx \\ &= \int_{x=0}^{x=2}(9x^2-\tfrac{9}{2})dx = 3x^3-\tfrac{9}{2}x\Big|_{x=0}^{x=2}=15.\end{aligned}$$

b. First integrate with respect to *x* and then integrate with respect to *y*:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dA &= \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)dx\,dy \\ &= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)dx\right)dy = \int_{y=0}^{y=3}\left[x^3-xy\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=3}(8-2y)dy = 8y-y^2\Big|_{y=0}^{y=3}=15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y) = 3x^2 - y$ over the region

$R = [0, 2] \times [0, 3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

Note:
Exercise:

Problem: Evaluate $\displaystyle\int_{y=-3}^{y=2} \int_{x=3}^{x=5} \left(2 - 3x^2 + y^2\right) dx\, dy.$

Solution:

$-\frac{1340}{3}$

Hint

Use Fubini’s theorem.

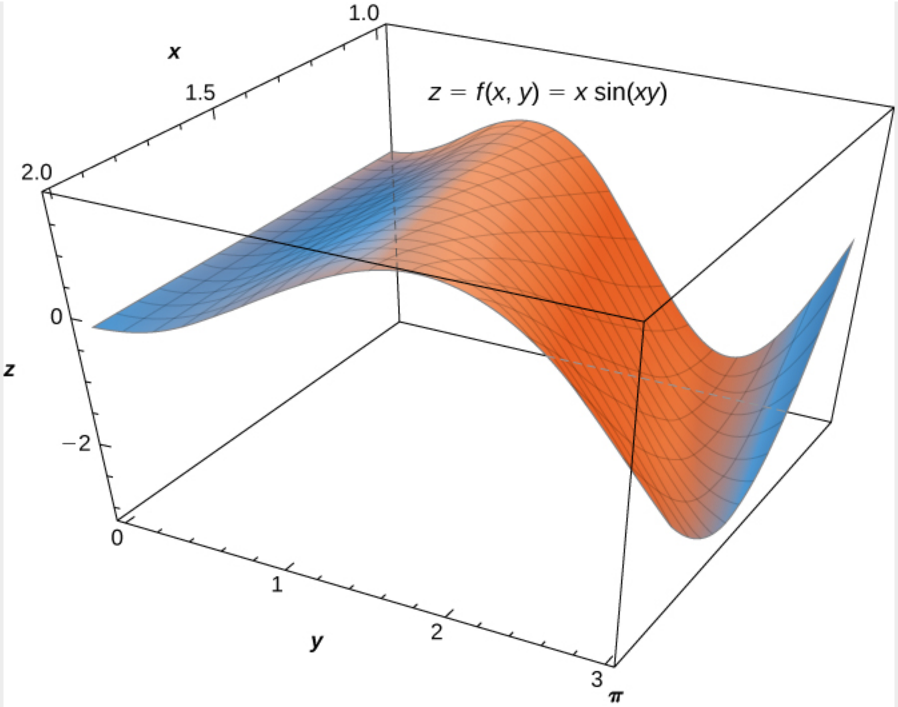
In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example:
Exercise:

Problem:
Switching the Order of Integration

Consider the double integral $\displaystyle\iint_R x \sin(xy) dA$ over the region $R = \{(x, y) | 0 \le x \le 3, 0 \le y \le 2\}$ ([link]).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.



The function $z = f(x, y) = x\sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

Solution:

a. We can express $\iint\limits_R x\sin(xy)dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.
Equation:

$$\begin{aligned} &\iint\limits_R x\sin(xy)dA \\ &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx \qquad \text{Integrate first with respect to } y. \\ &= \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)dx\,dy \qquad \text{Integrate first with respect to } x. \end{aligned}$$

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\,dy$. Hence the inner integral is simply $\int \sin u\,du$ and we can change the limits to be functions of x,
Equation:

$$\iint_R x\sin(xy)dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx = \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$.
Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so
Equation:

$$\iint_R x\sin(xy)dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.
Equation:

$$\begin{aligned}\iint_R x\sin(xy)dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u|_{u=x}^{u=2x}\right]dx = \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)dx \\ &= -\tfrac{1}{2}\sin 2x + \sin x|_{x=0}^{x=\pi} = 0.\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral $\iint_R xe^{xy}dA$ where $R = [0,1] \times [0,\ln 5]$.

Solution:

$\frac{4-\ln 5}{\ln 5}$

Hint

Integrate with respect to y first.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by

integrating the constant function $f(x,y)=1$ over the region R.

Note:
Definition
The area of the region R is given by $A(R)=\iint\limits_R 1dA.$

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example:
Exercise:

Problem:
Finding Area Using a Double Integral

Find the area of the region $R=\{(x,y)|0\le x\le 3, 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution:

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:
Equation:

$$A(R)=\int_0^2\int_0^3 1dx\,dy=\int_0^2\left[x|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y|_0^2=3(2)=6.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

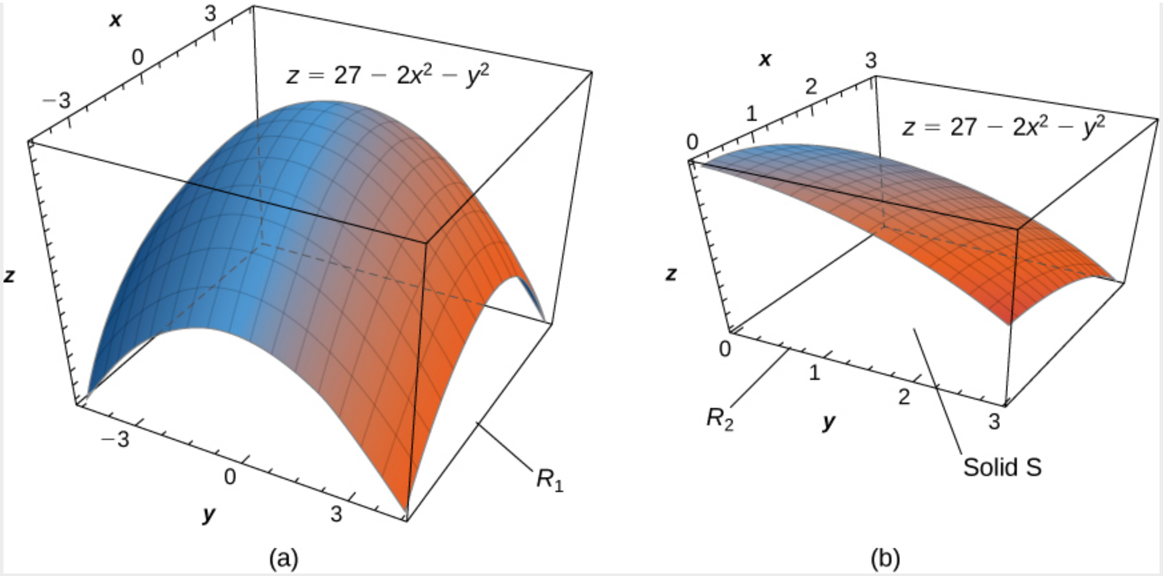
Example:
Exercise:

Problem:
Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2+y^2+z=27$, the planes $x=3$ and $y=3$, and the three coordinate planes.

Solution:

First notice the graph of the surface $z=27-2x^2-y^2$ in [link](a) and above the square region $R_1=[-3,3]\times[-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2+y^2+z=27$, the planes $x=3$ and $y=3$, and the three coordinate planes.



(a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R_1 = [-3, 3] \times [-3, 3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R_2 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in [link](b). We determine the volume V by evaluating the double integral over R_2:

Equation:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R \left(27 - 2x^2 - y^2\right) dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} \left(27 - 2x^2 - y^2\right) dx\,dy && \text{Convert to iterated integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \tfrac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} \left(64 - 3y^2\right) dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

Note:
Exercise:

Problem:

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Solution:

$\frac{\pi}{2}$

Hint

Graph the function, set up the integral, and use an iterated integral.

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

Equation:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Similarly, we can define the average value of a function of two variables over a region *R*. The main difference is that we divide by an area instead of the width of an interval.

Note:
Definition
The average value of a function of two variables over a region R is

Equation:

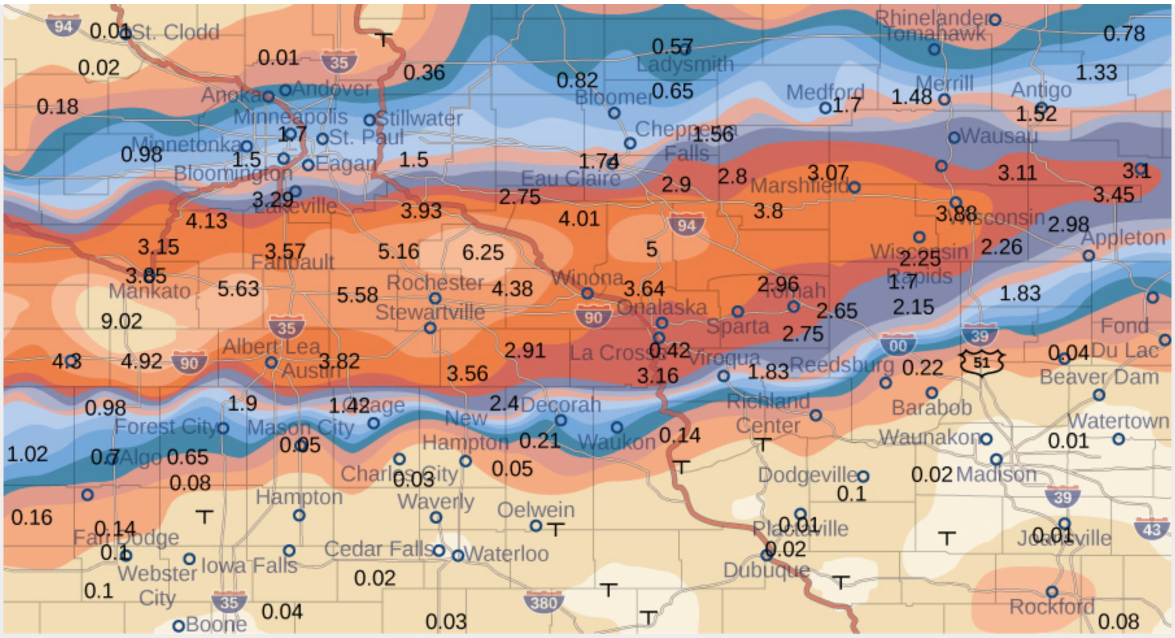
$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)dA.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example:
Exercise:

Problem:
Calculating Average Storm Rainfall

The weather map in [link] shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.



Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution:

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles $(m = 2$ and $n = 3)$, as shown in [link]. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

Equation:

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume (x_{ij}^*, y_{ij}^*) are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

At (x_{11}, y_{11}) the rainfall is 0.08.

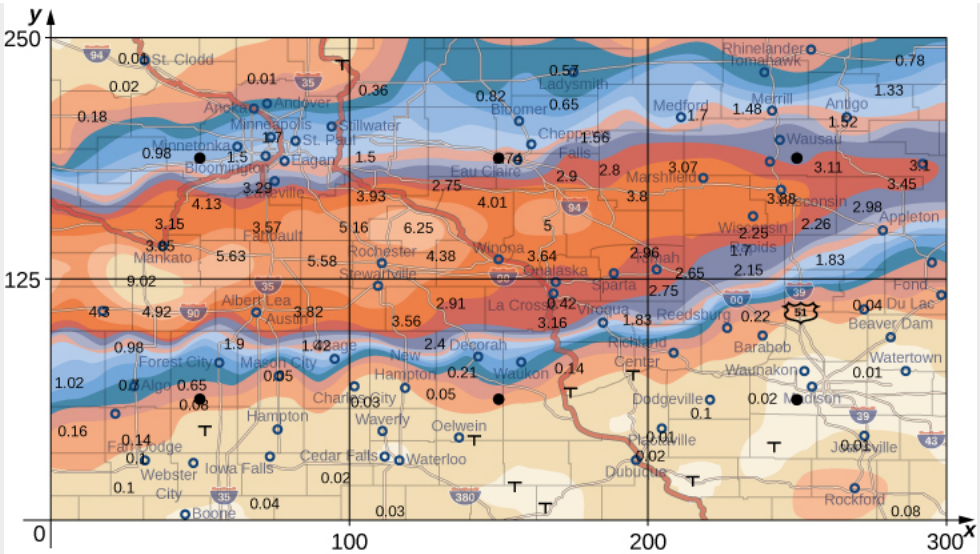
At (x_{12}, y_{12}) the rainfall is 0.08.

At (x_{13}, y_{13}) the rainfall is 0.01.

At (x_{21}, y_{21}) the rainfall is 1.70.

At (x_{22}, y_{22}) the rainfall is 1.74.

At (x_{23}, y_{23}) the rainfall is 3.00.



Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

According to our definition, the average storm rainfall in the entire area during those two days was

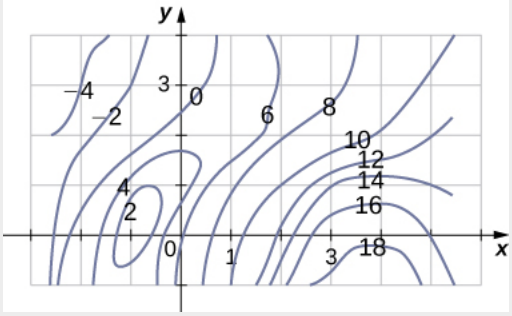
Equation:

$$
\begin{aligned}
f_{\text{ave}} &= \frac{1}{\text{Area } R}\iint\limits_R f(x,y)\,dx\,dy = \frac{1}{75000}\iint\limits_R f(x,y)\,dx\,dy \\
&\cong \frac{1}{75{,}000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&\cong \tfrac{1}{75{,}000}\left[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A\right. \\
&\quad \left. + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\right] \\
&\cong \tfrac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \\
&\cong \tfrac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \\
&\cong \tfrac{5}{30}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \\
&\cong 1.10.
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Note:
Exercise:

Problem: A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint\limits_R f(x, y)dA$.

b. Estimate the average value of the function $f(x, y)$.

Solution:

Answers to both parts a. and b. may vary.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x, y)$.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- **Double integral**
$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
- **Iterated integral**
$$\int_a^b \int_c^d f(x, y)dx\, dy = \int_a^b \left[\int_c^d f(x, y)dy\right] dx$$
or
$$\int_c^d \int_b^a f(x, y)dx\, dy = \int_c^d \left[\int_a^b f(x, y)dx\right] dy$$
- **Average value of a function of two variables**
$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint\limits_R f(x, y)dx\, dy$$

In the following exercises, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane

$z = 0$.
Exercise:

Problem: $f(x, y) = 4x + 2y + 8xy$

Solution:

27.

Exercise:

Problem: $f(x, y) = 16x^2 + \frac{y}{2}$

In the following exercises, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region *R* by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the subrectangles of the partition.
Exercise:

Problem: $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

Solution:

0.

Exercise:

Problem: $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times \left[0, \frac{\pi}{2}\right]$

Exercise:

Problem:

Use the midpoint rule with $m = n = 2$ to estimate $\iint\limits_R f(x, y)dA$, where the values of the function *f* on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Solution:

21.3.

Exercise:

Problem:

The values of the function *f* on the rectangle $R = [0,2] \times [7,9]$ are given in the following table. Estimate the double integral $\iint\limits_R f(x,y)dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the subsquares of *R*.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

Exercise:

Problem:

The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0,4] \times [0,4]$.
b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Solution:

a. 28 ft^3 b. 1.75 ft.

Exercise:

Problem:

The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the subsquares of *R*.
b. Find the average depth of the hole.

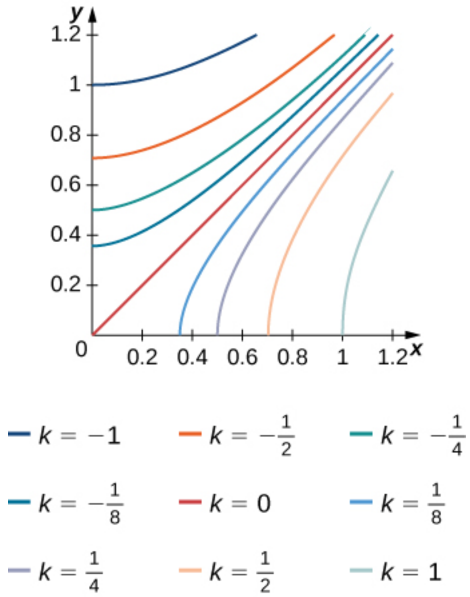
	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

Exercise:

Problem:

The level curves $f(x, y) = k$ of the function *f* are given in the following graph, where *k* is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint\limits_R f(x, y)dA$, where $R = [0.2, 1] \times [0, 0.8]$.
b. Estimate the average value of the function *f* on *R*.



Solution:

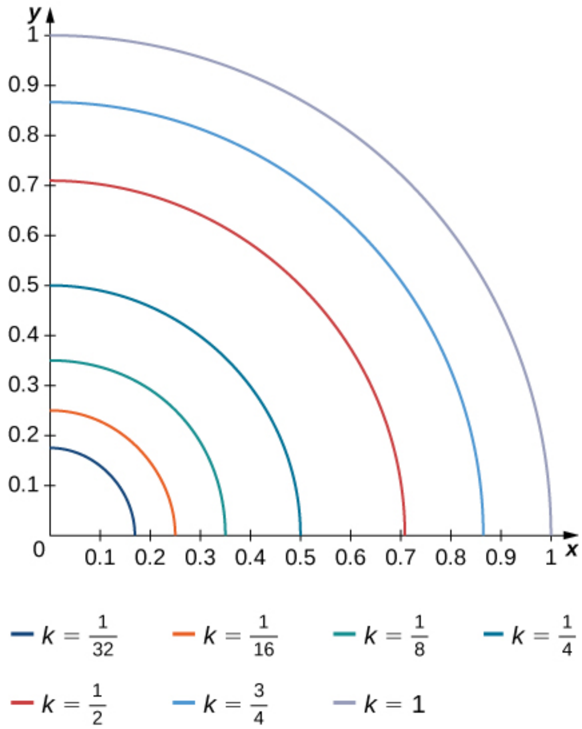
a. 0.112 b. $f_{\text{ave}} \simeq 0.175$; here $f(0.4, 0.2) \simeq 0.1$, $f(0.2, 0.6) \simeq -0.2$, $f(0.8, 0.2) \simeq 0.6$, and $f(0.8, 0.6) \simeq 0.2$.

Exercise:

Problem:

The level curves $f(x, y) = k$ of the function *f* are given in the following graph, where *k* is a constant.

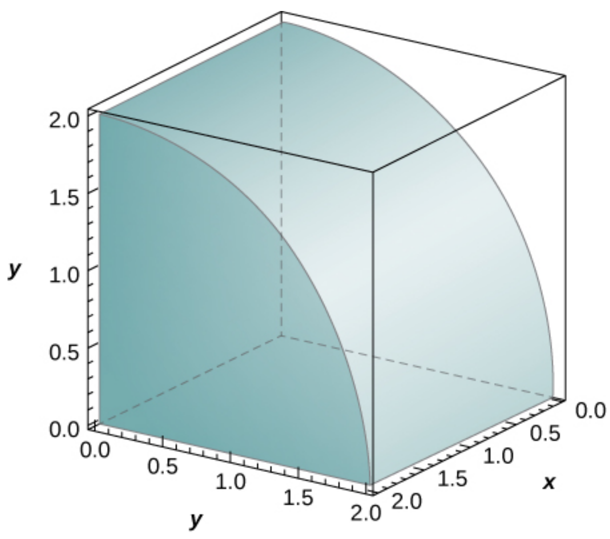
a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint\limits_R f(x, y)dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
b. Estimate the average value of the function *f* on *R*.



Exercise:

Problem:

The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)dA$, where $f(x, y) = \sqrt{4 - y^2}$, by finding the volume of the corresponding solid.



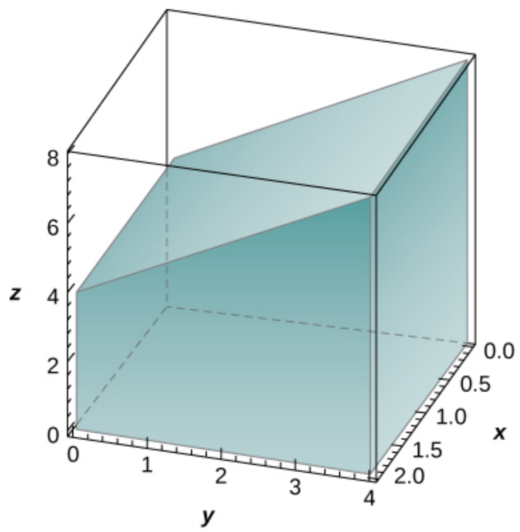
Solution:

$2\pi.$

Exercise:

Problem:

The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint\limits_R f(x, y)dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the following exercises, calculate the integrals by interchanging the order of integration.

Exercise:

Problem: $\displaystyle\int_{-1}^{1}\left(\int_{-2}^{2}(2x + 3y + 5)dx\right)dy$

Solution:

40.

Exercise:

Problem: $\displaystyle\int_{0}^{2}\left(\int_{0}^{1}(x + 2e^y - 3)dx\right)dy$

Exercise:

Problem: $\displaystyle\int_{1}^{27}\left(\int_{1}^{2}(\sqrt[3]{x} + \sqrt[3]{y})dy\right)dx$

Solution:

$\frac{81}{2} + 39\sqrt[3]{2}.$

Exercise:

Problem: $\int\limits_{1}^{16}\left(\int\limits_{1}^{8}\left(\sqrt[4]{x}+2\sqrt[3]{y}\right)dy\right)dx$

Exercise:

Problem: $\int\limits_{\ln 2}^{\ln 3}\left(\int\limits_{0}^{1}e^{x+y}dy\right)dx$

Solution:

$e-1.$

Exercise:

Problem: $\int\limits_{0}^{2}\left(\int\limits_{0}^{1}3^{x+y}dy\right)dx$

Exercise:

Problem: $\int\limits_{1}^{6}\left(\int\limits_{2}^{9}\frac{\sqrt{y}}{x^2}dy\right)dx$

Solution:

$15-\frac{10\sqrt{2}}{9}.$

Exercise:

Problem: $\int\limits_{1}^{9}\left(\int\limits_{4}^{2}\frac{\sqrt{x}}{y^2}dy\right)dx$

In the following exercises, evaluate the iterated integrals by choosing the order of integration.

Exercise:

Problem: $\int\limits_{0}^{\pi}\int\limits_{0}^{\pi/2}\sin(2x)\cos(3y)dx\,dy$

Solution:

$0.$

Exercise:

Problem: $\int\limits_{\pi/12}^{\pi/8} \int\limits_{\pi/4}^{\pi/3} [\cot x + \tan(2y)] dx\, dy$

Exercise:

Problem: $\int\limits_{1}^{e} \int\limits_{1}^{e} \left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y)\right] dx\, dy$

Solution:

$(e-1)(1+\sin 1-\cos 1).$

Exercise:

Problem: $\int\limits_{1}^{e} \int\limits_{1}^{e} \frac{\sin(\ln x)\cos(\ln y)}{xy} dx\, dy$

Exercise:

Problem: $\int\limits_{1}^{2} \int\limits_{1}^{2} \left(\frac{\ln y}{x} + \frac{x}{2y+1}\right) dy\, dx$

Solution:

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2\ln^2 2 - \ln 2.$

Exercise:

Problem: $\int\limits_{1}^{e} \int\limits_{1}^{2} x^2 \ln(x) dy\, dx$

Exercise:

Problem: $\int\limits_{1}^{\sqrt{3}} \int\limits_{1}^{2} y \arctan\left(\frac{1}{x}\right) dy\, dx$

Solution:

$\frac{1}{8}\left[\left(2\sqrt{3}-3\right)\pi + 6\ln 2\right].$

Exercise:

Problem: $\int\limits_{0}^{1} \int\limits_{0}^{1/2} (\arcsin x + \arcsin y) dy\, dx$

Exercise:

Problem: $\int_0^1 \int_1^2 xe^{x+4y} dy\, dx$

Solution:

$\frac{1}{4}e^4\left(e^4-1\right).$

Exercise:

Problem: $\int_1^2 \int_0^1 xe^{x-y} dy\, dx$

Exercise:

Problem: $\int_1^e \int_1^e \left(\frac{\ln y}{\sqrt{y}}+\frac{\ln x}{\sqrt{x}}\right) dy\, dx$

Solution:

$4(e-1)\left(2-\sqrt{e}\right).$

Exercise:

Problem: $\int_1^e \int_1^e \left(\frac{x \ln y}{\sqrt{y}}+\frac{y \ln x}{\sqrt{x}}\right) dy\, dx$

Exercise:

Problem: $\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\, dx$

Solution:

$-\frac{\pi}{4}+\ln\left(\frac{5}{4}\right)-\frac{1}{2}\ln 2+\arctan 2.$

Exercise:

Problem: $\int_0^1 \int_1^2 \frac{y}{x+y^2} dy\, dx$

In the following exercises, find the average value of the function over the given rectangles.

Exercise:

Problem: $f(x,y)=-x+2y,\ R=[0,1]\times[0,1]$

Solution:

$\frac{1}{2}$.

Exercise:

Problem: $f(x, y) = x^4 + 2y^3$, $R = [1, 2] \times [2, 3]$

Exercise:

Problem: $f(x, y) = \sinh x + \sinh y$, $R = [0, 1] \times [0, 2]$

Solution:

$\frac{1}{2}(2\cosh 1 + \cosh 2 - 3)$.

Exercise:

Problem: $f(x, y) = \arctan(xy)$, $R = [0, 1] \times [0, 1]$

Exercise:

Problem:

Let f and g be two continuous functions such that $0 \leq m_1 \leq f(x) \leq M_1$ for any $x \in [a, b]$ and $0 \leq m_2 \leq g(y) \leq M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \leq \int_a^b \int_c^d f(x) g(y) dy\, dx \leq M_1 M_2 (b-a)(c-d).$$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

Exercise:

Problem: $\frac{1}{e^2} \leq \iint\limits_R e^{-x^2-y^2} dA \leq 1$, where $R = [0, 1] \times [0, 1]$

Exercise:

Problem: $\frac{\pi^2}{144} \leq \iint\limits_R \sin x \cos y \, dA \leq \frac{\pi^2}{48}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Exercise:

Problem: $0 \leq \iint\limits_R e^{-y} \cos x \, dA \leq \frac{\pi}{2}$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

Exercise:

Problem: $0 \leq \iint\limits_R (\ln x)(\ln y) dA \leq (e-1)^2$, where $R = [1, e] \times [1, e]$

Exercise:

Problem:

Let *f* and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$$(m_1 + m_2)(b-a)(c-d) \le \int_a^b \int_c^d [f(x) + g(y)]dy\,dx \le (M_1 + M_2)(b-a)(c-d).$$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

Exercise:

Problem: $\frac{2}{e} \le \iint_R \left(e^{-x^2} + e^{-y^2}\right) dA \le 2$, where $R = [0, 1] \times [0, 1]$

Exercise:

Problem: $\frac{\pi^2}{36} \le \iint_R (\sin x + \cos y) dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Exercise:

Problem: $\frac{\pi}{2} e^{-\pi/2} \le \iint_R \left(\cos x + e^{-y}\right) dA \le \pi$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

Exercise:

Problem: $\frac{1}{e} \le \iint_R \left(e^{-y} - \ln x\right) dA \le 2$, where $R = [0, 1] \times [0, 1]$

In the following exercises, the function *f* is given in terms of double integrals.

a. Determine the explicit form of the function *f*.
b. Find the volume of the solid under the surface $z = f(x, y)$ and above the region *R*.
c. Find the average value of the function *f* on *R*.
d. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{\text{ave}}$ in the same system of coordinates.

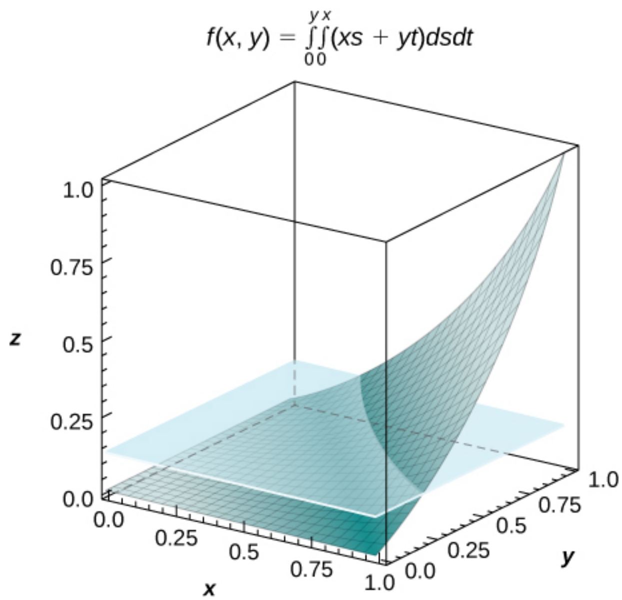
Exercise:

Problem: **[T]** $f(x, y) = \int_0^y \int_0^x (xs + yt) ds\, dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$

Solution:

a. $f(x, y) = \frac{1}{2} xy \left(x^2 + y^2\right)$ b. $V = \int_0^1 \int_0^1 f(x, y) dx\, dy = \frac{1}{8}$ c. $f_{\text{ave}} = \frac{1}{8}$;

d.



Exercise:

Problem: **[T]** $f(x,y) = \int_0^x \int_0^y [\cos(s) + \cos(t)] dt\, ds$, where $(x,y) \in R = [0,3] \times [0,3]$

Exercise:

Problem: Show that if f and g are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d [f(x) + g(y)] dy\, dx = (d-c) \int_a^b f(x) dx$$

$$+ \int_a^b \int_c^d g(y) dy\, dx = (b-a) \int_c^d g(y) dy + \int_c^d \int_a^b f(x) dx\, dy.$$

Exercise:

Problem:

Show that $\int_a^b \int_c^d yf(x) + xg(y) dy\, dx = \frac{1}{2}(d^2 - c^2)\left(\int_a^b f(x) dx\right) + \frac{1}{2}(b^2 - a^2)\left(\int_c^d g(y) dy\right)$.

Exercise:

Problem: **[T]** Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$.

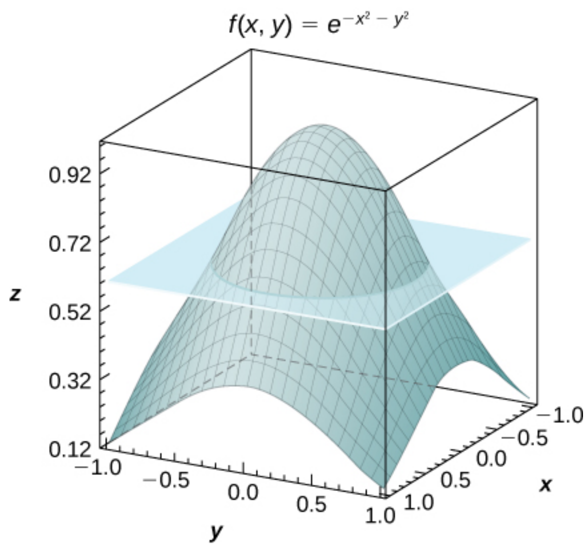
a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$.

Round your answers to the nearest hundredths.

b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{\text{ave}}$.

Solution:

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$ b. $f_{\text{ave}} = e^{-0.5} \simeq 0.61$;
c.



Exercise:

Problem: [T] Consider the function $f(x, y) = \sin\left(x^2\right)\cos\left(y^2\right)$, where $(x, y) \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin\left(x^2\right)\cos\left(y^2\right) dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin\left(x^2\right)\cos\left(y^2\right) dA$ and the plane $z = f_{\text{ave}}$.

In the following exercises, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x, y)$ and above the region *R*.
b. Determine the limit of the volumes of the solids S_n as *n* increases without bound.

Exercise:

Problem: $f(x, y) = x^n + y^n + xy, (x, y) \in R = [0, 1] \times [0, 1]$

Solution:

a. $\frac{2}{n+1} + \frac{1}{4}$ b. $\frac{1}{4}$

Exercise:

Problem: $f(x, y) = \frac{1}{x^n} + \frac{1}{y^n}, (x, y) \in R = [1, 2] \times [1, 2]$

Exercise:

Problem:

Show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is $f_{\text{ave}} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)$, where $\left(x_{ij}^*, y_{ij}^*\right)$ are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Exercise:

Problem:

Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is approximated by

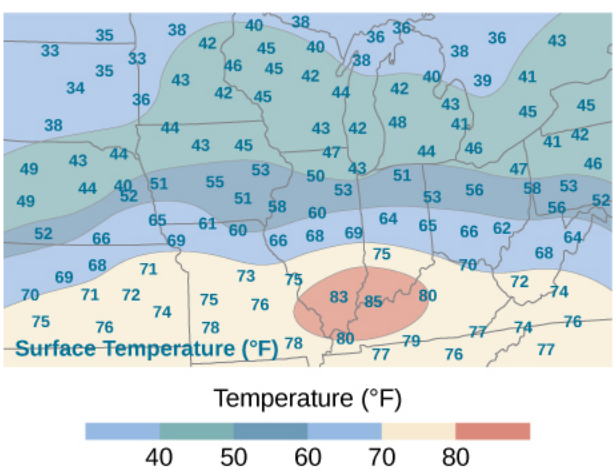
Equation:

$$f_{\text{ave}} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i-1} + x_i), \frac{1}{2}(y_{j-1} + y_j)\right).$$

Exercise:

Problem:

An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.



Solution:

56.5° F; here $f(x_1^*, y_1^*) = 71$, $f(x_2^*, y_1^*) = 72$, $f(x_2^*, y_1^*) = 40$, $f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d]$, respectively.

Glossary

double integral

of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum, $\iint\limits_R f(x,y)dA = \lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\Delta A.$

double Riemann sum

of the function $f(x,y)$ over a rectangular region R is $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\Delta A$ where R is divided into smaller subrectangles R_{ij} and (x_{ij}^{*}, y_{ij}^{*}) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$, then the double integral of f over the region equals an iterated integral, $\iint\limits_R f(x,y)dy\,dx = \int_a^b \int_c^d f(x,y)dx\,dy = \int_c^d \int_a^b f(x,y)dx\,dy$

iterated integral

for a function $f(x,y)$ over the region R is

a. $\int_a^b \int_c^d f(x,y)dx\,dy = \int_a^b \left[\int_c^d f(x,y)dy\right]dx,$

b. $\int_c^d \int_b^a f(x,y)dx\,dy = \int_c^d \left[\int_a^b f(x,y)dx\right]dy,$

where $a, b, c,$ and d are any real numbers and $R = [a,b] \times [c,d]$

Double Integrals over General Regions

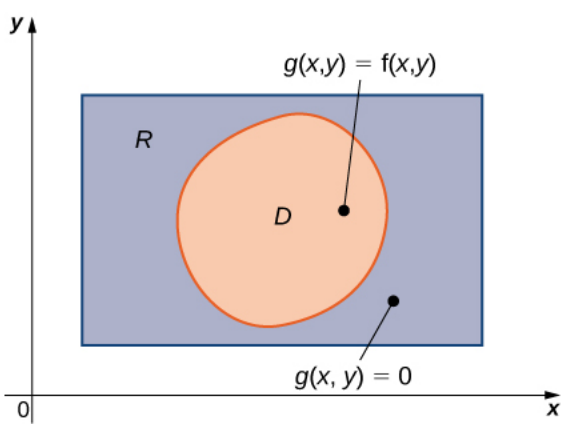
- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

In Double Integrals over Rectangular Regions, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in [link]. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D, that is, a rectangular region R exists such that D is a subset of $R\,(D \subseteq R)$.



For a region D that is a subset of R, we can define a function $g(x, y)$ to equal $f(x, y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f\,(x, y)$ is defined on a general planar bounded region D as in [link]. In order to develop double integrals of f over D, we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g\,(x, y)$ on R as follows:

Equation:

$$g(x,y)=\begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in Double Integrals over Rectangular Regions used an integrable function $f(x,y)$, we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

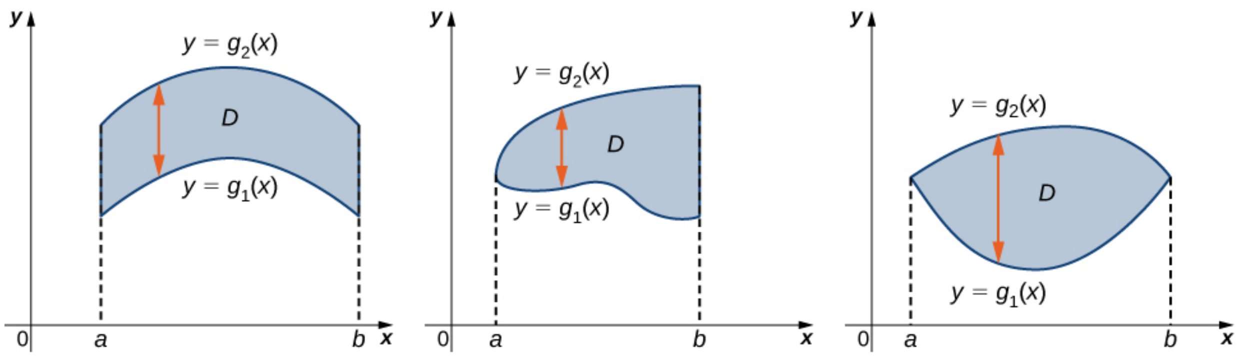
We consider two types of planar bounded regions.

Note:
Definition
A region D in the (x,y)-plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is ([link]),
Equation:

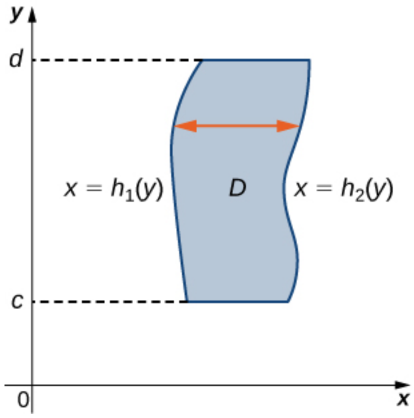
$$D=\{(x,y)|a\le x\le b, g_1(x)\le y\le g_2(x)\}.$$

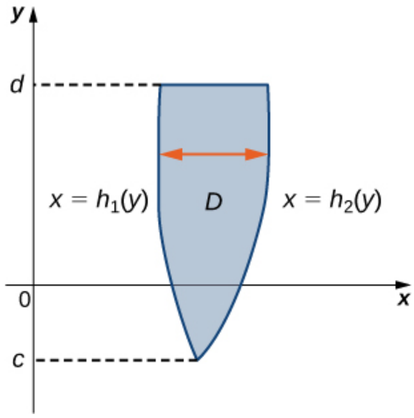
A region D in the xy plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is ([link]),
Equation:

$$D=\{(x,y)|c\le y\le d, h_1(y)\le x\le h_2(y)\}.$$



A Type I region lies between two vertical lines and the graphs of two functions of x.



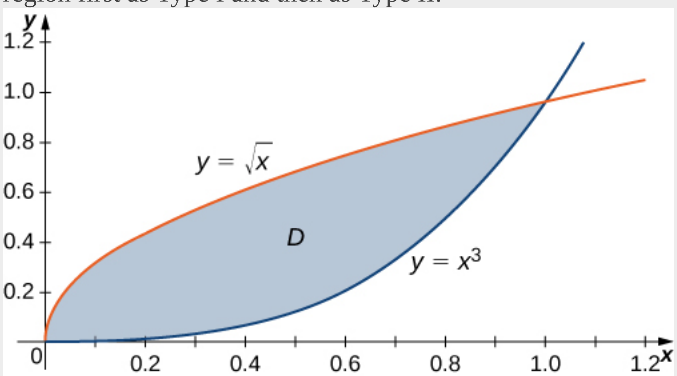


A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example:
Exercise:

Problem:
Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ ([link]). Describe the region first as Type I and then as Type II.



Region D can be described as Type I or as Type II.

Solution:

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) | 0 \le x \le 1, x^3 \le y \le \sqrt{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by

$x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\left\{(x, y)|0 \le y \le 1, y^2 \le x \le \sqrt[3]{y}\right\}$.

Note:
Exercise:

Problem:

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Solution:

Type I and Type II are expressed as $\left\{(x, y)|0 \le x \le 2, x^2 \le y \le 2x\right\}$ and $\left\{(x, y)|0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\right\}$, respectively.

Hint

Graph the functions, and draw vertical and horizontal lines.

Double Integrals over Nonrectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Note:
Double Integrals over Nonrectangular Regions
Suppose $g\,(x, y)$ is the extension to the rectangle R of the function $f\,(x, y)$ defined on the regions D and R as shown in [link] inside R. Then $g\,(x, y)$ is integrable and we define the double integral of $f\,(x, y)$ over D by
Equation:

$$\iint\limits_{D} f\,(x, y) dA = \iint\limits_{R} g\,(x, y) dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint\limits_{R} g\,(x, y) dA$ has been discussed in the preceding section. Also, the equality works because the values of $g\,(x, y)$ are 0 for any point (x, y) that lies outside D, and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Note:
Fubini’s Theorem (Strong Form)
For a function $f\left(x,y\right)$ that is continuous on a region D of Type I, we have
Equation:

$$\iint\limits_D f(x,y)dA = \iint\limits_D f(x,y)dy\,dx = \int\limits_a^b \left[\int\limits_{g_1(x)}^{g_2(x)} f(x,y)dy\right]dx.$$

Similarly, for a function $f\left(x,y\right)$ that is continuous on a region D of Type II, we have
Equation:

$$\iint\limits_D f(x,y)dA = \iint\limits_D f(x,y)dx\,dy = \int\limits_c^d \left[\int\limits_{h_1(y)}^{h_2(y)} f(x,y)dx\right]dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f\left(x,y\right)$ with x being held constant and the limits of integration being $g_1\left(x\right)$ and $g_2\left(x\right)$. In the inner integral in the second expression, we integrate $f\left(x,y\right)$ with y being held constant and the limits of integration are $h_1\left(x\right)$ and $h_2\left(x\right)$.

Example:
Exercise:

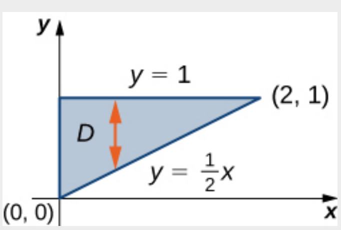
Problem:
Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint\limits_D x^2 e^{xy} dA$ where D is shown in [link].

Solution:

First construct the region D as a Type I region ([link]). Here $D = \left\{(x,y)|0 \le x \le 2, \frac{1}{2}x \le y \le 1\right\}$. Then we have
Equation:

$$\iint\limits_D x^2 e^{xy} dA = \int\limits_{x=0}^{x=2} \int\limits_{y=1/2x}^{y=1} x^2 e^{xy} dy\,dx.$$



We can express region D

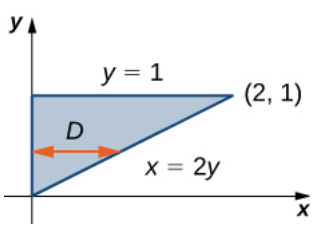
as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$, between the lines $x = 0$ and $x = 2$.

Therefore, we have

Equation:

$$\begin{aligned}
\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy} dy\, dx &= \int_{x=0}^{x=2} \left[\int_{y=1/2x}^{y=1} x^2 e^{xy} dy \right] dx && \text{Iterated integral for a Type I region.} \\
&= \int_{x=0}^{x=2} \left[x^2 \frac{e^{xy}}{x} \right] \Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \text{ using } u\text{-substitution with } u = xy \text{ where } x \text{ is held constant.} \\
&= \int_{x=0}^{x=2} \left[xe^x - xe^{x^2/2} \right] dx && \text{Integrate with respect to } x \text{ using } u\text{-substitution with } u = \frac{1}{2}x^2. \\
&= \left[xe^x - e^x - e^{\frac{1}{2}x^2} \right] \Big|_{x=0}^{x=2} = 2
\end{aligned}$$

In [link], we could have looked at the region in another way, such as $D = \{(x, y) | 0 \le y \le 1, 0 \le x \le 2y\}$ ([link]).



This is a Type II region and the integral would then look like

Equation:

$$\iint_D x^2 e^{xy} dA = \int_{y=0}^{y=1} \int_{x=0}^{x=2y} x^2 e^{xy} dx\, dy.$$

However, if we integrate first with respect to x, this integral is lengthy to compute because we have to use integration by parts twice.

Example:

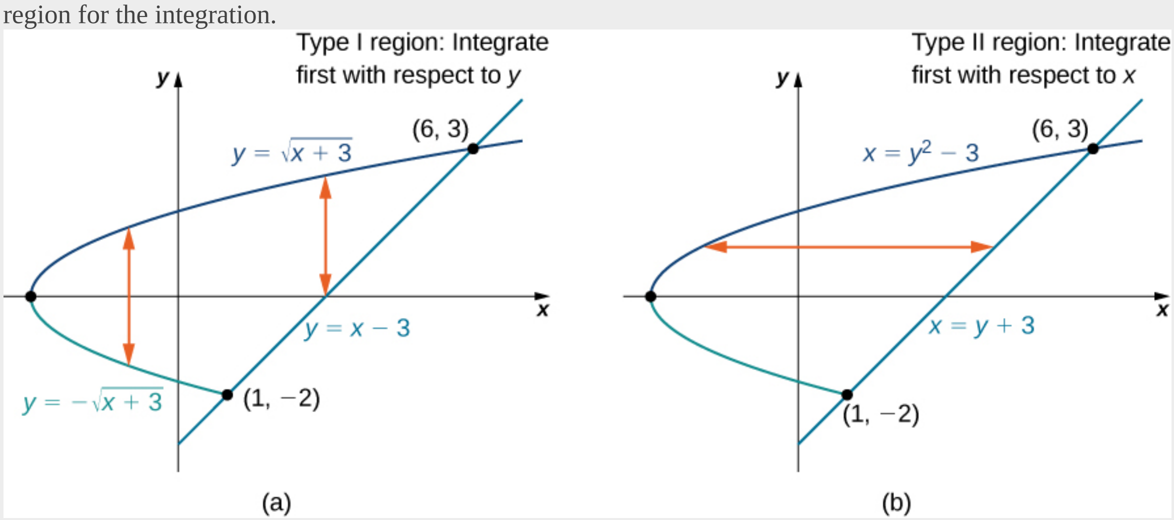
Exercise:

Problem:
Evaluating an Iterated Integral over a Type II Region

Evaluate the integral $\iint\limits_D (3x^2 + y^2)\,dA$ where $= \{(x, y) | -2 \le y \le 3, y^2 - 3 \le x \le y + 3\}$.

Solution:

Notice that D can be seen as either a Type I or a Type II region, as shown in [link]. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.



The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have
Equation:

$$\begin{aligned}
\iint\limits_D (3x^2 + y^2)\,dA &= \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\,dx\,dy && \text{Iterated integral, T} \\
&= \int_{y=-2}^{y=3} (x^3 + xy^2)\Big|_{y^2-3}^{y+3}\,dy && \text{Integrate with resp} \\
&= \int_{y=-2}^{y=3} \left((y+3)^3 + (y+3)y^2 - (y^2-3)^3 - (y^2-3)y^2\right)dy \\
&= \int_{-2}^{3} (54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6)\,dy && \text{Integrate with resp} \\
&= \left[54y + \tfrac{27y^2}{2} - 4y^3 + \tfrac{y^4}{2} + \tfrac{8y^5}{5} - \tfrac{y^7}{7}\right]\Big|_{-2}^{3} \\
&= \tfrac{2375}{7}.
\end{aligned}$$

Note:
Exercise:

Problem:

Sketch the region D and evaluate the iterated integral $\iint\limits_D xy\,dy\,dx$ where D is the region bounded by the curves $y = \cos x$ and $y = \sin x$ in the interval $[-3\pi/4, \pi/4]$.

Solution:

$\pi/4$

Hint

Express D as a Type I region, and integrate with respect to y first.

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a nonrectangular bounded region on a plane. In particular, property 3 states:

If $R = S \cup T$ and $S \cap T = \emptyset$ except at their boundaries, then
Equation:

$$\iint\limits_R f(x, y) dA = \iint\limits_S f(x, y) dA + \iint\limits_T f(x, y) dA.$$

Similarly, we have the following property of double integrals over a nonrectangular bounded region on a plane.

Note:
Decomposing Regions into Smaller Regions
Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then
Equation:

$$\iint\limits_D f(x, y) dA = \iint\limits_{D_1} f(x, y) dA + \iint\limits_{D_2} f(x, y) dA.$$

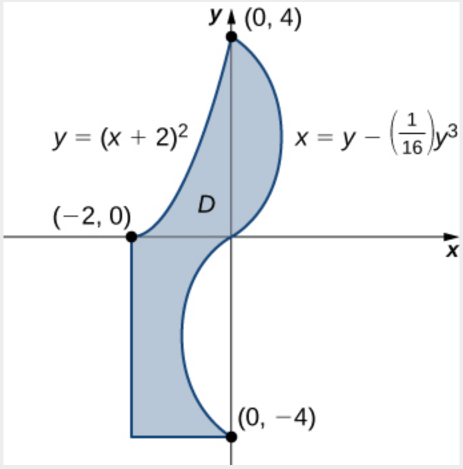
This theorem is particularly useful for nonrectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example:
Exercise:

Problem:
Decomposing Regions

Express the region D shown in [link] as a union of regions of Type I or Type II, and evaluate the integral
Equation:

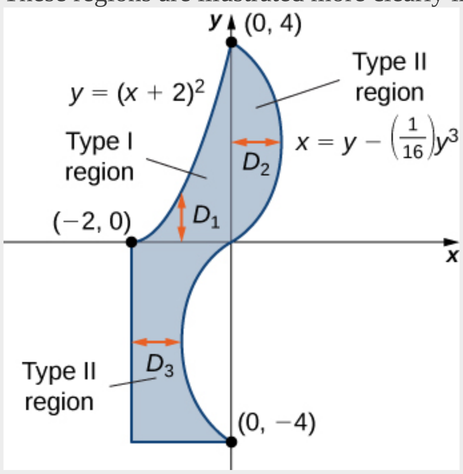
$$\iint_D (2x + 5y)dA.$$



This region can be decomposed into a union of three regions of Type I or Type II.

Solution:

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where,
$D_1 = \left\{(x, y)| - 2 \le x \le 0, 0 \le y \le (x + 2)^2\right\}, D_2 = \left\{(x, y)|0 \le y \le 4, 0 \le x \le \left(y - \frac{1}{16}y^3\right)\right\}.$
These regions are illustrated more clearly in [link].



Breaking the region into three

subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

Equation:

$$\begin{aligned}
\iint\limits_D (2x+5y)dA &= \iint\limits_{D_1}(2x+5y)dA + \iint\limits_{D_2}(2x+5y)dA + \iint\limits_{D_3}(2x+5y)dA \\
&= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2}(2x+5y)dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3}(2+5y)dx\,dy + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3}(\\
&= \int_{x=-2}^{x=0}\left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right] + \int_{y=0}^{y=4}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] \\
&\quad + \int_{y=-4}^{y=0}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] \\
&= \tfrac{40}{3} + \tfrac{1664}{35} - \tfrac{1696}{35} = \tfrac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see the Checkpoint).

Note:
Exercise:

Problem:

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Solution:

$\{(x,y)|0 \le y \le 1, 1 \le x \le e^y\} \cup \{(x,y)|1 \le y \le e, 1 \le x \le 2\} \cup \{(x,y)|e \le y \le e^2, \ln y \le x \le 2\}$

Hint

Sketch the region, and split it into three regions to set it up.

Note:
Exercise:

Problem: Redo [link] using a union of two Type II regions.

Solution:

Same as in the example shown.

Hint

$\{(x,y)|0 \le y \le 4, 2+\sqrt{y} \le x \le \left(y - \frac{1}{16}y^3\right)\} \cup \{(x,y)| -4 \le y \le 0, -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

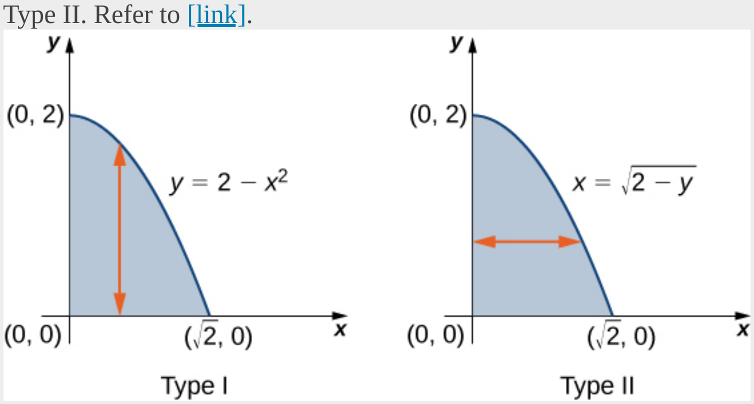
Example:
Exercise:

Problem:
Changing the Order of Integration

Reverse the order of integration in the iterated integral $\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} xe^{x^2}\, dy\, dx$. Then evaluate the new iterated integral.

Solution:

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to [link].



Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$, where x is in the interval $\left[0, \sqrt{2}\right]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2-y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2-y}$.

Hence
Equation:

$$\int_0^{\sqrt{2}}\int_0^{2-x^2} xe^{x^2}\,dy\,dx = \int_0^2\int_0^{\sqrt{2-y}} xe^{x^2}\,dx\,dy$$

Reverse the orc integration the substitution.

$$= \int_0^2 \left[\frac{1}{2}ex^2\Big|_0^{\sqrt{2-y}}\right]dy = \int_0^2 \frac{1}{2}\left(e^{2-y}-1\right)dy = -\tfrac{1}{2}\left(e^{2-y}+y\right)\Big|_0^2$$

$$= \tfrac{1}{2}\left(e^2-3\right).$$

Example:
Exercise:

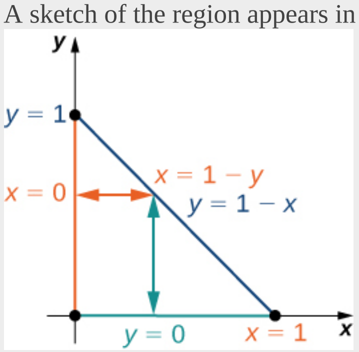
Problem:
Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral $\iint_R f(x,y)\,dx\,dy$ where $z = f(x,y) = x - 2y$ over a triangular region R that has sides on $x = 0, y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution:

A sketch of the region appears in [link].



A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dx\,dy &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)dy\,dx = \int_{x=0}^{x=1}\left[xy-2y^2\right]_{y=0}^{y=1-x}dx \\ &= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]dx = \int_{x=0}^{x=1}\left[-1+3x-2x^2\right]dx = \left[-x+\frac{3}{2}x^2-\frac{2}{3}\right.\end{aligned}$$

b. The other way to do this problem is by first integrating x from $x=0$ to $x=1-y$ horizontally and then integrating y from $y=0$ to $y=1$:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dx\,dy &= \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy \\ &= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]dy \\ &= \left[\tfrac{1}{2}y-\tfrac{3}{2}y^2+\tfrac{5}{6}y^3\right]_{y=0}^{y=1} = -\tfrac{1}{6}.\end{aligned}$$

Note:
Exercise:

Problem:

Evaluate the iterated integral $\iint\limits_D (x^2+y^2)dA$ over the region D in the first quadrant between the functions $y=2x$ and $y=x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Solution:

$\frac{216}{35}$

Hint

Sketch the region and follow [link].

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

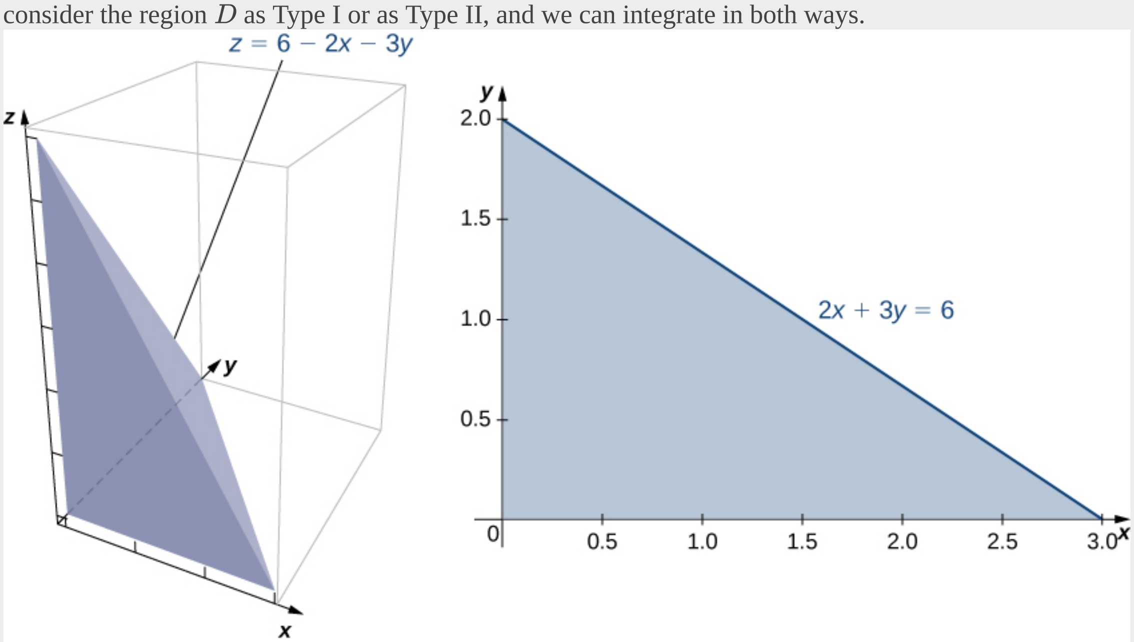
Example:
Exercise:

Problem:
Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x=0, y=0, z=0$, and $2x+3y+z=6$.

Solution:

The solid is a tetrahedron with the base on the xy-plane and a height $z=6-2x-3y$. The base is the region D bounded by the lines, $x=0, y=0$ and $2x+3y=6$ where $z=0$ ([link]). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.



A tetrahedron consisting of the three coordinate planes and the plane $z=6-2x-3y$, with the base bound by $x=0, y=0$, and $2x+3y=6$.

First, consider D as a Type I region, and hence $D=\left\{(x,y)|0\le x\le 3, 0\le y\le 2-\frac{2}{3}x\right\}$.

Therefore, the volume is
Equation:

$$\begin{aligned} V &= \int_{x=0}^{x=3}\int_{y=0}^{y=2-(2x/3)}(6-2x-3y)dy\,dx = \int_{x=0}^{x=3}\left[\left(6y-2xy-\frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)}\right]dx \\ &= \int_{x=0}^{x=3}\left[\frac{2}{3}(x-3)^2\right]dx = 6. \end{aligned}$$

Now consider D as a Type II region, so $D=\left\{(x,y)\middle|0\le y\le 2, 0\le x\le 3-\frac{3}{2}y\right\}$. In this calculation, the volume is
Equation:

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)dx\, dy = \int_{y=0}^{y=2} \left[\left(6x - x^2 - 3xy\right)\Big|_{x=0}^{x=3-(3y/2)} \right] dy$$
$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

Note:
Exercise:

Problem:

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where x is in the interval $[0, 1]$.

Solution:

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Hint

Sketch the region, and describe it as Type I.

Finding the area of a rectangular region is easy, but finding the area of a nonrectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general nonrectangular region, as stated in the next definition.

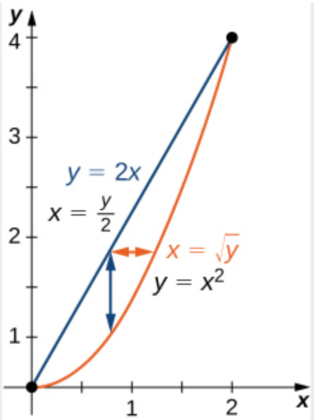
Note:
Definition
The area of a plane-bounded region D is defined as the double integral $\iint_D 1 dA$.

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example:
Exercise:

Problem:
Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant ([link]).



The region bounded by $y = x^2$ and $y = 2x$.

Solution:

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ or $\int_{y=0}^{x=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

Equation:

$$A = \iint_D 1 dx\, dy = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1 dy\, dx = \int_{x=0}^{x=2} \left[y|_{y=x^2}^{y=2x} \right] dx = \int_{x=0}^{x=2} (2x - x^2) dx = x^2 - \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}.$$

Note:
Exercise:

Problem:

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Solution:

$\frac{81}{4}$ square units

Hint

Sketch the region.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Note:
Definition
If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is
Equation:

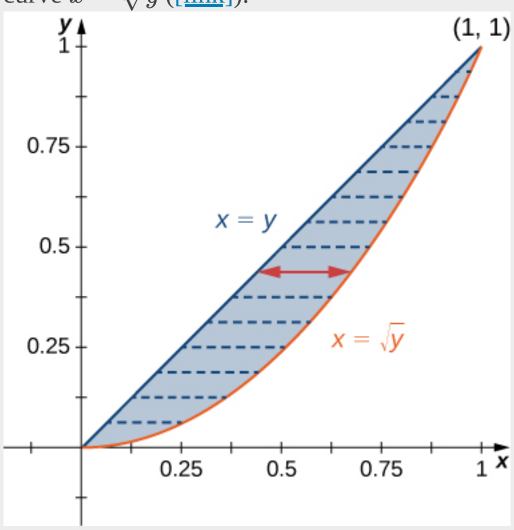
$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y) dA.$$

Note that the area is $A(D) = \iint_D 1 dA.$

Example:
Exercise:

Problem:
Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ ([link]).



The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution:

First find the area $A(D)$ where the region D is given by the figure. We have
Equation:

$$A(D) = \iint_D 1 dA = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1 dx\, dy = \int_{y=0}^{y=1} \left[x \Big|_{x=y}^{x=\sqrt{y}} \right] dy = \int_{y=0}^{y=1} (\sqrt{y} - y) dy = \frac{2}{3} y^{3/2} - \frac{y^2}{2} \Big|_0^1 = \frac{1}{6}.$$

Then the average value of the given function over this region is

Equation:

$$\begin{aligned} f_{ave} &= \tfrac{1}{A(D)} \iint\limits_D f(x,y) dA = \frac{1}{A(D)} \int\limits_{y=0}^{y=1} \int\limits_{x=y}^{x=\sqrt{y}} 7xy^2 dx\, dy = \frac{1}{1/6} \int\limits_{y=0}^{y=1} \left[\frac{7}{2} x^2 y^2 \Big|_{x=y}^{x=\sqrt{y}} \right] dy \\ &= 6 \int\limits_{y=0}^{y=1} \left[\frac{7}{2} y^2 \left(y - y^2 \right) \right] dy = 6 \int\limits_{y=0}^{y=1} \left[\frac{7}{2} \left(y^3 - y^4 \right) \right] dy = \frac{42}{2} \left(\tfrac{y^4}{4} - \tfrac{y^5}{5} \right) \Big|_0^1 = \tfrac{42}{40} = \tfrac{21}{20}. \end{aligned}$$

Note:

Exercise:

Problem:

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Solution:

$\frac{3}{4}$

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Improper Double Integrals

An **improper double integral** is an integral $\iint\limits_D f\, dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y) \mid |x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/\left(1 - x^2 - 2y^2\right)$ over the ellipse $x^2 + 3y^2 \leq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $\iint\limits_D xy\, dA$ where $D = \{(x,y) \mid |x-y| \geq 2\}$;

ii. $\iint\limits_D \frac{1}{1 - x^2 - 2y^2} dA$ where $D = \left\{(x,y) \mid x^2 + 3y^2 \leq 1\right\}$.

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini’s theorem does apply for some types of improper integrals.

Note:

Fubini’s Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by $\{(x,y)\colon a \leq x \leq b, g(x) \leq y \leq h(x)\}$ and also by $\{(x,y)\colon c \leq y \leq d, j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D, then

Equation:

$$\iint_D f\,dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)dy\,dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x,y)dx\,dy.$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example:
Exercise:

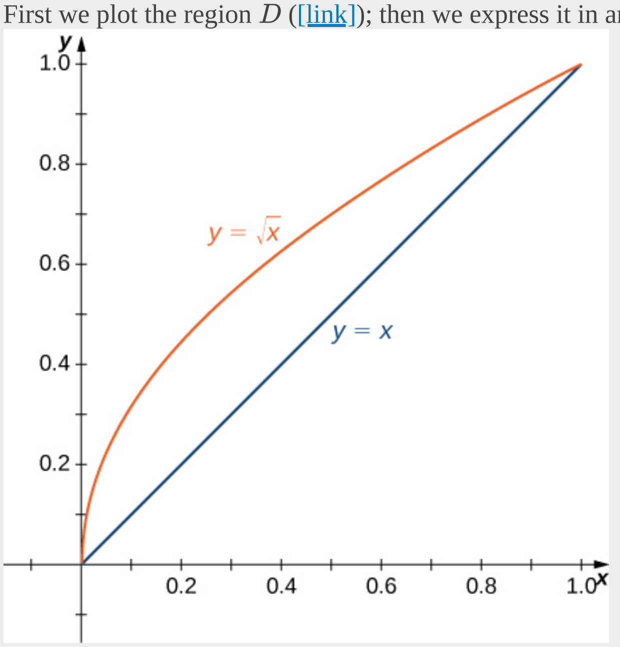
Problem:
Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y)\colon 0 \le x \le 1, x \le y \le \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution:

First we plot the region D ([link]); then we express it in another way.



The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is
Equation:

$$D = \left\{(x, y) \colon 0 \le y \le 1, y^2 \le x \le y\right\}.$$

Thus we can use Fubini’s theorem for improper integrals and evaluate the integral as
Equation:

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy.$$

Therefore, we have
Equation:

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy = \int_{y=0}^{y=1} \frac{e^y}{y} x\Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{ey}{y} \left(y - y^2\right) dy = \int_0^1 (ey - ye^y) dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Note:
Improper Integrals on an Unbounded Region
If R is an unbounded rectangle such as $R = \{(x, y) \colon a \le x \le \infty, c \le y \le \infty\}$, then when the limit exists, we have $\iint\limits_R f(x, y) dA = \lim\limits_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x, y) dy\right) dx = \lim\limits_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x, y) dy\right) dy.$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example:
Exercise:

Problem:
Evaluating a Double Improper Integral

Evaluate the integral $\iint\limits_R xye^{-x^2-y^2} dA$ where R is the first quadrant of the plane.

Solution:

The region R is the first quadrant of the plane, which is unbounded. So
Equation:

$$\iint_R xye^{-x^2-y^2}dA = \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2}dy \right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_{y=0}^{y=d} \left(\int_{x=0}^{x=b} xye^{-x^2-y^2}dy \right) dy$$

$$= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right) = \frac{1}{4}$$

Thus, $\iint_R xye^{-x^2-y^2}dA$ is convergent and the value is $\frac{1}{4}$.

Note:
Exercise:

Problem:

Evaluate the improper integral $\iint_D \frac{y}{\sqrt{1-x^2-y^2}}dA$ where $D = \left\{(x,y)x \geq 0, y \geq 0, x^2+y^2 \leq 1\right\}$.

Solution:

$\frac{\pi}{4}$

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D = \left\{(x,y)\colon 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\right\}$ and integrate using the method of substitution.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Note:
Definition
Consider a pair of continuous random variables X and Y, such as the birthdays of two people or the number of sunny and rainy days in a month. The joint density function f of X and Y satisfies the probability that (X,Y) lies in a certain region D:
Equation:

$$P\left((X,Y) \in D\right) = \iint_D f(x,y)dA.$$

Since the probabilities can never be negative and must lie between 0 and 1, the joint density function satisfies the following inequality and equation:
Equation:

$$f(x,y) \geq 0 \text{ and } \iint_{R^2} f(x,y)dA = 1.$$

Note:
Definition
The variables X and Y are said to be independent random variables if their joint density function is the product of their individual density functions:
Equation:

$$f\left(x,y\right)=f_1\left(x\right)f_2\left(y\right).$$

Example:
Exercise:

Problem:
Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution:

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as
Equation:

$$f(t)=\begin{cases}0 & \text{if } t<0,\\ \frac{1}{m}e^{-t/m} & \text{if } t\geq 0.\end{cases}$$

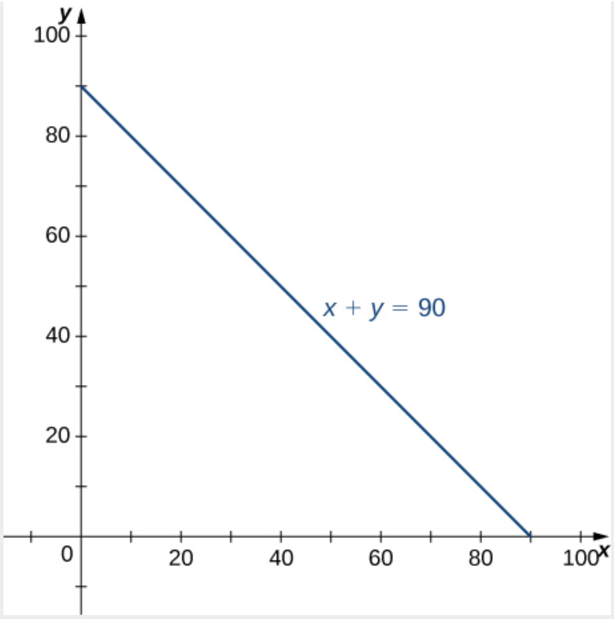
If X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,
Equation:

$$f_1(x)=\begin{cases}0 & \text{if } x<0,\\ \frac{1}{15}e^{-x/15} & \text{if } x\geq 0.\end{cases}\text{and } f_2(y)=\begin{cases}0 & \text{if } y<0,\\ \frac{1}{40}e^{-y/40} & \text{if } y\geq 0.\end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions
Equation:

$$f(x,y)=f_1(x)f_2(y)=\begin{cases}0 & \text{if } x<0 \text{ or } y<0,\\ \frac{1}{600}e^{-x/15}e^{-y/60} & \text{if } x,y\geq 0.\end{cases}$$

We want to find the probability that the combined time $X+Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x+y=90$ ([link]).



The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is
Equation:

$$P(X+Y \leq 90) = P((X,Y) \in D) = \iint\limits_D f(x,y)dA = \iint\limits_D \frac{1}{600} e^{-x/15} e^{-y/40} dA.$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so
Equation:

$$\begin{aligned} D &= \{(x,y) | 0 \leq x \leq 90, 0 \leq y \leq 90 - x\}, \\ P(X+Y \leq 90) &= \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-x/15} e^{-y/40} dx\, dy = \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-x/15} e^{-y/40} dx\, dy \\ &= \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-(x/15 + y/40)} dx\, dy = 0.8328. \end{aligned}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Note:
Definition
In probability theory, we denote the expected values $E(X)$ and $E(Y)$, respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by
Equation:

$$E(X) = \iint_S x f(x, y) dA \text{ and } E(Y) = \iint_S y f(x, y) dA,$$

where S is the sample space of the random variables X and Y.

Example:
Exercise:

Problem:
Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in [link].

Solution:

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is
Equation:

$$\begin{aligned} E(X) &= \iint_S x \frac{1}{600} e^{-x/15} e^{-y/40} dA = \frac{1}{600} \int_{x=0}^{x=\infty} \int_{y=0}^{y=\infty} x e^{-x/15} e^{-y/40} dA \\ &= \frac{1}{600} \lim_{(a,b)\to(\infty,\infty)} \int_{x=0}^{x=a} \int_{y=0}^{y=b} x e^{-x/15} e^{-y/40} dx\, dy \\ &= \frac{1}{600} \left(\lim_{a\to\infty} \int_{x=0}^{x=a} x e^{-x/15} dx \right) \left(\lim_{b\to\infty} \int_{y=0}^{y=b} e^{-y/40} dy \right) \\ &= \frac{1}{600} \left(\left(\lim_{a\to\infty} \left(-15 e^{-x/15} (x + 15)\right) \right) \Big|_{x=0}^{x=a} \right) \left(\left(\lim_{b\to\infty} \left(-40 e^{-y/40}\right) \right) \Big|_{y=0}^{y=b} \right) \\ &= \frac{1}{600} \left(\lim_{a\to\infty} \left(-15 e^{-a/15} (x + 15) + 225\right) \right) \left(\lim_{b\to\infty} \left(-40 e^{-b/40} + 40\right) \right) \\ &= \frac{1}{600} (225)(40) \\ &= 15. \end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Note:
Exercise:

Problem: The joint density function for two random variables X and Y is given by
Equation:

$$f(x,y)=\begin{cases}\frac{1}{600}\left(x^2+y^2\right) & \text{if } 0\le x\le 15, 0\le y\le 10\\ 0 & \text{otherwise}\end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Solution:

$\frac{55}{72}\approx 0.7638$

Hint

Compute the probability $P\left(X\le 10, Y\ge 5\right)=\int\limits_{x=-\infty}^{10}\int\limits_{y=5}^{y=10}\frac{1}{6000}\left(x^2+y^2\right)dy\,dx.$

Key Concepts

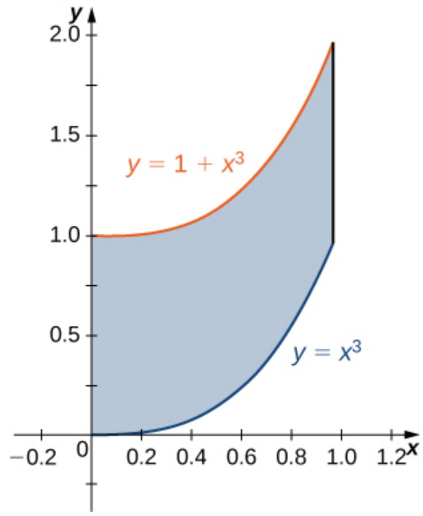
- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**
$$\iint\limits_D f(x,y)dA=\iint\limits_D f(x,y)dy\,dx=\int\limits_a^b\left[\int\limits_{g_1(x)}^{g_2(x)}f(x,y)dy\right]dx$$
- **Iterated integral over a Type II region**
$$\iint\limits_D f(x,y)dA=\iint\limits_D f(x,y)dx\,dy=\int\limits_c^d\left[\int\limits_{h_1(y)}^{h_2(y)}f(x,y)dx\right]dy$$

In the following exercises, specify whether the region is of Type I or Type II.
Exercise:

Problem: The region D bounded by $y=x^3$, $y=x^3+1$, $x=0$, and $x=1$ as given in the following figure.



Exercise:

Problem:

Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Solution:

$\frac{27}{20}$

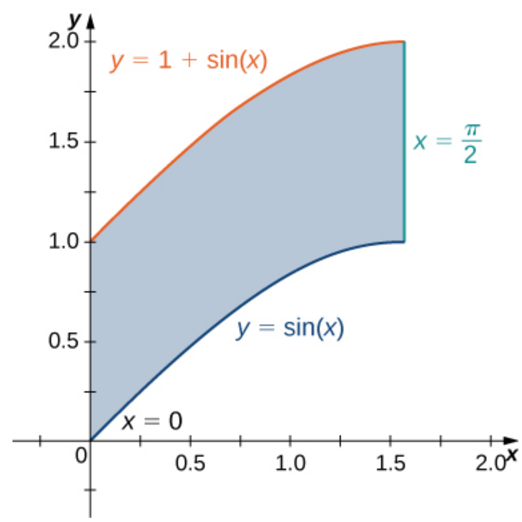
Exercise:

Problem: Find the area of the region D given in the previous exercise.

Exercise:

Problem:

The region D bounded by $y = \sin x, y = 1 + \sin x, x = 0,$ and $x = \frac{\pi}{2}$ as given in the following figure.



Solution:

Type I but not Type II

Exercise:

Problem:

Find the average value of the function $f\left(x,y\right)=\cos x$ on the region graphed in the previous exercise.

Exercise:

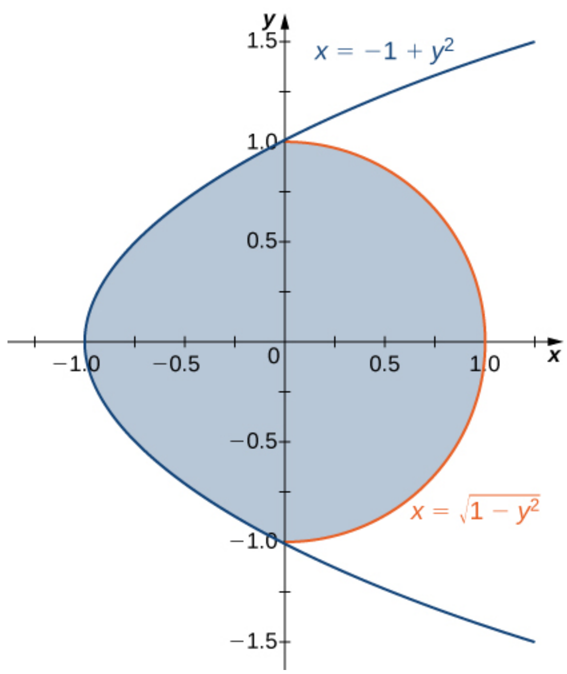
Problem: Find the area of the region D given in the previous exercise.

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: The region D bounded by $x={y}^{2}-1$ and $x=\sqrt{1-{y}^{2}}$ as given in the following figure.



Exercise:

Problem:

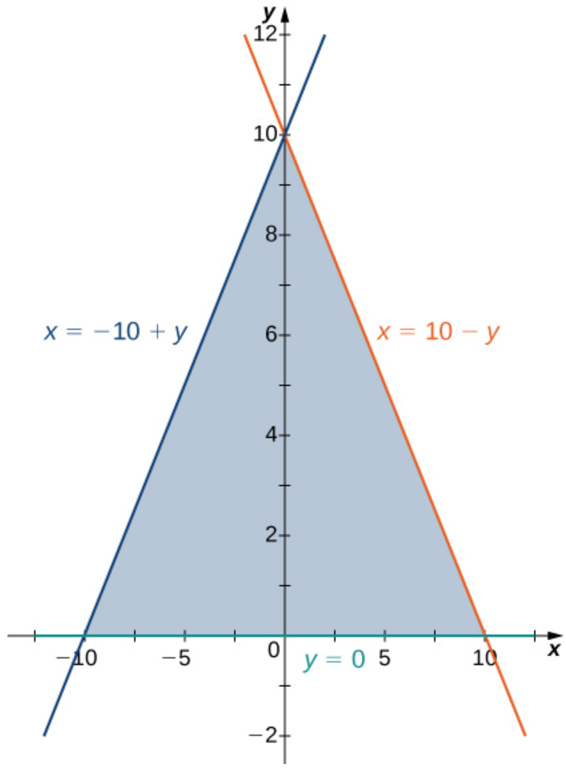
Find the volume of the solid under the graph of the function $f\left(x,y\right)=xy+1$ and above the region in the figure in the previous exercise.

Solution:

$\frac{1}{6}\left(8+3\pi\right)$

Exercise:

Problem: The region D bounded by $y=0,x=-10+y,$ and $x=10-y$ as given in the following figure.



Exercise:

Problem:

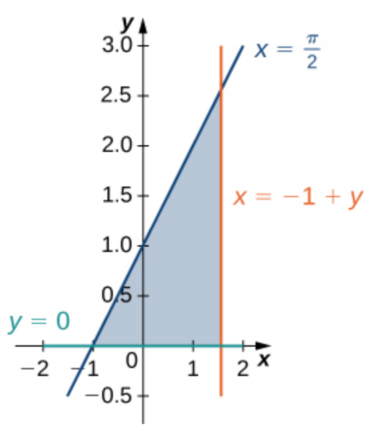
Find the volume of the solid under the graph of the function $f(x, y) = x + y$ and above the region in the figure from the previous exercise.

Solution:

$\frac{1000}{3}$

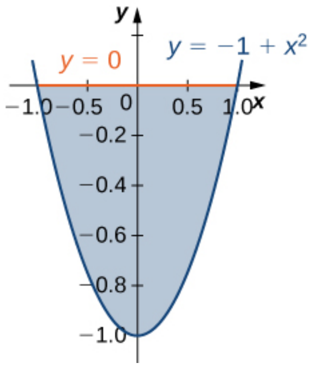
Exercise:

Problem: The region D bounded by $y = 0, x = y - 1, x = \frac{\pi}{2}$ as given in the following figure.



Exercise:

Problem: The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Solution:

Type I and Type II

Exercise:

Problem:

Let D be the region bounded by the curves of equations $y = x, y = -x$, and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

Exercise:

Problem:

Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Solution:

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region D is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

In the following exercises, evaluate the double integral $\iint\limits_D f(x, y) dA$ over the region D.

Exercise:

Problem: $f(x, y) = 2x + 5y$ and $D = \left\{(x, y) | 0 \le x \le 1, x^3 \le y \le x^3 + 1\right\}$

Exercise:

Problem: $f(x, y) = 1$ and $D = \left\{(x, y) | 0 \le x \le \frac{\pi}{2}, \sin x \le y \le 1 + \sin x\right\}$

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: $f(x, y) = 2$ and $D = \{(x, y) | 0 \le y \le 1, y - 1 \le x \le \arccos y\}$

Exercise:

Problem: $f(x,y) = xy$ and $D = \left\{(x,y) | -1 \le y \le 1, y^2 - 1 \le x \le \sqrt{1-y^2}\right\}$

Solution:

0

Exercise:

Problem: $f(x,y) = \sin y$ and D is the triangular region with vertices $(0,0), (0,3), \text{ and } (3,0)$

Exercise:

Problem: $f(x,y) = -x + 1$ and D is the triangular region with vertices $(0,0), (0,2), \text{ and } (2,2)$

Solution:

$\frac{2}{3}$

Evaluate the iterated integrals.

Exercise:

Problem: $\int_0^1 \int_{2x}^{3x} \left(x + y^2\right) dy\, dx$

Exercise:

Problem: $\int_0^1 \int_{2\sqrt{x}}^{2\sqrt{x}+1} (xy + 1) dy\, dx$

Solution:

$\frac{41}{20}$

Exercise:

Problem: $\int_e^{e^2} \int_{\ln u}^{2} (v + \ln u) dv\, du$

Exercise:

Problem: $\int_1^2 \int_{-u^2-1}^{-u} (8uv) dv\, du$

Solution:

-63

Exercise:

Problem: $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2x+4x^3)\,dx\,dy$

Exercise:

Problem: $\int_0^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} 4dx\,dy$

Solution:

π

Exercise:

Problem: Let D be the region bounded by $y=1-x^2$, $y=4-x^2$, and the x- and y-axes.

a. Show that $\iint_D x\,dA = \int_0^1 \int_{1-x^2}^{4-x^2} x\,dy\,dx + \int_1^2 \int_0^{4-x^2} x\,dy\,dx$ by dividing the region D into two regions of Type I.
b. Evaluate the integral $\iint_D x\,dA$.

Exercise:

Problem: Let D be the region bounded by $y=1$, $y=x$, $y=\ln x$, and the x-axis.

a. Show that $\iint_D y\,dA = \int_0^1 \int_0^x y\,dy\,dx + \int_1^e \int_{\ln x}^1 y\,dy\,dx$ by dividing D into two regions of Type I.
b. Evaluate the integral $\iint_D y\,dA$.

Solution:

a. Answers may vary; b. $\frac{2}{3}$

Exercise:

Problem:

a. Show that $\iint_D y^2 dA = \int_{-1}^0 \int_{-x}^{2-x^2} y^2 dy\,dx + \int_0^1 \int_x^{2-x^2} y^2 dy\,dx$ by dividing the region D into two regions of Type I, where $D=\{(x,y)\,|\,y\ge x, y\ge -x, y\le 2-x^2\}$.

b. Evaluate the integral $\iint\limits_D y^2 dA$.

Exercise:

Problem: Let D be the region bounded by $y = x^2, y = x + 2$, and $y = -x$.

a. Show that $\iint\limits_D x\, dA = \int\limits_0^1 \int\limits_{-y}^{\sqrt{y}} x\, dx\, dy + \int\limits_1^2 \int\limits_{y-2}^{\sqrt{y}} x\, dx\, dy$ by dividing the region D into two regions of Type II, where $D = \{(x, y) | y \geq x^2, y \geq -x, y \leq x + 2\}$.

b. Evaluate the integral $\iint\limits_D x\, dA$.

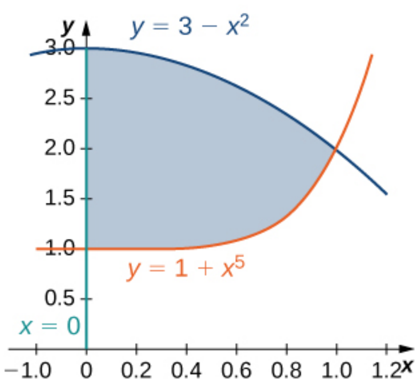
Solution:

a. Answers may vary; b. $\frac{8}{12}$

Exercise:

Problem:

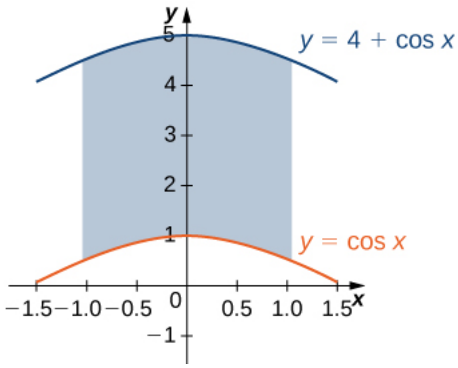
The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



Exercise:

Problem:

The region D bounded by $y = \cos x, y = 4 \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



Solution:

$\frac{8\pi}{3}$

Exercise:

Problem: Find the area $A(D)$ of the region $D = \left\{(x,y)|y \geq 1 - x^2, y \leq 4 - x^2, y \geq 0, x \geq 0\right\}$.

Exercise:

Problem:

Let D be the region bounded by $y = 1, y = x, y = \ln x$, and the x-axis. Find the area $A\left(D\right)$ of the region D.

Solution:

$e - \frac{3}{2}$

Exercise:

Problem:

Find the average value of the function $f\left(x,y\right) = \sin y$ on the triangular region with vertices $(0,0)$, $(0,3)$, and $(3,0)$.

Exercise:

Problem:

Find the average value of the function $f\left(x,y\right) = -x + 1$ on the triangular region with vertices $(0,0)$, $(0,2)$, and $(2,2)$.

Solution:

$\frac{2}{3}$

In the following exercises, change the order of integration and evaluate the integral.

Exercise:

Problem: $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx$

Exercise:

Problem: $\int\limits_{0}^{1} \int\limits_{x-1}^{1-x} x \, dy \, dx$

Solution:

$$\int\limits_{0}^{1} \int\limits_{x-1}^{1-x} x \, dy \, dx = \int\limits_{-1}^{0} \int\limits_{0}^{y+1} x \, dx \, dy + \int\limits_{0}^{1} \int\limits_{0}^{1-y} x \, dx dy = \frac{1}{3}$$

Exercise:

Problem: $\int\limits_{-1}^{0} \int\limits_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx \, dy$

Exercise:

Problem: $\int\limits_{-1/2}^{1/2} \int\limits_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \, dx \, dy$

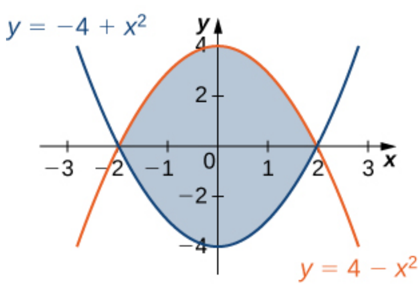
Solution:

$$\int\limits_{-1/2}^{1/2} \int\limits_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \, dx \, dy = \int\limits_{1}^{2} \int\limits_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y \, dy \, dx = 0$$

Exercise:

Problem:

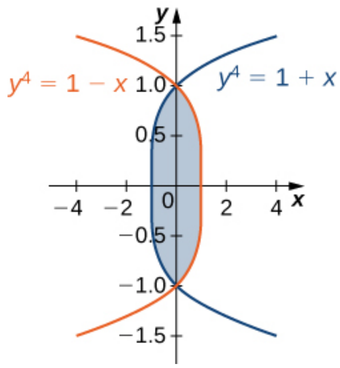
The region D is shown in the following figure. Evaluate the double integral $\iint\limits_{D} \left(x^2 + y\right) dA$ by using the easier order of integration.



Exercise:

Problem:

The region D is given in the following figure. Evaluate the double integral $\iint\limits_D (x^2 - y^2)\,dA$ by using the easier order of integration.



Solution:

$$\iint\limits_D (x^2 - y^2)\,dA = \int_{-1}^{1} \int_{y^4-1}^{1-y^4} (x^2 - y^2)\,dx\,dy = \frac{464}{4095}$$

Exercise:

Problem:

Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

Exercise:

Problem:

Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Solution:

$\frac{4}{5}$

Exercise:

Problem:

Find the volume of the solid under the plane $z = x - y$ and above the region bounded by $x = \tan y$, $x = -\tan y$, and $x = 1$.

Exercise:

Problem:

Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y$, $x = -\sin y$, and $x = 1$ for values of y between $y = \frac{-\pi}{2}$ and $y = \frac{\pi}{2}$.

Solution:

$\frac{5\pi}{32}$

Exercise:

Problem:

Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g\prime(x)$ and above the region bounded by $y = 0$, $y = g(x)$, $x = a$, and $x = b$ is given by $\frac{1}{2}\left(g^2(b) - g^2(a)\right)$.

Exercise:

Problem:

Let g be a positive, increasing, and differentiable function on the interval $[a, b]$, and let k be a positive real number. Show that the volume of the solid under the surface $z = g\prime(x)$ and above the region bounded by $y = g(x), y = g(x) + k, x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

Exercise:

Problem:

Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0, x + y = 1, x = 0,$ and $y = 0$.

Exercise:

Problem:

Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0, y = 0, z = 4,$ and $z = 0$.

Solution:

1

Exercise:

Problem:

Find the volume of the solid bounded by the planes $x + y = 1, x - y = 1, x = 0, z = 0$, and $z = 10$.

Exercise:

Problem:

Find the volume of the solid bounded by the planes $x + y = 1, x - y = 1, x + y = -1$, $x - y = -1, z = 1$ and $z = 0$.

Solution:

2

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$, respectively, and let S be the solid situated between $S_1, S_2, x = 0,$ and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$, respectively, and let S be the solid situated between $S_1, S_2, x = 0,$ and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Solution:

a. $\frac{1}{3}$; b. $\frac{1}{6}$; c. $\frac{1}{6}$

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$, determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

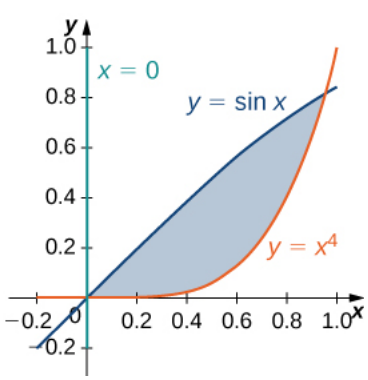
Solution:

a. $\frac{4}{3}$; b. 2π; c. $\frac{6\pi-4}{3}$

Exercise:

Problem:

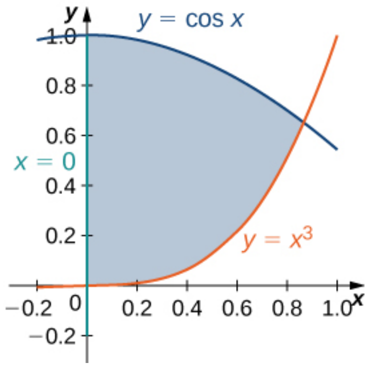
[T] The following figure shows the region D bounded by the curves $y = \sin x$, $x = 0$, and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Exercise:

Problem:

[T] The region D bounded by the curves $y = \cos x, x = 0,$ and $y = x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the *x*-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Solution:

0 and 0.865474; $A\left(D\right) = 0.621135$

Exercise:

Problem:

Suppose that $\left(X, Y\right)$ is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of $\left(X, Y\right)$ being in D is defined as $P\left[\left(X, Y\right) \in D\right] = \iint\limits_D p(x, y)dx\,dy$, where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint\limits_S p(x, y)dx\,dy = 1$. Assume that a point $\left(X, Y\right)$ is chosen arbitrarily in the square $\left[0, 3\right] \times \left[0, 3\right]$ with the probability density

$$p(x, y) = \begin{cases} \frac{1}{9} & (x, y) \in \left[0, 3\right] \times \left[0, 3\right], \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that the point $\left(X, Y\right)$ is inside the unit square and interpret the result.

Exercise:

Problem:

Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p(x, y) = p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad p_2(y) = \begin{cases} \frac{1}{5}e^{-y/5} & y \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x,y)dx\,dy$, where $D = \{(x,y)\}|x \geq 0, y \geq 0, x + y \leq 6\}$. Find $P[X + Y \leq 6]$ and interpret the result.

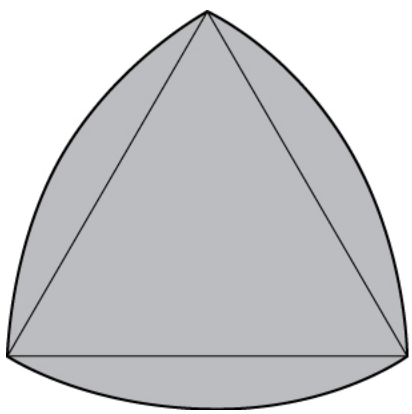
Solution:

$P[X + Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

Exercise:

Problem:

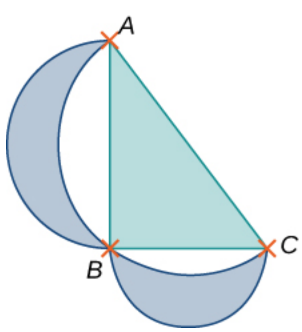
[T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius *s* centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}\left(\pi - \sqrt{3}\right)$.



Exercise:

Problem:

[T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle *ABC*. The outer boundaries of the lunes are semicircles of diameters AB and AC, respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



Glossary

improper double integral
: a double integral over an unbounded region or of an unbounded function

Type I
: a region D in the xy-plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II
: a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$

Double Integrals in Polar Coordinates

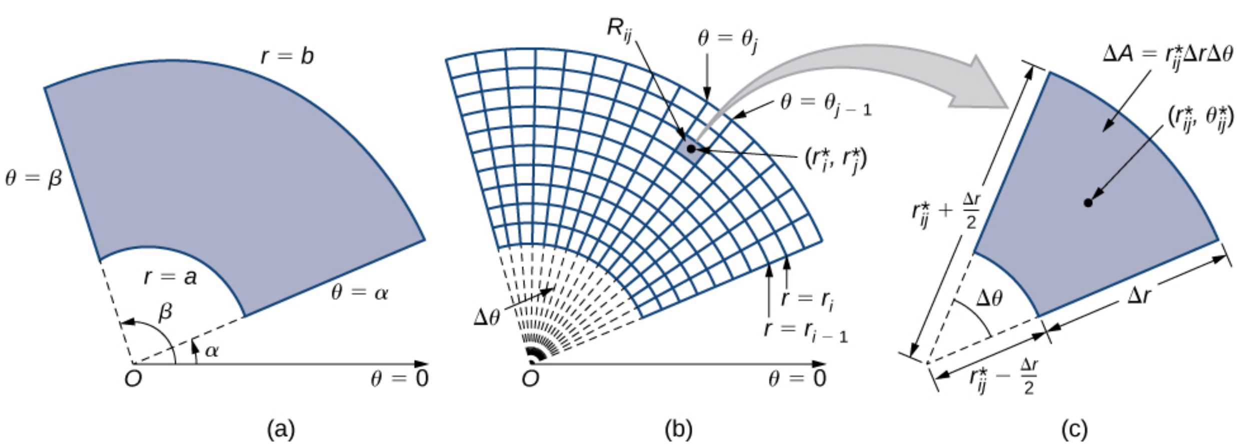
- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a **polar rectangle**, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in [link](a), with $R=\{(r,\theta)|a\le r\le b,\alpha\le\theta\le\beta\}$.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r,\theta)$ over a polar rectangle R. We divide the interval $[a,b]$ into m subintervals $[r_{i-1},r_i]$ of length $\Delta r=(b-a)/m$ and divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{i-1},\theta_i]$ of width $\Delta\theta=(\beta-\alpha)/n$. This means that the circles $r=r_i$ and rays $\theta=\theta_i$ for $1\le i\le m$ and $1\le j\le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} ([link](b)).



(a) A polar rectangle R (b) divided into subrectangles R_{ij}. (c) Close-up of a subrectangle.

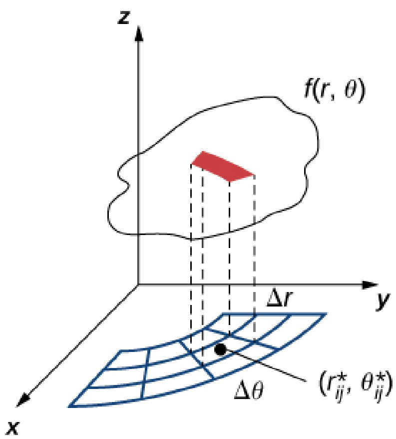
As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the “polar” volume of the thin box above R_{ij}. Recall that, in a circle of radius r, the length s of an arc subtended by a central angle of θ radians is $s=r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is
Equation:

$$\Delta A = \frac{1}{2}\Delta r\,(r_{i-1}\Delta\theta + r_1\Delta\theta).$$

Simplifying and letting $r_{ij}^{*} = \frac{1}{2}(r_{i-1} + r_i)$, we have $\Delta A = r_{ij}^{*}\Delta r\Delta\theta$. Therefore, the polar volume of the thin box above R_{ij} ([link]) is

Equation:

$$f(r_{ij}^{*}, \theta_{ij}^{*})\Delta A = f(r_{ij}^{*}, \theta_{ij}^{*})r_{ij}^{*}\Delta r\Delta\theta.$$



Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

Equation:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*})r_{ij}^{*}\Delta r\Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

Equation:

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*})r_{ij}^{*}\Delta r\Delta\theta.$$

This becomes the expression for the double integral.

Note:
Definition
The double integral of the function $f(r,\theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as
Equation:

$$\iint\limits_{R} f(r,\theta)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*},\theta_{ij}^{*})\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*},\theta_{ij}^{*})r_{ij}^{*}\Delta r\Delta\theta.$$

Again, just as in Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,
Equation:

$$\iint\limits_{R} f(r,\theta)dA = \iint\limits_{R} f(r,\theta)r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)r\,dr\,d\theta.$$

Notice that the expression for dA is replaced by $r\,dr\,d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y, using $x = r\cos\theta, y = r\sin\theta,$ and $dA = r\,dr\,d\theta$ changes it to
Equation:

$$\iint\limits_{R} f(x,y)dA = \iint\limits_{R} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Note that all the properties listed in Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

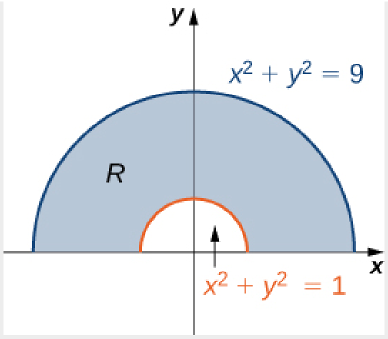
Example:
Exercise:

Problem:
Sketching a Polar Rectangular Region

Sketch the polar rectangular region $R = \{(r,\theta)|1 \le r \le 3, 0 \le \theta \le \pi\}$.

Solution:

As we can see from [link], $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.



The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example:
Exercise:

Problem:
Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint\limits_R 3x\,dA$ over the region $R = \{(r,\theta) | 1 \le r \le 2, 0 \le \theta \le \pi\}$.

Solution:

First we sketch a figure similar to [link] but with outer radius 2. From the figure we can see that we have

Equation:

$$\begin{aligned}
\iint\limits_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta r\,dr\,d\theta && \text{Use an iterated integral with correct limits of integration.} \\
&= \int_{\theta=0}^{\theta=\pi} \cos\theta \left[r^3 \Big|_{r=1}^{r=2}\right] d\theta && \text{Integrate first with respect to } r. \\
&= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta = 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.
\end{aligned}$$

Note:
Exercise:

Problem: Sketch the region $R = \left\{(r,\theta)|1 \le r \le 2, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$, and evaluate $\iint\limits_R x\,dA$.

Solution:

$\frac{14}{3}$

Hint

Follow the steps in [link].

Example:
Exercise:

Problem:
Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint\limits_R \left(1 - x^2 - y^2\right) dA$ where R is the unit circle on the xy-plane.

Solution:

The region R is a unit circle, so we can describe it as $R = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta, y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have
Equation:

$$\begin{aligned}
\iint\limits_R \left(1 - x^2 - y^2\right) dA &= \int_0^{2\pi}\int_0^1 \left(1 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}.
\end{aligned}$$

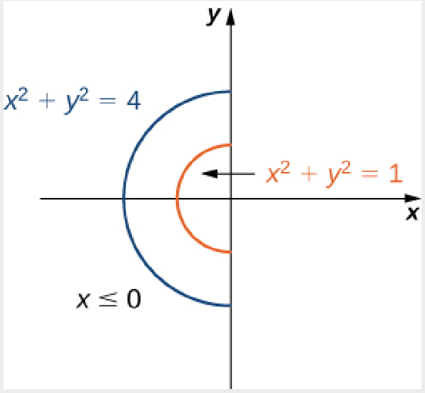
Example:
Exercise:

Problem:
Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint\limits_R (x + y) dA$ where $R = \left\{(x,y)|1 \le x^2 + y^2 \le 4, x \le 0\right\}$.

Solution:

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \left\{(r, \theta) | 1 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$ (see the following graph).



The annular region of integration R.

Hence, using the conversion $x = r\cos\theta, y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

Equation:

$$\begin{aligned}\iint\limits_R (x+y)dA &= \int\limits_{\theta=\pi/2}^{\theta=3\pi/2} \int\limits_{r=1}^{r=2} (r\cos\theta + r\sin\theta) r\,dr\,d\theta \\ &= \left(\int\limits_{r=1}^{r=2} r^2 dr\right)\left(\int\limits_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta) d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Note:
Exercise:

Problem:

Evaluate the integral $\iint\limits_R \left(4 - x^2 - y^2\right) dA$ where R is the circle of radius 2 on the xy-plane.

Solution:

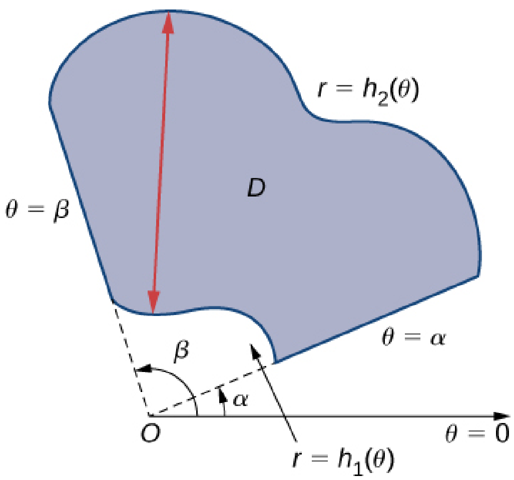
8π

Hint

Follow the steps in the previous example.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ (see the following figure).



A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

Note:
Double Integrals over General Polar Regions
If $f(r,\theta)$ is continuous on a general polar region D as described above, then
Equation:

$$\iint_D f(r,\theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) r \, dr \, d\theta$$

Example:
Exercise:

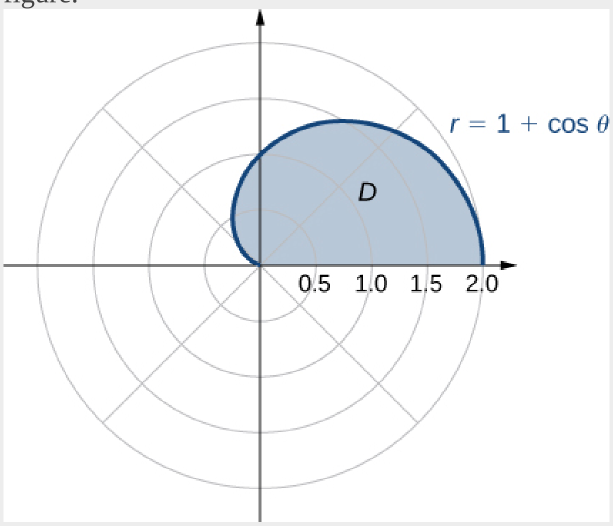
Problem:

Evaluating a Double Integral over a General Polar Region

Evaluate the integral $\iint\limits_D r^2 \sin\theta r\,dr\,d\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution:

We can describe the region D as $\{(r,\theta) | 0 \le \theta \le \pi, 0 \le r \le 1 + \cos\theta\}$ as shown in the following figure.



The region D is the top half of a cardioid.

Hence, we have

Equation:

$$\begin{aligned}
\iint\limits_D r^2 \sin\theta r\,dr\,d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} \left(r^2 \sin\theta\right) r\,dr\,d\theta \\
&= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} \left[r^4\right]_{r=0}^{r=1+\cos\theta} \sin\theta\,d\theta \\
&= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\,d\theta \\
&= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.
\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral
Equation:

$$\iint_D r^2 \sin^2 2\theta r\,dr\,d\theta \text{ where } D = \left\{(r,\theta)|0 \le \theta \le \pi, 0 \le r \le 2\sqrt{\cos 2\theta}\right\}.$$

Solution:

$\pi/8$

Hint

Graph the region and follow the steps in the previous example.

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a, r = b, \theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as
Equation:

$$V = \iint_R f(r,\theta) r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta) r\,dr\,d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes
Equation:

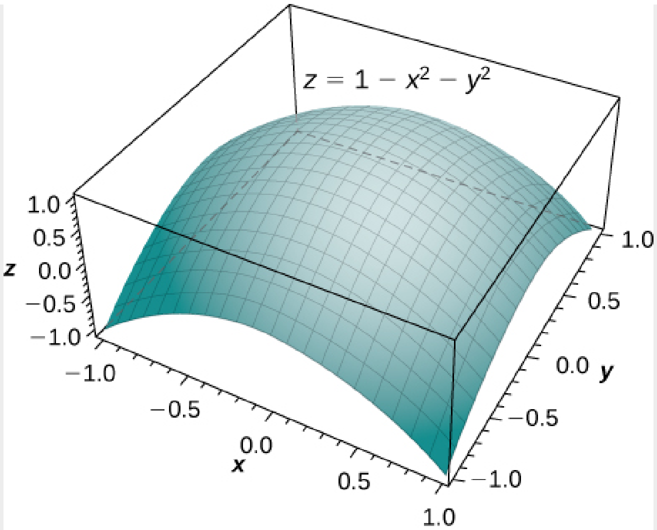
$$V = \iint_D f(r,\theta) r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) r\,dr\,d\theta.$$

We illustrate this idea with some examples.

Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (see the following figure).



The paraboloid $z=1-x^2-y^2$.

Solution:

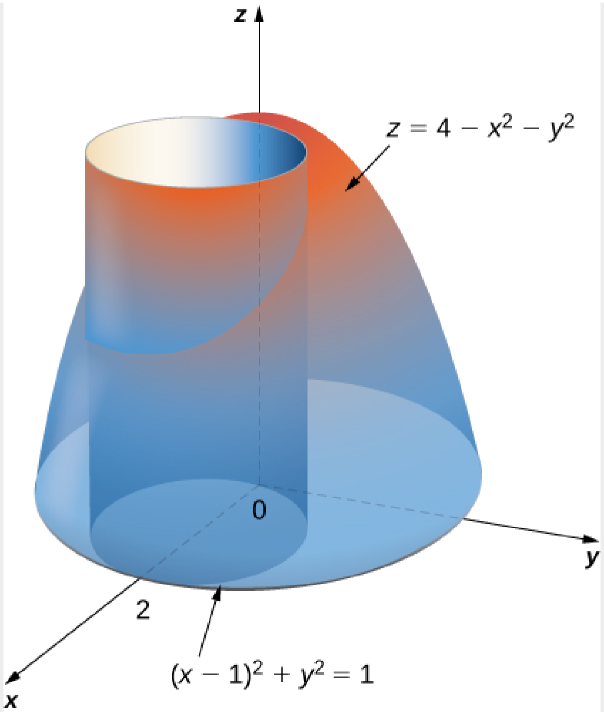
By the method of double integration, we can see that the volume is the iterated integral of the form $\iint\limits_R \left(1-x^2-y^2\right)dA$ where $R=\{(r,\theta)|0\le r\le 1, 0\le\theta\le 2\pi\}$.

This integration was shown before in [link], so the volume is $\frac{\pi}{2}$ cubic units.

Example:
Exercise:

Problem:
Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z=4-x^2-y^2$ and above the disk $(x-1)^2+y^2=1$ on the xy-plane. See the paraboloid in [link] intersecting the cylinder $(x-1)^2+y^2=1$ above the xy-plane.



Finding the volume of a solid with a paraboloid cap and a circular base.

Solution:

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\cos\theta$ and then either $r=0$ or $r=2\cos\theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy-plane as the region
Equation:

$$D=\{(r,\theta)|0\le\theta\le\pi, 0\le r\le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\cos\theta$ is
Equation:

$$\begin{aligned} V &= \iint\limits_D f(r,\theta) r\,dr\,d\theta = \int\limits_{\theta=0}^{\theta=\pi} \int\limits_{r=0}^{r=2\cos\theta} \left(4 - r^2\right) r\,dr\,d\theta \\ &= \int\limits_{\theta=0}^{\theta=\pi} \left[4\frac{r^2}{2} - \frac{r^4}{4} \Big|_0^{2\cos\theta} \right] d\theta \\ &= \int\limits_0^{\pi} \left[8\cos^2\theta - 4\cos^2\theta \right] d\theta = \left[\tfrac{5}{2}\theta + \tfrac{5}{2}\sin\theta\cos\theta - \sin\theta\cos^3\theta \right]_0^{\pi} = \tfrac{5}{2}\pi. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

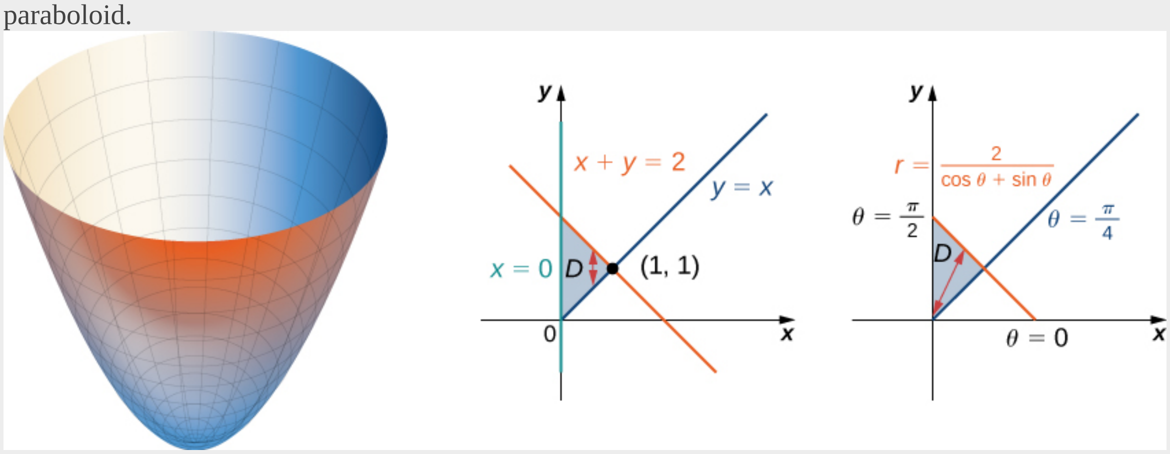
Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane ([link]).

Solution:

First examine the region over which we need to set up the double integral and the accompanying paraboloid.



Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x, y) | 0 \leq x \leq 1, x \leq y \leq 2 - x\}$. Converting the lines $y = x, x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ, we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\theta + \sin\theta)$, respectively. Graphing the region on the xy-plane, we see that it looks like $D = \{(r, \theta) | \pi/4 \leq \theta \leq \pi/2, 0 \leq r \leq 2/(\cos\theta + \sin\theta)\}$. Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

Equation:

$$\begin{aligned} V &= \iint_D f(r, \theta) r \, dr \, d\theta = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\theta+\sin\theta)} r^2 r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta + \sin\theta}\right)^4 d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta = 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

Equation:

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx.$$

Evaluating gives

Equation:

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3} \, dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Bigg|_0^1 = \frac{4}{3}. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

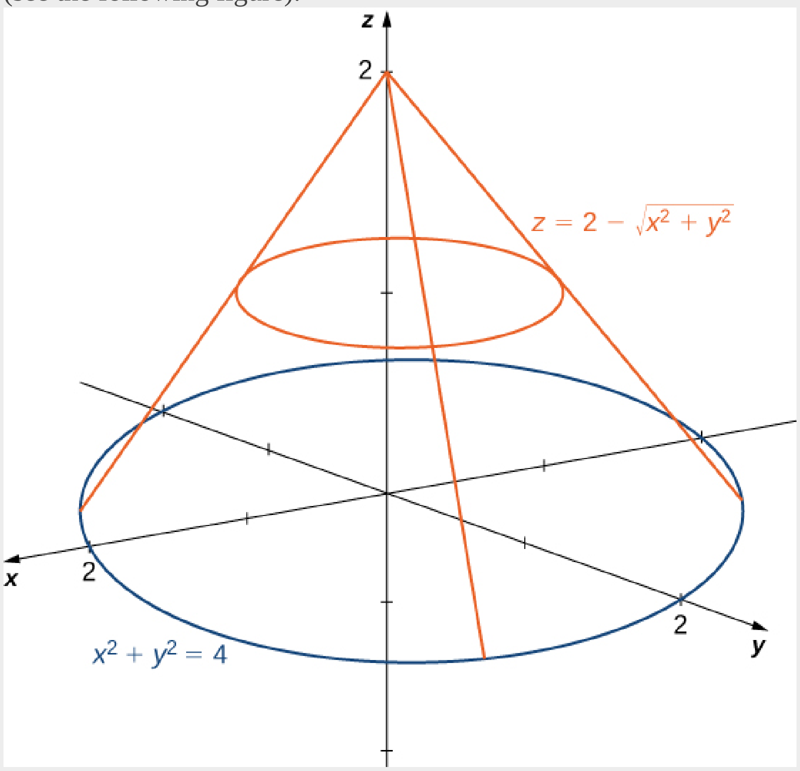
Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution:

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (see the following figure).



Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:
Equation:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2, so for the integration we have $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2-r)r \, dr \, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius a units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use [link] and set up the integral as $\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r \, dr \, d\theta$.

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Note:
Exercise:

Problem:

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Solution:

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(16 - 2r^2\right) r \, dr \, d\theta = 64\pi \text{ cubic units}$$

Hint

Sketching the graphs can help.

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

Equation:

$$\text{Area } A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} 1r \, dr \, d\theta.$$

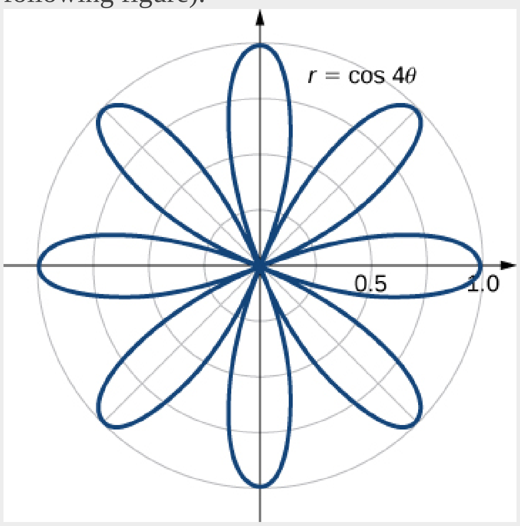
Example:
Exercise:

Problem:
Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution:

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).



Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos 4\theta$ is

Equation:

$$\begin{aligned} A &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \int_{r=0}^{r=\cos 4\theta} 1r\, dr\, d\theta \\ &= 8 \int_{-\pi/8}^{\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta} \right] d\theta = 8 \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta\, d\theta = 8 \left[\tfrac{1}{4}\theta + \tfrac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8} \right] = 8 \left[\tfrac{\pi}{16} \right] = \tfrac{\pi}{2}. \end{aligned}$$

Example:
Exercise:

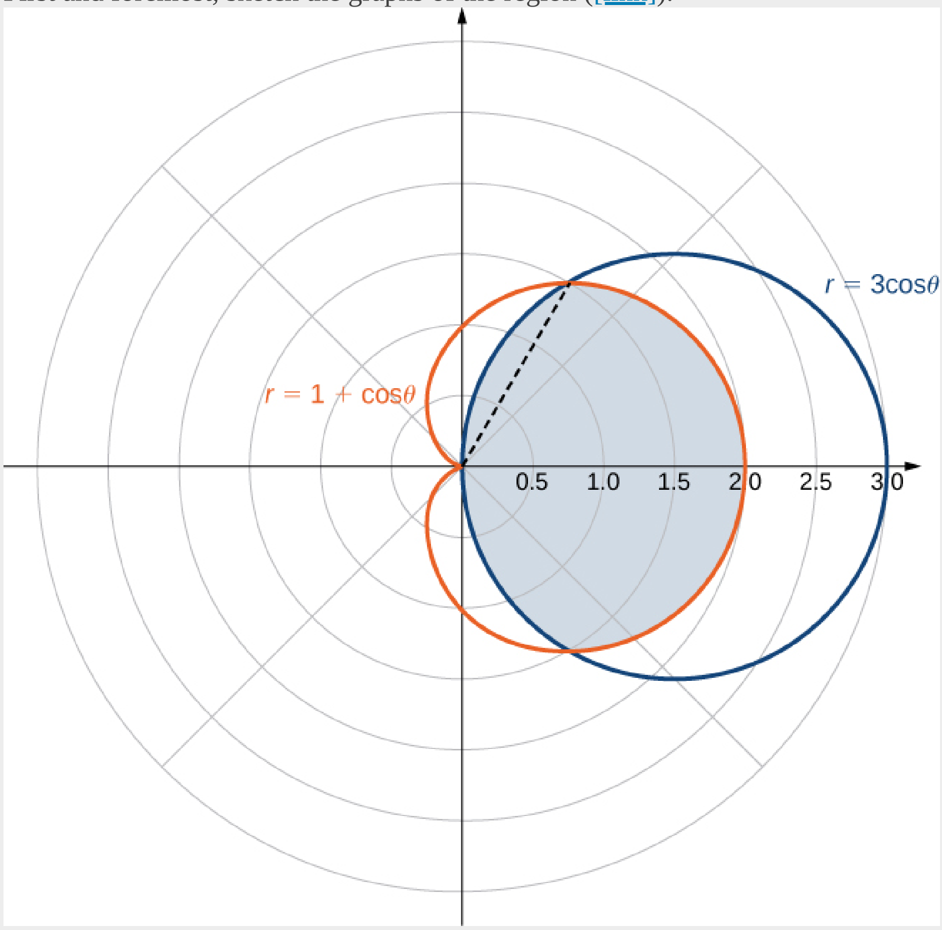
Problem:

Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

Solution:

First and foremost, sketch the graphs of the region ([link]).



Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

Equation:

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

Equation:

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is
Equation:

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

Note:
Exercise:

Problem:

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Solution:

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\,dr\,d\theta = 8\pi + 9\sqrt{3}$$

Hint

Sketch the graph, and solve for the points of intersection.

Example:
Exercise:

Problem:
Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral $\iint\limits_{\mathrm{R}^2} e^{-10\left(x^2+y^2\right)}\,dx\,dy.$

Solution:

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have
Equation:

$$\begin{aligned}
\iint\limits_{\mathrm{R}^2} e^{-10\left(x^2+y^2\right)} dx\, dy &= \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{r=0}^{r=\infty} e^{-10r^2} r\, dr\, d\theta = \int\limits_{\theta=0}^{\theta=2\pi} \left(\lim_{a\to\infty} \int\limits_{r=0}^{r=a} e^{-10r^2} r\, dr \right) d\theta \\
&= \left(\int\limits_{\theta=0}^{\theta=2\pi} d\theta \right) \left(\lim_{a\to\infty} \int\limits_{r=0}^{r=a} e^{-10r^2} r\, dr \right) \\
&= 2\pi \left(\lim_{a\to\infty} \int\limits_{r=0}^{r=a} e^{-10r^2} r\, dr \right) \\
&= 2\pi \lim_{a\to\infty} \left(-\tfrac{1}{20}\right) \left(e^{-10r^2} \Big|_0^a \right) \\
&= 2\pi \left(-\tfrac{1}{20}\right) \lim_{a\to\infty} \left(e^{-10a^2} - 1 \right) \\
&= \tfrac{\pi}{10}.
\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral $\iint\limits_{\mathrm{R}^2} e^{-4\left(x^2+y^2\right)} dx\, dy.$

Solution:

$\frac{\pi}{4}$

Hint

Convert to the polar coordinate system.

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\, dr\, d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r, \theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region** R
$$\iint\limits_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}{*}, \theta_{ij}{*})\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}{*}, \theta_{ij}{*})r_{ij}{*}\Delta r\Delta\theta$$
- **Double integral over a general polar region**
$$\iint\limits_D f(r,\theta)r\,dr\,d\theta = \int\limits_{\theta=\alpha}^{\theta=\beta}\int\limits_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)r\,dr\,d\theta$$

In the following exercises, express the region D in polar coordinates.

Exercise:

Problem: D is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

Exercise:

Problem:

D is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Solution:

$D = \left\{(r,\theta)|4 \le r \le 5, \frac{\pi}{2} \le \theta \le \pi\right\}$

Exercise:

Problem: D is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

Exercise:

Problem: D is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

Solution:

$D = \left\{(r,\theta)|0 \le r \le \sqrt{2}, 0 \le \theta \le \pi\right\}$

Exercise:

Problem: $D = \left\{(x,y)|x^2+y^2 \le 4x\right\}$

Exercise:

Problem: $D = \left\{(x,y)|x^2+y^2 \le 4y\right\}$

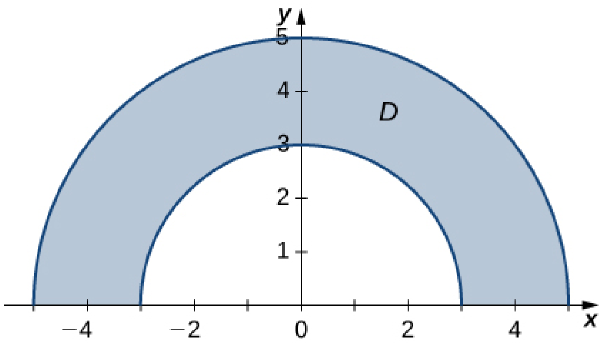
Solution:

$D = \{(r,\theta)|0 \le r \le 4\sin\theta, 0 \le \theta \le \pi\}$

In the following exercises, the graph of the polar rectangular region D is given. Express D in polar coordinates.

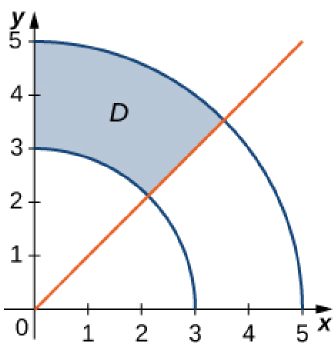
Exercise:

Problem:



Exercise:

Problem:

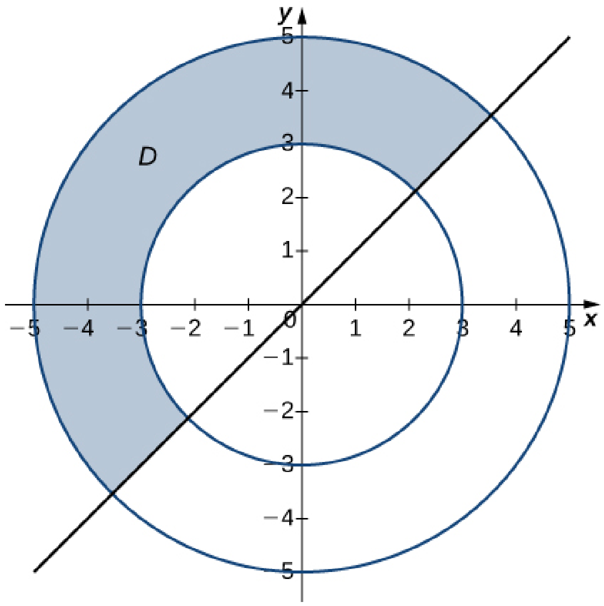


Solution:

$D = \left\{(r, \theta) | 3 \leq r \leq 5, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\right\}$

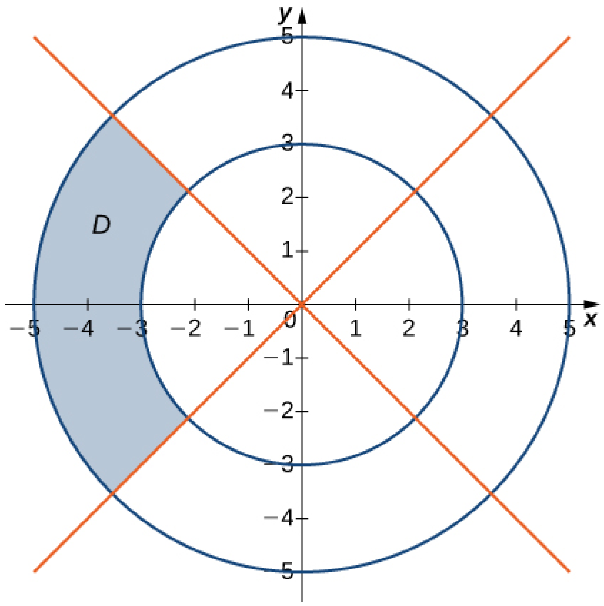
Exercise:

Problem:



Exercise:

Problem:



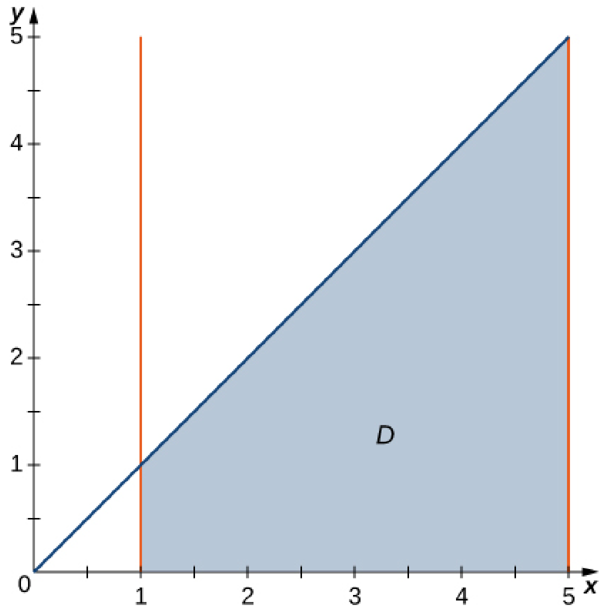
Solution:

$D = \left\{(r, \theta) | 3 \leq r \leq 5, \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\right\}$

Exercise:

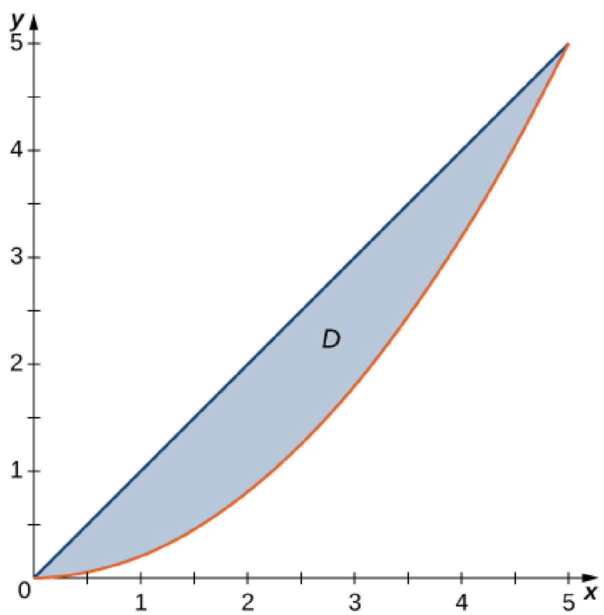
Problem:

In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1, x = 5$, and $y = 0$.



Exercise:

Problem: In the following graph, the region D is bounded by $y = x$ and $y = x^2$.



Solution:

$D = \left\{(r, \theta) | 0 \leq r \leq \tan\theta \sec\theta, 0 \leq \theta \leq \frac{\pi}{4}\right\}$

In the following exercises, evaluate the double integral $\iint\limits_R f(x, y) dA$ over the polar rectangular region D.

Exercise:

Problem: $f(x,y) = x^2 + y^2, D = \{(r,\theta)|3 \le r \le 5, 0 \le \theta \le 2\pi\}$

Exercise:

Problem: $f(x,y) = x + y,\ \ D = \{(r,\theta)|3 \le r \le 5, 0 \le \theta \le 2\pi\}$

Solution:

0

Exercise:

Problem: $f(x,y) = x^2 + xy, D = \{(r,\theta)|1 \le r \le 2, \pi \le \theta \le 2\pi\}$

Exercise:

Problem: $f(x,y) = x^4 + y^4, D = \left\{(r,\theta)|1 \le r \le 2, \frac{3\pi}{2} \le \theta \le 2\pi\right\}$

Solution:

$\frac{63\pi}{16}$

Exercise:

Problem: $f(x,y) = \sqrt[3]{x^2 + y^2}$, where $D = \left\{(r,\theta)|0 \le r \le 1, \frac{\pi}{2} \le \theta \le \pi\right\}$.

Exercise:

Problem: $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $D = \left\{(r,\theta)|3 \le r \le 4, \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\right\}$.

Solution:

$\frac{3367\pi}{18}$

Exercise:

Problem: $f(x,y) = \sin\left(\arctan\frac{y}{x}\right)$, where $D = \left\{(r,\theta)\middle|1 \le r \le 2, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$

Exercise:

Problem: $f(x,y) = \arctan\left(\frac{y}{x}\right)$, where $D = \left\{(r,\theta)\middle|2 \le r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

Solution:

$\frac{35\pi^2}{576}$

Exercise:

Problem: $\iint\limits_D e^{x^2+y^2}\left[1 + 2\arctan\left(\frac{y}{x}\right)\right]dA, D = \left\{(r,\theta)|1 \le r \le 2, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$

Exercise:

Problem: $\iint\limits_D \left(e^{x^2+y^2}+x^4+2x^2y^2+y^4\right)\arctan\left(\frac{y}{x}\right)dA, D=\left\{(r,\theta)|1\le r\le 2, \frac{\pi}{4}\le\theta\le\frac{\pi}{3}\right\}$

Solution:

$\frac{7}{576}\pi^2\left(21-e+e^4\right)$

In the following exercises, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

Exercise:

Problem: $\int\limits_1^2\int\limits_0^x\left(x^2+y^2\right)dy\,dx=\int\limits_0^{\frac{\pi}{4}}\int\limits_{\sec\theta}^{2\sec\theta}r^3dr\,d\theta$

Exercise:

Problem: $\int\limits_2^3\int\limits_0^x\frac{x}{\sqrt{x^2+y^2}}dy\,dx=\int\limits_0^{\pi/4}\int\limits_0^{\tan\theta\sec\theta}r\cos\theta\,dr\,d\theta$

Solution:

$\frac{5}{4}\ln\left(3+2\sqrt{2}\right)$

Exercise:

Problem: $\int\limits_0^1\int\limits_{x^2}^x\frac{1}{\sqrt{x^2+y^2}}dy\,dx=\int\limits_0^{\pi/4}\int\limits_0^{\tan\theta\sec\theta}dr\,d\theta$

Exercise:

Problem: $\int\limits_0^1\int\limits_{x^2}^x\frac{y}{\sqrt{x^2+y^2}}dy\,dx=\int\limits_0^{\pi/4}\int\limits_0^{\tan\theta\sec\theta}r\sin\theta\,dr\,d\theta$

Solution:

$\frac{1}{6}\left(2-\sqrt{2}\right)$

In the following exercises, convert the integrals to polar coordinates and evaluate them.

Exercise:

Problem: $\int_0^3 \int_0^{\sqrt{9-y^2}} \left(x^2+y^2\right) dx\, dy$

Exercise:

Problem: $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^2 dx\, dy$

Solution:

$$\int_0^{\pi} \int_0^{2} r^5 dr\, d\theta = \frac{32\pi}{3}$$

Exercise:

Problem: $\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y) dy\, dx$

Exercise:

Problem: $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin\left(x^2+y^2\right) dy\, dx$

Solution:

$$\int_{-\pi/2}^{\pi/2} \int_0^{4} r \sin\left(r^2\right) dr\, d\theta = \pi \sin^2 8$$

Exercise:

Problem:

Evaluate the integral $\iint\limits_D r\, dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Exercise:

Problem:

Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

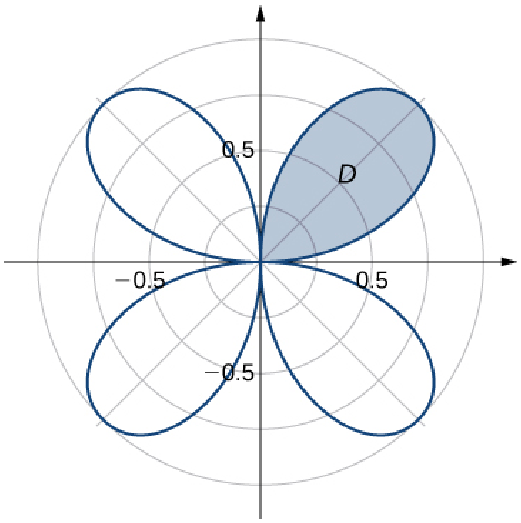
Solution:

$\frac{3\pi}{4}$

Exercise:

Problem:

Evaluate the integral $\iint\limits_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



Exercise:

Problem:

Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Solution:

$\frac{\pi}{2}$

Exercise:

Problem:

Find the area of the region D, which is the region bounded by $y = \sqrt{4 - x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

Exercise:

Problem:

Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

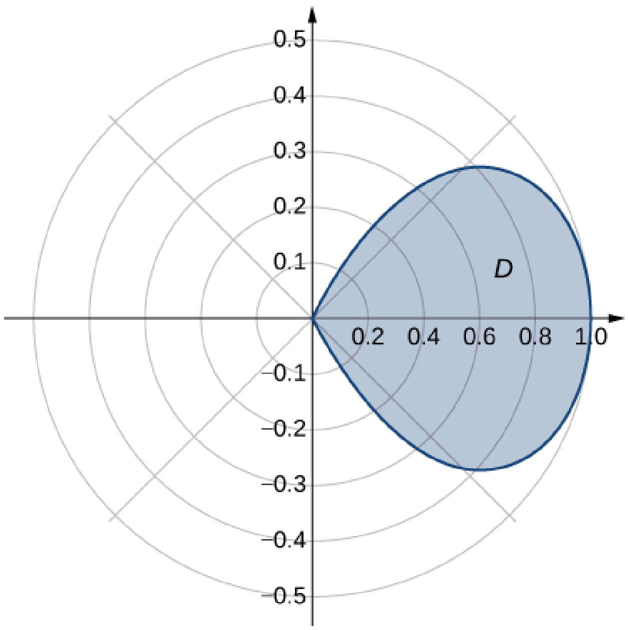
Solution:

$\frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$

Exercise:

Problem:

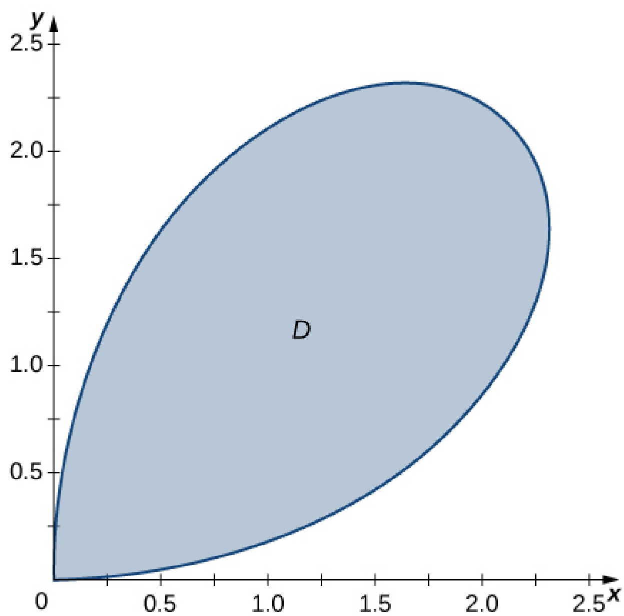
Determine the average value of the function $f\left(x,y\right)=x^2+y^2$ over the region D bounded by the polar curve $r=\cos 2\theta$, where $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ (see the following graph).



Exercise:

Problem:

Determine the average value of the function $f\left(x,y\right)=\sqrt{x^2+y^2}$ over the region D bounded by the polar curve $r=3\sin 2\theta$, where $0\le\theta\le\frac{\pi}{2}$ (see the following graph).



Solution:

$\frac{16}{3\pi}$

Exercise:

Problem:

Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0, y = 0$, and $z = 0$.

Exercise:

Problem:

Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Solution:

$\frac{\pi}{18}$

Exercise:

Problem:

a. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
b. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
c. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

Exercise:

Problem:

a. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.
b. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.
c. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

Exercise:

Problem:

If the sphere has radius 4 and the cylinder has radius 2, find the volume of the spherical ring.

Exercise:

Problem:

A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Solution:

$\frac{256\pi}{3}\ \text{cm}^3$

Exercise:

Problem:

Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

Exercise:

Problem:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

Solution:

$\frac{3\pi}{32}$

Exercise:

Problem:

Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

Exercise:

Problem:

Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Solution:

4π

Exercise:

Problem:

A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then $\iint\limits_D f(x, y) dA = (\theta_2 - \theta_1) [G(R_2) - G(R_1)]$, where $G\prime(r) = rg(r)$ and $(x, y) \in D = \{(r, \theta) | R_1 \leq r \leq R_2, 0 \leq \theta \leq 2\pi\}$, with $0 \leq R_1 < R_2$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$.

Exercise:

Problem:

Use the information from the preceding exercise to calculate the integral $\iint\limits_D (x^2 + y^2)^3 dA$, where D is the unit disk.

Solution:

$\frac{\pi}{4}$

Exercise:

Problem:

Let $f(x,y) = \frac{F\prime(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r,\theta)|R_1 \le r \le R_2, 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2+y^2}$, $0 < R_1 < R_2$, and F is a differentiable function. Show that $\iint\limits_D f(x,y) dA = 2\pi [F(R_2) - F(R_1)]$.

Exercise:

Problem:

Apply the preceding exercise to calculate the integral $\iint\limits_D \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} dx\, dy$, where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Solution:

$\frac{1}{2}\pi e(e-1)$

Exercise:

Problem:

Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = h(\theta)$, where $(x,y) \in D = \{(r,\theta)|R_1 \le r \le R_2, \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that $\iint\limits_D f(x,y) dA = \frac{1}{2}\left(R_2^2 - R_1^2\right)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

Exercise:

Problem:

Apply the preceding exercise to calculate the integral $\iint\limits_D \frac{y^2}{x^2} dA$, where $D = \left\{(r,\theta)\middle|1 \le r \le 2, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$.

Solution:

$\sqrt{3} - \frac{\pi}{4}$

Exercise:

Problem:

Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = g(r)h(\theta)$, where $(x,y) \in D = \{(r,\theta)|R_1 \le r \le R_2, \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that $\iint\limits_D f(x,y) dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)]$, where G and H are antiderivatives of g and h, respectively.

Exercise:

Problem: Evaluate $\iint\limits_D \arctan\left(\frac{y}{x}\right)\sqrt{x^2+y^2}dA$, where $D = \left\{(r,\theta)\middle|2 \le r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$.

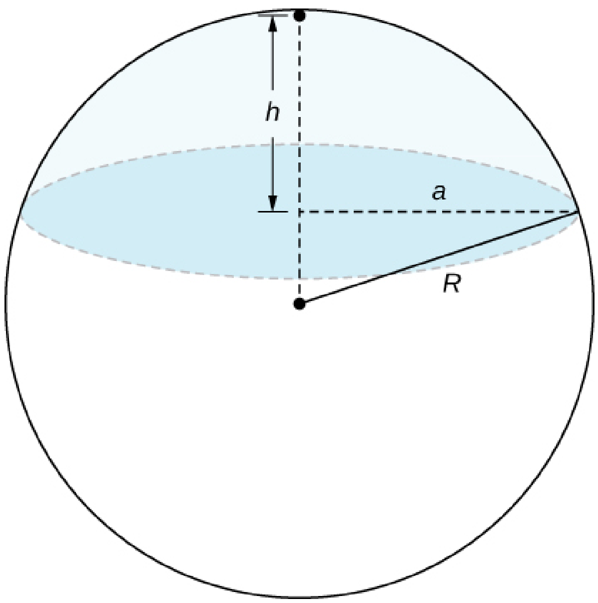
Solution:

$\frac{133\pi^3}{864}$

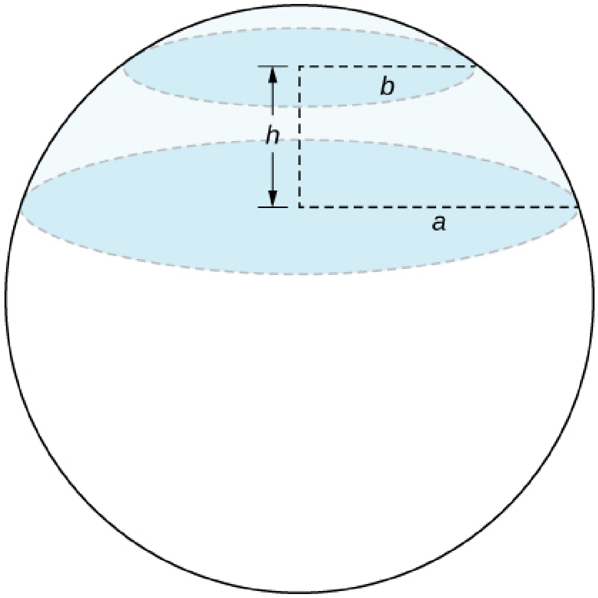
Exercise:

Problem: A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h\left(3a^2+h^2\right)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h, show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h\left(3a^2+3b^2+h^2\right)$.



Exercise:

Problem:

In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X, Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by $P\left[X^2 + Y^2 \le a^2\right] = \iint_D p(x,y)\,dy\,dx$, where D is the disk of radius *a* centered at the origin. Show that $P\left[X^2 + Y^2 \le a^2\right] = 1 - e^{-a^2/2\sigma^2}$.

Exercise:

Problem:

The double improper integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-x^2+y^2/2)}\,dy\,dx$ may be defined as the limit value of the double integrals $\iint_{D_a} e^{(-x^2+y^2/2)}\,dA$ over disks D_a of radii *a* centered at the origin, as *a* increases without bound; that is, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-x^2+y^2/2)}\,dy\,dx = \lim_{a\to\infty} \iint_{D_a} e^{(-x^2+y^2/2)}\,dA$.

a. Use polar coordinates to show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-x^2+y^2/2)}\,dy\,dx = 2\pi$.

b. Show that $\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$, by using the relation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-x^2+y^2/2)}dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right).$$

Glossary

polar rectangle
: the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta)|a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

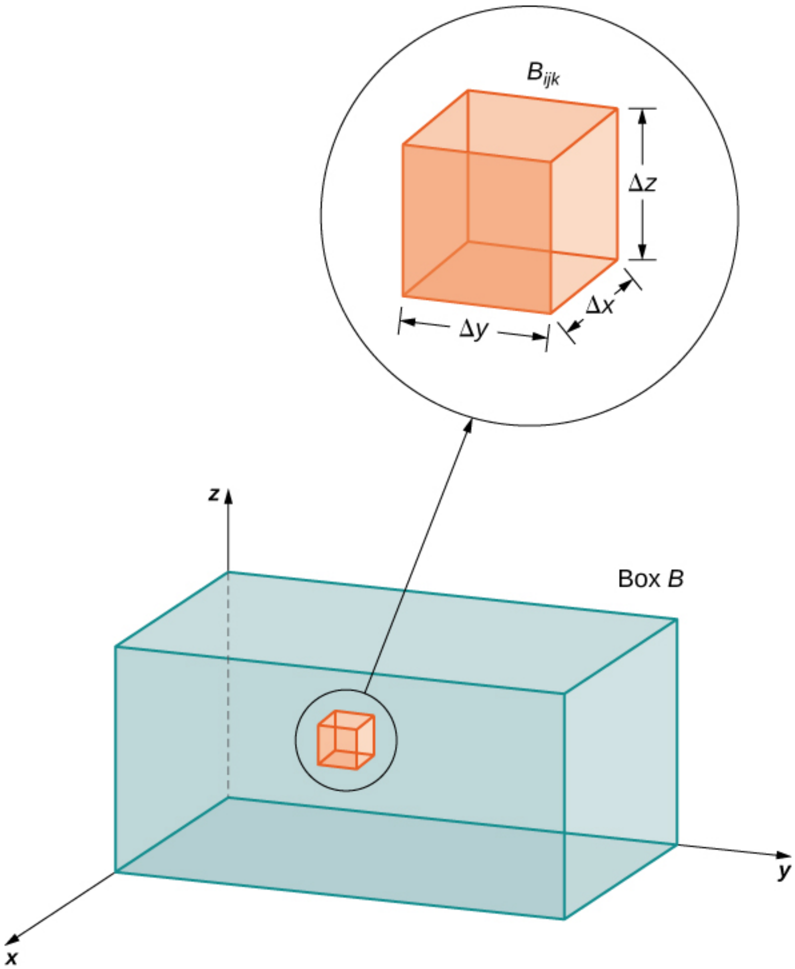
Triple Integrals

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

In Double Integrals over Rectangular Regions, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as $B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$. We follow a similar procedure to what we did in Double Integrals over Rectangular Regions. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{x_i - x_{i-1}}{l}$, divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length $\Delta y = \frac{y_j - y_{j-1}}{m}$, and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length $\Delta z = \frac{z_k - z_{k-1}}{n}$. Then the rectangular box B is subdivided into lmn subboxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i]$, as shown in [link].



A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each $i, j,$ and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

Equation:

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Note:
Definition
The **triple integral** of a function $f(x, y, z)$ over a rectangular box B is defined as
Equation:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta x\Delta y\Delta z = \iiint_B f(x,y,z)dV$$

if this limit exists.

When the triple integral exists on B, the function $f(x,y,z)$ is said to be integrable on B. Also, the triple integral exists if $f(x,y,z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of Fubini's thereom for triple integrals exists.

Note:
Fubini's Theorem for Triple Integrals
If $f(x,y,z)$ is continuous on a rectangular box $B=[a,b]\times[c,d]\times[e,f]$, then
Equation:

$$\iiint_B f(x,y,z)dV = \int_e^f\int_c^d\int_a^b f(x,y,z)dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a,b,c,d,e, and f real numbers, the iterated triple integral can be expressed in six different orderings:
Equation:

$$\begin{aligned}\int_e^f\int_c^d\int_a^b f(x,y,z)dx\,dy\,dz &= \int_e^f\left(\int_c^d\left(\int_a^b f(x,y,z)dx\right)dy\right)dz = \int_c^d\left(\int_e^f\left(\int_a^b f(x,y,z)dx\right)dz\right)dy\\ &= \int_a^b\left(\int_e^f\left(\int_c^d f(x,y,z)dy\right)dz\right)dx = \int_e^f\left(\int_a^b\left(\int_c^d f(x,y,z)dy\right)dx\right)dz\\ &= \int_c^e\left(\int_a^b\left(\int_e^f f(x,y,z)dz\right)dx\right)dy = \int_a^b\left(\int_c^e\left(\int_e^f f(x,y,z)dz\right)dy\right)dx.\end{aligned}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example:

Exercise:

Problem:
Evaluating a Triple Integral

Evaluate the triple integral $\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2) dx\, dy\, dz.$

Solution:

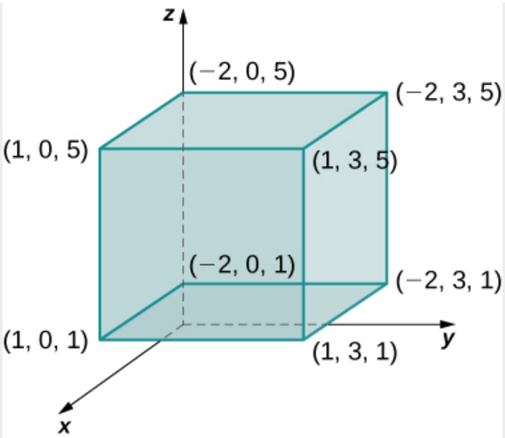
The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.
Equation:

$$\begin{aligned}
& \int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2) dx\, dy\, dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\, dz && \text{Integrate with respect to } x. \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\, dz && \text{Evaluate.} \\
&= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2} z^2 \Big|_{y=2}^{y=4} \right] dz && \text{Integrate with respect to } y. \\
&= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz && \text{Evaluate.} \\
&= \left[24z + 36\tfrac{z^3}{3} \right]_{z=0}^{z=1} = 36. && \text{Integrate with respect to } z.
\end{aligned}$$

Example:
Exercise:

Problem:
Evaluating a Triple Integral

Evaluate the triple integral $\iiint\limits_B x^2 yz\, dV$ where $B = \{(x, y, z) | -2 \leq x \leq 1, 0 \leq y \leq 3, 1 \leq z \leq 5\}$ as shown in the following figure.



Evaluating a triple integral over a given rectangular box.

Solution:

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate *y* first, then *x*, and then *z*.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 \left[x^2yz\right]dy\,dx\,dz = \int_1^5\int_{-2}^1\left[x^2\tfrac{y^2}{2}z\Big|_0^3\right]dx\,dz \\ &= \int_1^5\int_{-2}^1 \tfrac{9}{2}x^2z\,dx\,dz = \int_1^5\left[\tfrac{9}{2}\tfrac{x^3}{3}z\Big|_{-2}^1\right]dz = \int_1^5 \tfrac{27}{2}z\,dz = \tfrac{27}{2}\tfrac{z^2}{2}\Big|_1^5 = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, and then y.

Equation:

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 \left[x^2yz\right]dx\,dz\,dy = \int_0^3\int_1^5\left[\frac{x^3}{3}yz\Big|_{-2}^1\right]dz\,dy \\ &= \int_0^3\int_1^5 3yz\,dz\,dy = \int_0^3\left[3y\frac{z^2}{2}\Big|_1^5\right]dy = \int_0^3 36y\,dy = 36\tfrac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

Note:

Exercise:

Problem:

Evaluate the triple integral $\iiint\limits_B z \sin x \cos y \, dV$ where $B = \left\{(x, y, z) \middle| 0 \le x \le \pi, \frac{3\pi}{2} \le y \le 2\pi, 1 \le z \le 3\right\}$.

Solution:

$$\iiint\limits_B z \sin x \cos y \, dV = 8$$

Hint

Follow the steps in the previous example.

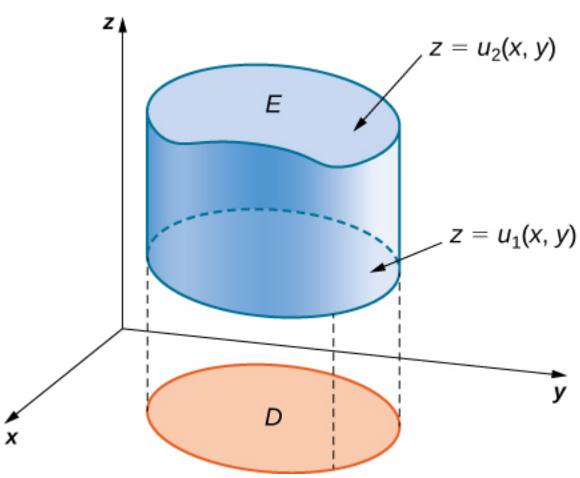
Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more general bounded region E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

Equation:

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}.$$

For two functions $z = u_1(x, y)$ and $z = u_2(x, y)$, such that $u_1(x, y) \le u_2(x, y)$ for all (x, y) in D as shown in the following figure.



We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Note:
Triple Integral over a General Region
The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region
Equation:

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as
Equation:

$$E = \{(x, y, z)|(x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA.$$

Finally, if D is a general bounded region in the yz-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \leq u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as
Equation:

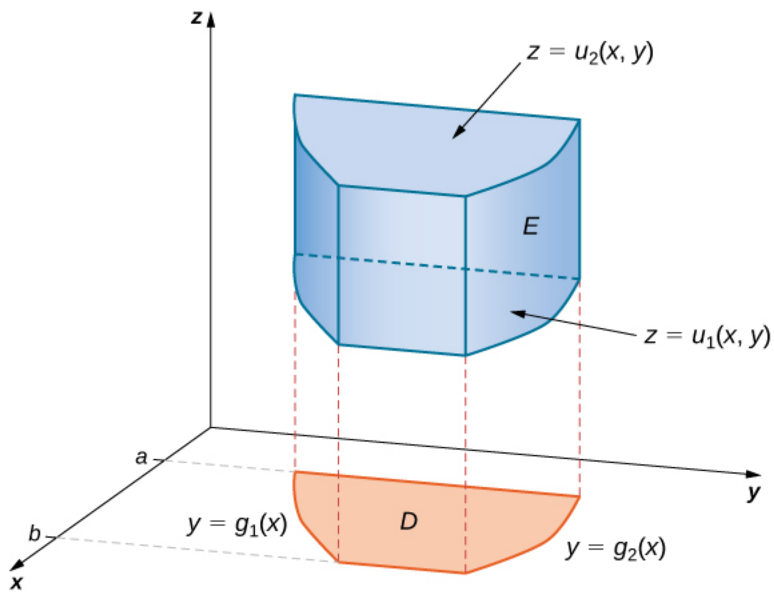
$$E = \{(x, y, z)|(y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane and the triple integral is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in Double Integrals over General Regions. If D in the xy-plane is of Type I ([link]), then
Equation:

$$E = \{(x, y, z)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}.$$



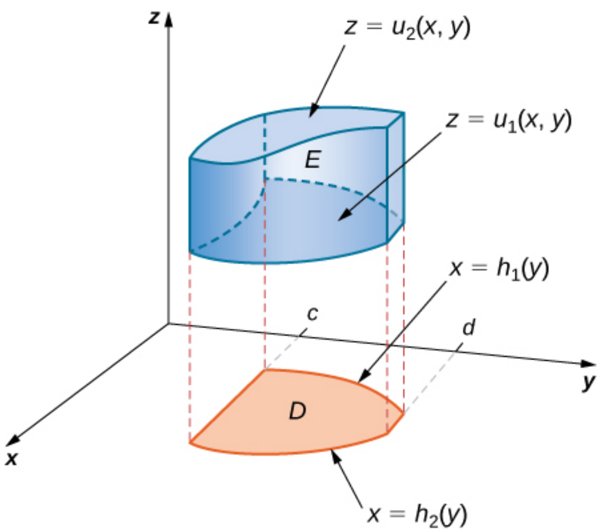
A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes
Equation:

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\,dy\,dx.$$

If D in the xy-plane is of Type II ([link]), then
Equation:

$$E = \{(x,y,z)|c \le x \le d, h_1(x) \le y \le h_2(x), u_1(x,y) \le z \le u_2(x,y)\}.$$



A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

Equation:

$$\iiint_E f(x,y,z)dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)dz\,dx\,dy.$$

Example:

Exercise:

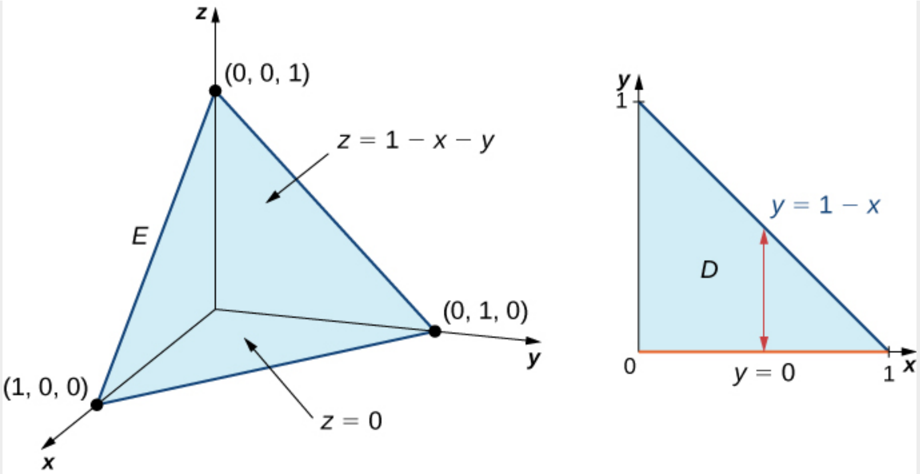
Problem:

Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Solution:

[link] shows the solid tetrahedron E and its projection D on the xy-plane.



The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as
Equation:

$$E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)dz\,dy\,dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)dz$. We have
Equation:

$$\int_{z=0}^{z=1-x-y} (5x - 3y)dz = (5x - 3y)\,(1 - x - y).$$

Now evaluate the integral $\int_{y=0}^{y=1-x} (5x - 3y)\,(1 - x - y)dy$, obtaining
Equation:

$$\int_{y=0}^{y=1-x} (5x - 3y)\,(1 - x - y)dy = \tfrac{1}{2}(x - 1)^2(6x - 1).$$

Finally, evaluate
Equation:

$$\int_{x=0}^{x=1} \tfrac{1}{2}(x - 1)^2(6x - 1)dx = \tfrac{1}{12}.$$

Putting it all together, we have
Equation:

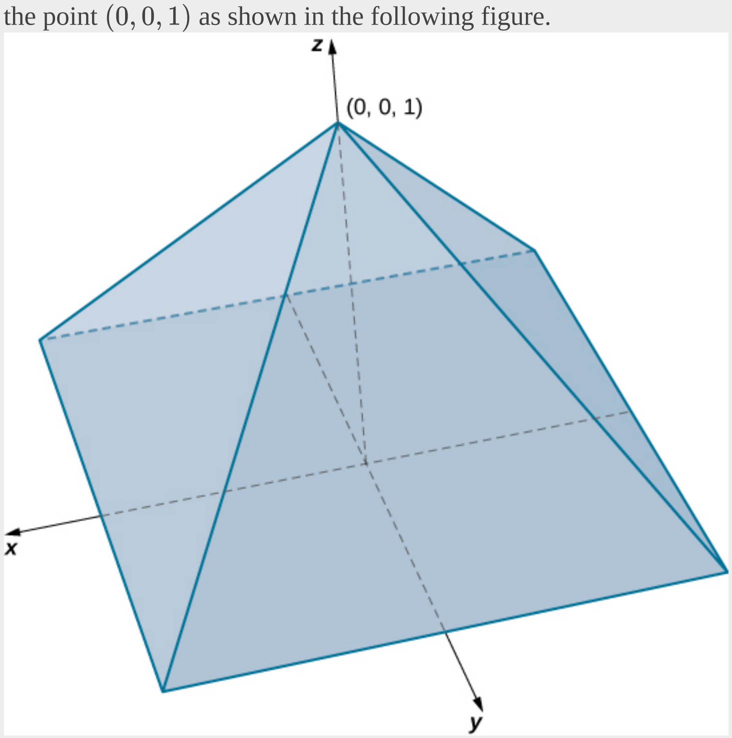
$$\iiint_E f(x,y,z)dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x-3y)dz\,dy\,dx = \tfrac{1}{12}.$$

Just as we used the double integral $\iint_D 1dA$ to find the area of a general bounded region D, we can use $\iiint_E 1dV$ to find the volume of a general solid bounded region E. The next example illustrates the method.

Example:
Exercise:

Problem:
Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1] \times [-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.



Finding the volume of a pyramid with a square base.

Solution:

In this pyramid the value of z changes from 0 to 1, and at each height z, the cross section of the pyramid for any value of z is the square $[-1+z, 1-z] \times [-1+z, 1-z]$. Hence, the volume of the pyramid is $\iiint\limits_E 1dV$ where

Equation:

$$E = \{(x,y,z)|0 \leq z \leq 1, -1+z \leq y \leq 1-z, -1+z \leq x \leq 1-z\}.$$

Thus, we have

Equation:

$$\iiint\limits_E 1dV = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} \int_{x=1+z}^{x=1-z} 1dx\,dy\,dz = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} (2-2z)dy\,dz = \int_{z=0}^{z=1} (2-2z)^2 dz = \frac{4}{3}.$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Note:
Exercise:

Problem:

Consider the solid sphere $E = \left\{(x,y,z)\middle|x^2+y^2+z^2=9\right\}$. Write the triple integral $\iiint\limits_E f(x,y,z)dV$ for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z)=1$. Notice that this gives the volume of a sphere using a triple integral.

Solution:

$$\iiint\limits_E 1dV = 8\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1dz\,dy\,dx = 36\pi.$$

Hint

Follow the steps in the previous example. Use symmetry.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example:
Exercise:

Problem:

Changing the Order of Integration

Consider the iterated integral

Equation:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx.$$

The order of integration here is first with respect to *z*, then *y*, and then *x*. Express this integral by changing the order of integration to be first with respect to *x*, then *z*, and then *y*. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution:

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

Equation:

$$E = \{(x,y,z)\,|\,0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}.$$

and

Equation:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

Equation:

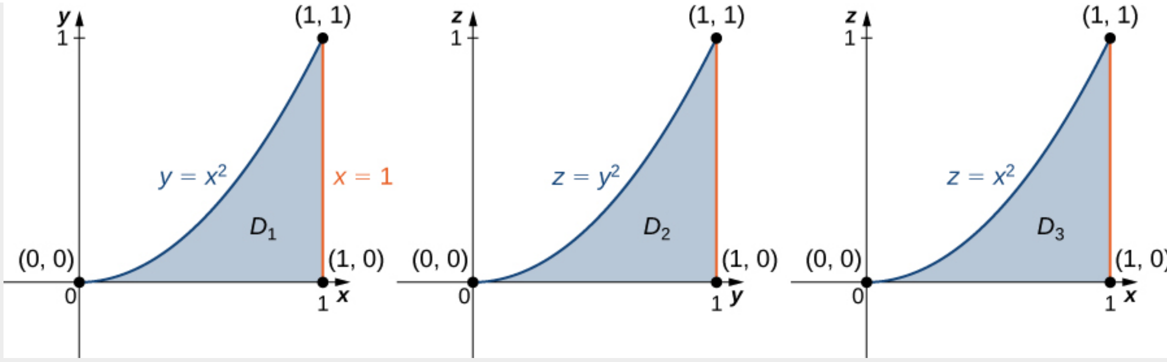
$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections ([link]):

on the xy-plane is $D_1 = \{(x,y)\,|\,0 \le x \le 1, 0 \le y \le x^2\} = \{(x,y)\,|\,0 \le y \le 1, \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z)\,|\,0 \le y \le 1, 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z)\,|\,0 \le x \le 1, 0 \le z \le x^2\}$.



The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\left\{(x, y, z) \middle| 0 \leq y \leq 1, 0 \leq z \leq y^2, \sqrt{y} \leq x \leq 1\right\}$, and consequently, the triple integral becomes

Equation:

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z) dx\, dz\, dy = \int_{y=0}^{y=1} \int_{z=0}^{z=x^2} \int_{x=\sqrt{y}}^{x=1} f(x, y, z) dx\, dz\, dy.$$

Now assume that $f(x, y, z) = xyz$ in each of the integrals. Then we have

Equation:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y^2} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left[xy\frac{z^2}{2}\Bigg|_{z=0}^{z=y^2}\right] dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left(x\frac{y^5}{2}\right) dy\, dx = \int_{x=0}^{x=1} \left[x\frac{y^6}{12}\Bigg|_{y=0}^{y=x^2}\right] dx = \int_{x=0}^{x=1} \frac{x^{13}}{12} dx = \tfrac{1}{168}$$

$$\int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \int_{x=\sqrt{y}}^{x=1} xyz\, dx\, dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left[yz\frac{x^2}{2}\Bigg|_{\sqrt{y}}^{1}\right] dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left(\frac{yz}{2} - \frac{y^2 z}{2}\right) dz\, dy = \int_{y=0}^{y=1} \left[\frac{yz^2}{4} - \frac{y^2 z^2}{4}\Bigg|_{z=0}^{z=y^2}\right] dy = \int_{y=0}^{y=1} \left(\frac{y^5}{4} - \frac{y^6}{4}\right) dy = \tfrac{1}{168}.$$

The answers match.

Note:
Exercise:

Problem: Write five different iterated integrals equal to the given integral
Equation:

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)dx\,dy\,dz.$$

Solution:

(i) $\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)dy\,dx\,dz$, (ii) $\int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)dx\,dz\,dy$, (iii) $\int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{z=4-y} f(x,y,z)dz\,dx\,dy$, (iv) $\int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)dz\,dy\,dx$, (v) $\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)dy\,dz\,dx$

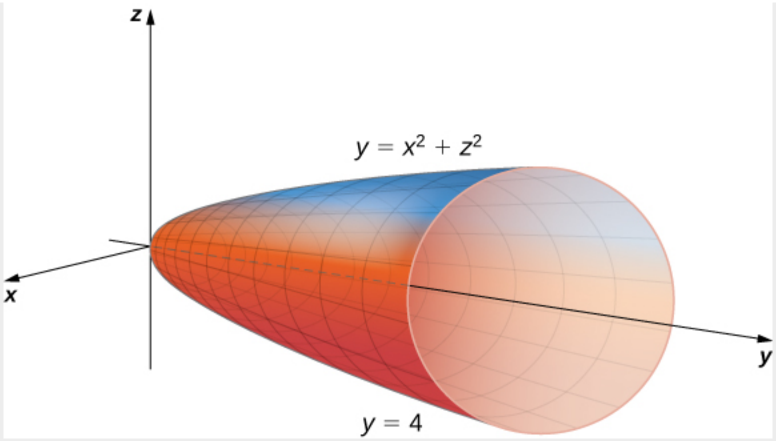
Hint

Follow the steps in the previous example, using the region E as $\{(x,y,z)|0\le z\le 4, 0\le y\le 4-z, 0\le x\le\sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

Example:
Exercise:

Problem:
Changing Integration Order and Coordinate Systems

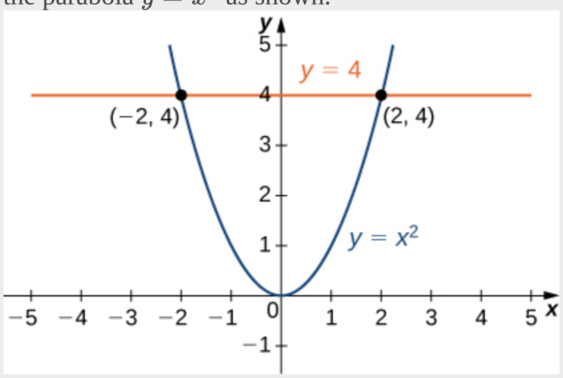
Evaluate the triple integral $\iiint\limits_E \sqrt{x^2+z^2}dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ ([link]) and the plane $y=4$.



Integrating a triple integral over a paraboloid.

Solution:

The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.



Cross section in the xy-plane of the paraboloid in [link].

Thus, we have

Equation:

$$E = \left\{(x, y, z) | -2 \le x \le 2, x^2 \le y \le 4, -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\right\}.$$

The triple integral becomes

Equation:

$$\iiint\limits_E \sqrt{x^2 + z^2} dV = \int\limits_{x=-2}^{x=2} \int\limits_{y=x^2}^{y=4} \int\limits_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2} dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2+z^2 \le 4$. So we obtain

Equation:

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dz\,dx.$$

Here the order of integration changes from being first with respect to z, then y, and then x to being first with respect to y, then to z, and then to x. It will soon be clear how this change can be beneficial for computation. We have

Equation:

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dz\,dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\,dx = r\,dr\,d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2+z^2$,

Equation:

$$\begin{aligned}\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(4-r^2\right) r r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\,d\theta = \frac{128\pi}{15}.\end{aligned}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Note:

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

Equation:

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x,y,z)\,dV.$$

Note that the volume is $V(E) = \iiint_E 1 dV$.

Example:
Exercise:

Problem:
Finding an Average Temperature

The temperature at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) = (xy + 8z + 20)^\circ$C. Find the average temperature over the solid.

Solution:

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x + y + z = 1$ has intercepts $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The region E looks like
Equation:

$$E = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}.$$

Hence the triple integral of the temperature is
Equation:

$$\iiint_E f(x, y, z) dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20) dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is $V(E) = \iiint_E 1 dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1 dz\, dy\, dx = \frac{1}{6}$.

Hence the average value is $T_{\text{ave}} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}$ degrees Celsius.

Note:
Exercise:

Problem:

Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Solution:

$f_{\text{ave}} = 8$

Hint

Follow the steps in the previous example.

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then
  Equation:
  $$\iiint\limits_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx\, dy\, dz$$
  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral
  Equation:
  $$V(E) = \iiint\limits_E 1 dV.$$
- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use
  Equation:
  $$f_{\text{ave}} = \frac{1}{V(E)} \iiint\limits_E f(x, y, z) dV.$$

Key Equations

- **Triple integral**
  $$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z = \iiint\limits_B f(x, y, z) dV$$

In the following exercises, evaluate the triple integrals over the rectangular solid box B.

Exercise:

Problem: $\iiint\limits_B \left(2x + 3y^2 + 4z^3\right) dV$, where $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$

Solution:

192

Exercise:

Problem: $\iiint\limits_B (xy + yz + xz) dV$, where $B = \{(x, y, z) | 1 \le x \le 2, 0 \le y \le 2, 1 \le z \le 3\}$

Exercise:

Problem: $\iiint\limits_B (x\cos y + z)dV$, where $B = \{(x,y,z)|0 \le x \le 1, 0 \le y \le \pi, -1 \le z \le 1\}$

Solution:

0

Exercise:

Problem: $\iiint\limits_B \left(z\sin x + y^2\right)dV$, where $B = \{(x,y,z)|0 \le x \le \pi, 0 \le y \le 1, -1 \le z \le 2\}$

In the following exercises, change the order of integration by integrating first with respect to z, then x, then y.
Exercise:

Problem: $\int\limits_0^1 \int\limits_1^2 \int\limits_2^3 \left(x^2 + \ln y + z\right)dx\, dy\, dz$

Solution:

$$\int\limits_1^2 \int\limits_2^3 \int\limits_0^1 \left(x^2 + \ln y + z\right)dz\, dx\, dy = \tfrac{35}{6} + 2\ln 2$$

Exercise:

Problem: $\int\limits_0^1 \int\limits_{-1}^1 \int\limits_0^3 (ze^x + 2y)dx\, dy\, dz$

Exercise:

Problem: $\int\limits_{-1}^2 \int\limits_1^3 \int\limits_0^4 \left(x^2 z + \frac{1}{y}\right)dx\, dy\, dz$

Solution:

$$\int\limits_1^3 \int\limits_0^4 \int\limits_{-1}^2 \left(x^2 z + \frac{1}{y}\right)dz\, dx\, dy = 64 + 12\ln 3$$

Exercise:

Problem: $\int\limits_1^2 \int\limits_{-2}^{-1} \int\limits_0^1 \frac{x+y}{z}dx\, dy\, dz$

Exercise:

Problem:

Let $F, G,$ and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f$. Show that

Equation:

$$\int_a^b \int_c^d \int_e^f F(x)G(y)H(z)dz\,dy\,dx = \left(\int_a^b F(x)dx\right)\left(\int_c^d G(y)dy\right)\left(\int_e^f H(z)dz\right).$$

Exercise:

Problem:

Let $F, G,$ and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f$. Show that

Equation:

$$\int_a^b \int_c^d \int_e^f F'(x)G'(y)H'(z)dz\,dy\,dx = [F(b) - F(a)]\ [G(d) - G(c)]\ [H(f) - H(e)].$$

In the following exercises, evaluate the triple integrals over the bounded region $E = \{(x, y, z) | a \le x \le b, h_1(x) \le y \le h_2(x), e \le z \le f\}$.

Exercise:

Problem: $\iiint\limits_E (2x + 5y + 7z)dV$, where $E = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le -x + 1, 1 \le z \le 2\}$

Solution:

$\frac{77}{12}$

Exercise:

Problem: $\iiint\limits_E (y \ln x + z)dV$, where $E = \{(x, y, z) | 1 \le x \le e, 0 \le y \le \ln x, 0 \le z \le 1\}$

Exercise:

Problem: $\iiint\limits_E (\sin x + \sin y)dV$, where $E = \left\{(x, y, z) | 0 \le x \le \frac{\pi}{2}, -\cos x \le y \le \cos x, -1 \le z \le 1\right\}$

Solution:

2

Exercise:

Problem: $\iiint\limits_E (xy + yz + xz)dV$, where $E = \left\{(x, y, z) | 0 \le x \le 1, -x^2 \le y \le x^2, 0 \le z \le 1\right\}$

In the following exercises, evaluate the triple integrals over the indicated bounded region E.
Exercise:

Problem: $\iiint\limits_E (x+2yz)dV$, where $E=\{(x,y,z)|0\le x\le 1, 0\le y\le x, 0\le z\le 5-x-y\}$

Solution:

$\frac{439}{120}$

Exercise:

Problem: $\iiint\limits_E \left(x^3+y^3+z^3\right)dV$, where $E=\{(x,y,z)|0\le x\le 2, 0\le y\le 2x, 0\le z\le 4-x-y\}$

Exercise:

Problem:

$\iiint\limits_E y\,dV$, where $E=\left\{(x,y,z)\middle|-1\le x\le 1, -\sqrt{1-x^2}\le y\le \sqrt{1-x^2}, 0\le z\le 1-x^2-y^2\right\}$

Solution:

0

Exercise:

Problem:

$\iiint\limits_E x\,dV$, where $E=\left\{(x,y,z)\middle|-2\le x\le 2, -4\sqrt{1-x^2}\le y\le \sqrt{4-x^2}, 0\le z\le 4-x^2-y^2\right\}$

In the following exercises, evaluate the triple integrals over the bounded region E of the form $E=\{(x,y,z)|g_1(y)\le x\le g_2(y), c\le y\le d, e\le z\le f\}$.
Exercise:

Problem: $\iiint\limits_E x^2dV$, where $E=\left\{(x,y,z)|1-y^2\le x\le y^2-1, -1\le y\le 1, 1\le z\le 2\right\}$

Solution:

$-\frac{64}{105}$

Exercise:

Problem: $\iiint\limits_E (\sin x+y)dV$, where $E=\left\{(x,y,z)\middle|-y^4\le x\le y^4, 0\le y\le 2, 0\le z\le 4\right\}$

Exercise:

Problem: $\iiint\limits_E (x-yz)dV$, where $E=\left\{(x,y,z)\middle|-y^6\le x\le \sqrt{y}, 0\le y\le 1x, -1\le z\le 1\right\}$

Solution:

$\frac{11}{26}$

Exercise:

Problem: $\iiint\limits_E zdV$, where $E = \left\{(x, y, z) | 2 - 2y \le x \le 2 + \sqrt{y}, 0 \le y \le 1x, 2 \le z \le 3\right\}$

In the following exercises, evaluate the triple integrals over the bounded region

Equation:

$$E = \{(x, y, z) | g_1(y) \le x \le g_2(y), c \le y \le d, u_1(x, y) \le z \le u_2(x, y)\}.$$

Exercise:

Problem: $\iiint\limits_E zdV$, where $E = \left\{(x, y, z) \middle| -y \le x \le y, 0 \le y \le 1, 0 \le z \le 1 - x^4 - y^4\right\}$

Solution:

$\frac{113}{450}$

Exercise:

Problem: $\iiint\limits_E (xz + 1)dV$, where $E = \left\{(x, y, z) \middle| 0 \le x \le \sqrt{y}, 0 \le y \le 2, 0 \le z \le 1 - x^2 - y^2\right\}$

Exercise:

Problem:

$\iiint\limits_E (x - z)dV$, where $E = \left\{(x, y, z) \middle| -\sqrt{1 - y^2} \le x \le y, 0 \le y \le \frac{1}{2}x, 0 \le z \le 1 - x^2 - y^2\right\}$

Solution:

$\frac{1}{160}\left(6\sqrt{3} - 41\right)$

Exercise:

Problem: $\iiint\limits_E (x + y)dV$, where $E = \left\{(x, y, z) \middle| 0 \le x \le \sqrt{1 - y^2}, 0 \le y \le 1x, 0 \le z \le 1 - x\right\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | (x, y) \in D, u_1(x, y)x \le z \le u_2(x, y)\}$, where D is the projection of E onto the xy-plane.

Exercise:

Problem: $\iint\limits_D \left(\int_1^2 (x + z)dz\right) dA$, where $D = \left\{(x, y) | x^2 + y^2 \le 1\right\}$

Solution:

$\frac{3\pi}{2}$

Exercise:

Problem: $\iint\limits_{D}\left(\int\limits_{1}^{3} x\,(z+1)dz\right)dA$, where $D=\left\{(x,y)\middle|x^2-y^2\geq 1, x\leq\sqrt{5}\right\}$

Exercise:

Problem: $\iint\limits_{D}\left(\int\limits_{0}^{10-x-y}(x+2z)dz\right)dA$, where $D=\{(x,y)|y\geq 0, x\geq 0, x+y\leq 10\}$

Solution:

1250

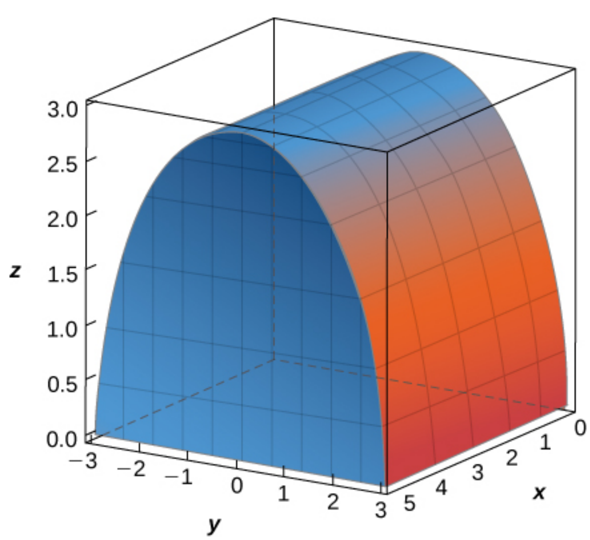
Exercise:

Problem: $\iint\limits_{D}\left(\int\limits_{0}^{4x^2+4y^2} y\,dz\right)dA$, where $D=\left\{(x,y)\middle|x^2+y^2\leq 4, y\geq 1, x\geq 0\right\}$

Exercise:

Problem:

The solid E bounded by $y^2+z^2=9, z=0, x=0$, and $x=5$ is shown in the following figure. Evaluate the integral $\iiint\limits_{E} z\,dV$ by integrating first with respect to z, then y, and then x.



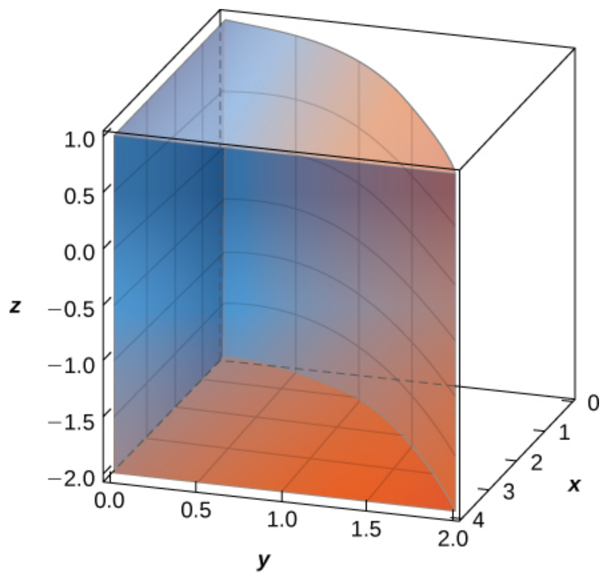
Solution:

$$\int_0^5 \int_{-3}^3 \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx = 90$$

Exercise:

Problem:

The solid E bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, and $z = 1$ is given in the following figure. Evaluate the integral $\iiint\limits_E xyz\, dV$ by integrating first with respect to x, then y, and then z.



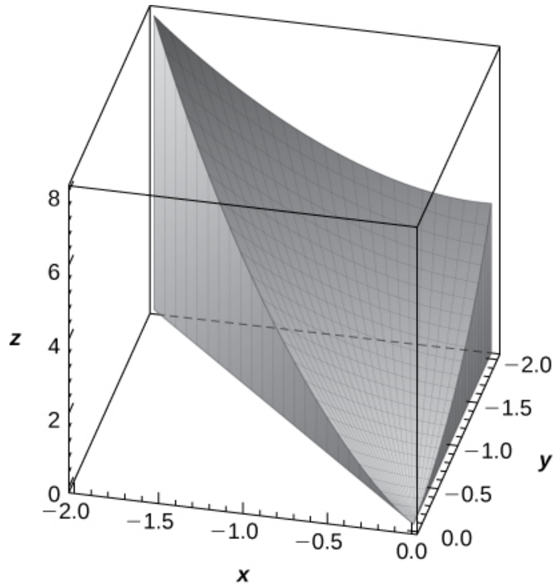
Exercise:

Problem:

[T] The volume of a solid E is given by the integral $\int_{-2}^0 \int_x^0 \int_0^{x^2+y^2} dz\, dy\, dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Solution:

$V = 5.33$



Exercise:

Problem:

[T] The volume of a solid E is given by the integral $\int_{-1}^{0}\int_{-x^2}^{0}\int_{0}^{1+\sqrt{x^2+y^2}} dz\,dy\,dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In the following exercises, use two circular permutations of the variables $x, y,$ and z to write new integrals whose values equal the value of the original integral. A circular permutation of $x, y,$ and z is the arrangement of the numbers in one of the following orders: $y, z,$ and x or $z, x,$ and y.

Exercise:

Problem: $\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(x^2z^2+1\right)dx\,dy\,dz$

Solution:

$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(y^2z^2+1\right)dz\,dx\,dy;\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(x^2y^2+1\right)dy\,dz\,dx$

Exercise:

Problem: $\int_{1}^{3}\int_{0}^{1}\int_{0}^{-x+1}(2x+5y+7z)dy\,dx\,dz$

Exercise:

Problem: $\int_{0}^{1}\int_{-y}^{y}\int_{0}^{1-x^4-y^4} \ln x \, dz \, dx \, dy$

Exercise:

Problem: $\int_{-1}^{1}\int_{0}^{1}\int_{-y^6}^{\sqrt{y}} (x+yz) dx \, dy \, dz$

Exercise:

Problem:

Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

Solution:

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{\sqrt{x^2+z^2}}^{a^2} dy \, dx \, dz$$

Exercise:

Problem:

Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Exercise:

Problem:

Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x = 0, x = 1, y = 0, y = 3, z = 0$, and $z = 5$.

Solution:

$\frac{9}{2}$

Exercise:

Problem:

Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Exercise:

Problem:

Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = \sqrt{y-1}, x = 0$, and $x + y = 7$.

Solution:

$\frac{156}{5}$

Exercise:

Problem:

Find the volume of the solid E that lies under the plane $2x + y + z = 8$ and whose projection onto the xy-plane is bounded by $x = \sin^{-1} y, y = 0$, and $x = \frac{\pi}{2}$.

Exercise:

Problem:

Consider the pyramid with the base in the xy-plane of $[-2, 2] \times [-2, 2]$ and the vertex at the point $(0, 0, 8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8$, $4y - z = -8$, $4x + z = 8$, and $-4x + z = 8$.
b. Find the volume of the pyramid.

Solution:

a. Answers may vary; b. $\frac{128}{3}$

Exercise:

Problem:

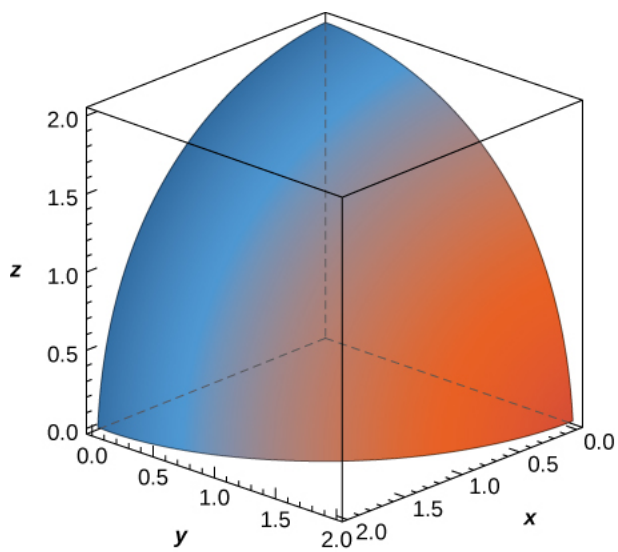
Consider the pyramid with the base in the xy-plane of $[-3, 3] \times [-3, 3]$ and the vertex at the point $(0, 0, 9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9$, $3y + z = 9$, $y = 0$ and $x = 0$.
b. Find the volume of the pyramid.

Exercise:

Problem:

The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

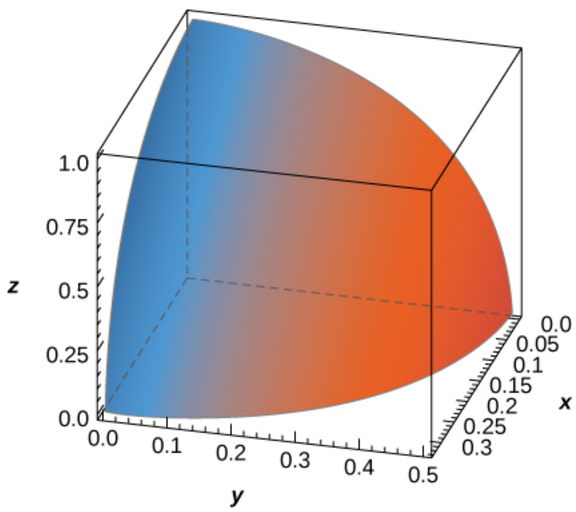
Solution:

a. $\int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\,dy\,dx$; b. $\int_0^2 \int_0^{\sqrt{r^2-y^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\,dx\,dy$, $\int_0^r \int_0^{\sqrt{r^2-z^2}} \int_0^{\sqrt{r^2-x^2-z^2}} dy\,dx\,dz$, $\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-z^2}} dy\,dz\,dx$, $\int_0^r \int_0^{\sqrt{r^2-z^2}} \int_0^{\sqrt{r^2-y^2-z^2}} dx\,dy\,dz$, $\int_0^r \int_0^{\sqrt{r^2-y^2}} \int_0^{\sqrt{r^2-y^2-z^2}} dx\,dz\,dy$

Exercise:

Problem:

The solid E bounded by the equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Exercise:

Problem:

Find the volume of the prism with vertices $(0, 0, 0), (2, 0, 0), (2, 3, 0), (0, 3, 0), (0, 0, 1),$ and $(2, 0, 1)$.

Solution:

3

Exercise:

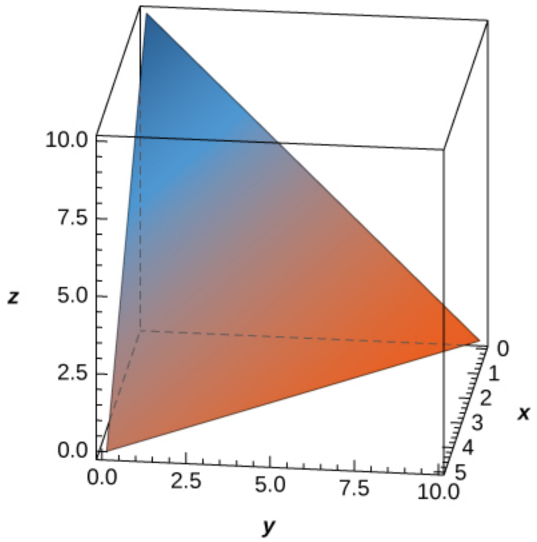
Problem:

Find the volume of the prism with vertices $(0, 0, 0), (4, 0, 0), (4, 6, 0), (0, 6, 0), (0, 0, 1),$ and $(4, 0, 1)$.

Exercise:

Problem:

The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.



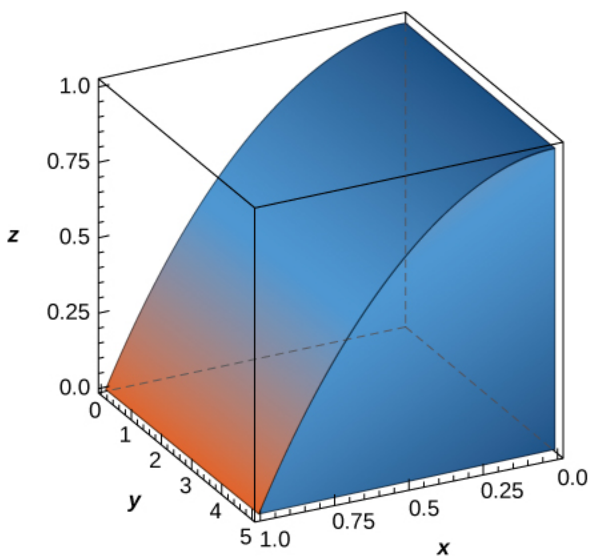
Solution:

$\frac{250}{3}$

Exercise:

Problem:

The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Exercise:

Problem:

The midpoint rule for the triple integral $\iiint\limits_B f(x, y, z) dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\overline{x_i}, \overline{y_j}, \overline{z_k}) \Delta V$, where $(\overline{x_i}, \overline{y_j}, \overline{z_k})$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate $\iiint\limits_B x^2 dV$ over the solid $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Solution:

$\frac{5}{16} \approx 0.313$

Exercise:

Problem: **[T]**

a. Apply the midpoint rule to approximate $\iiint\limits_B e^{-x^2} dV$ over the solid $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.
b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4, \ldots, 10$.

Exercise:

Problem:

Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = xz + 5z + 10$. Find the average temperature over the solid.

Solution:

$\frac{35}{2}$

Exercise:

Problem:

Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = x + y + xy$. Find the average temperature over the solid.

Exercise:

Problem:

Show that the volume of a right square pyramid of height h and side length a is $v = \frac{ha^2}{3}$ by using triple integrals.

Exercise:

Problem:

Show that the volume of a regular right hexagonal prism of edge length a is $\frac{3a^3\sqrt{3}}{2}$ by using triple integrals.

Exercise:

Problem:

Show that the volume of a regular right hexagonal pyramid of edge length a is $\frac{a^3\sqrt{3}}{2}$ by using triple integrals.

Exercise:

Problem:

If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint\limits_E \rho(x, y, z)dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Glossary

triple integral
: the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

Triple Integrals in Cylindrical and Spherical Coordinates

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

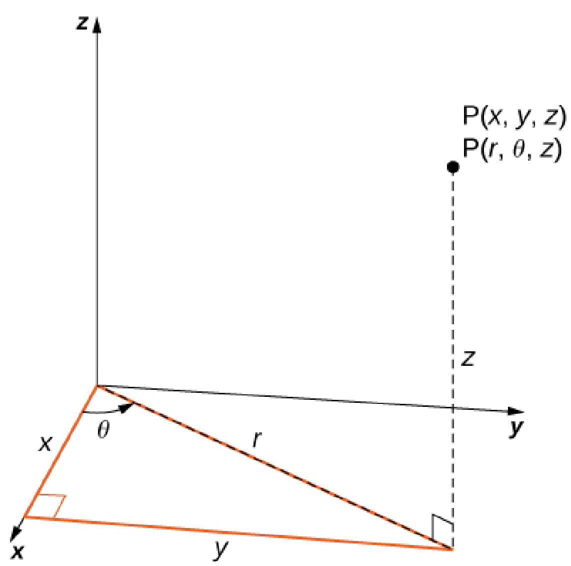
Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter opener, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$, a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$, a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the xy-plane as shown in the following figure.



Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion $x = r\cos\theta$ and $y = r\sin\theta$. To convert from cylindrical to rectangular coordinates, we use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. The z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z), the equations $x = k, y = l$, and $z = m$, where k, l, and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c, \theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

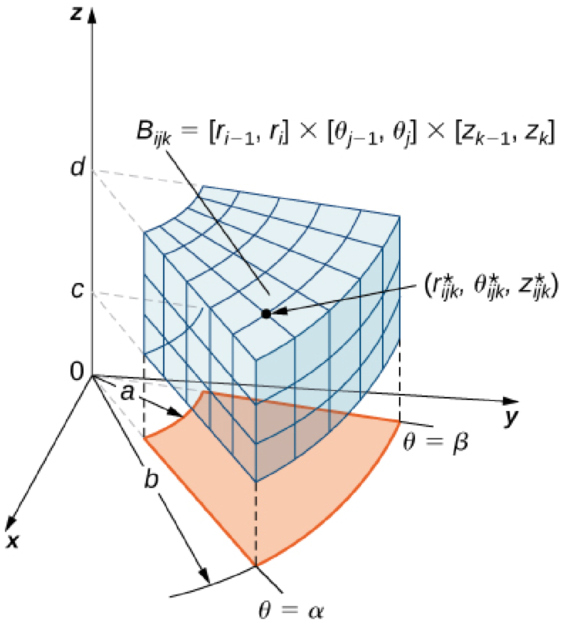
Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in [link]. These equations will become handy as we proceed with solving problems using triple integrals.

	Circular cylinder	**Circular cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2\left(x^2 + y^2\right)$	$x^2 + y^2 + z^2 = c^2$	$z = c\left(x^2 + y^2\right)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

Equations of Some Common Shapes

As before, we start with the simplest bounded region B in $\mathbb{R}^3$, to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$ ([link]). Suppose we divide each interval into l, m and n subdivisions such that $\Delta r = \frac{b-a}{l}, \Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.



A cylindrical box B described by cylindrical coordinates.

Note:
Definition
Consider the cylindrical box (expressed in cylindrical coordinates)
Equation:

$$B = \{(r, \theta, z) | a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k]$ ([link]), then we can define the **triple integral in cylindrical coordinates** as the limit of a triple Riemann sum, provided the following limit exists:
Equation:

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z.$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral $\iiint\limits_B g(x, y, z) dV$ is equal to the triple integral $\iiint\limits_B g(r \cos\theta, r \sin\theta, z) r\, dr\, d\theta\, dz$ and we have

Equation:

$$\iiint\limits_B g(x,y,z)dV = \iiint\limits_B g(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz = \iiint\limits_B f(r,\theta,z)r\,dr\,d\theta\,dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Note:
Fubini's Theorem in Cylindrical Coordinates
Suppose that $g(x,y,z)$ is continuous on a rectangular box B, which when described in cylindrical coordinates looks like $B = \{(r,\theta,z)|a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$.
Then $g(x,y,z) = g(r\cos\theta, r\sin\theta, z) = f(r,\theta,z)$ and
Equation:

$$\iiint\limits_B g(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(r,\theta,z)r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example:
Exercise:

Problem:
Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral $\iiint\limits_B (zr\sin\theta)r\,dr\,d\theta\,dz$ where the cylindrical box B is $B = \{(r,\theta,z)|0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le z \le 4\}$.

Solution:

As stated in Fubini's theorem, we can write the triple integral as the iterated integral
Equation:

$$\iiint\limits_B (zr\sin\theta)r\,dr\,d\theta\,dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\sin\theta)r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

Equation:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)r\,dz\,dr\,d\theta$$
$$=\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right)=\left(-\cos\theta|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$

Note:

Exercise:

Problem: Evaluate the triple integral $\int\limits_{\theta=0}^{\theta=\pi}\int\limits_{r=0}^{r=1}\int\limits_{z=0}^{z=4} rz\sin\theta r\,dz\,dr\,d\theta.$

Solution:

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Hint

Follow the same steps as in the previous example.

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|\,(r,\theta)\in D,u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$, where D is the projection of E onto the $r\theta$-plane, is

Equation:

$$\iiint\limits_E f(r,\theta,z)r\,dr\,d\theta\,dz=\iint\limits_D\left[\int\limits_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|g_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

Equation:

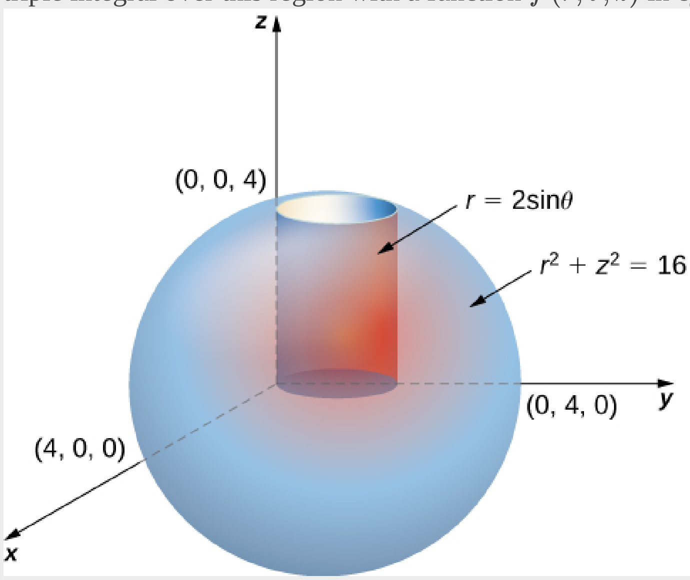
$$\iiint\limits_E f(r,\theta,z)r\,dr\,d\theta=\int\limits_{\theta=\alpha}^{\theta=\beta}\int\limits_{r=g_1(\theta)}^{r=g_2(\theta)}\int\limits_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin ([link]). Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.



Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution:

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is

Equation:

$$E = \left\{(r, \theta, z)|0 \leq \theta \leq \pi, 0 \leq r \leq 2\sin\theta, 0 \leq z \leq \sqrt{16 - r^2}\right\}.$$

Therefore, the triple integral is

Equation:

$$\iiint_E f(r,\theta,z) r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z) r\,dz\,dr\,d\theta.$$

Note:
Exercise:

Problem:

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Solution:

$$\iiint_E f(r,\theta,z) r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z) r\,dz\,dr\,d\theta.$$

Hint

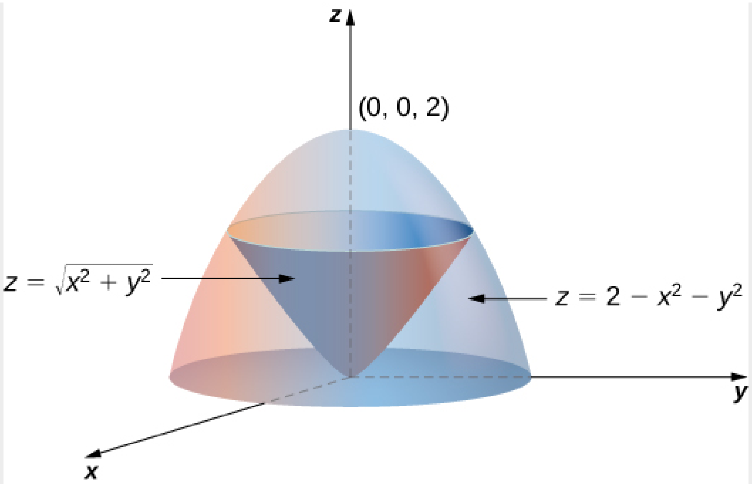
Analyze the region, and draw a sketch.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. ([link]). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$
b. $dr\,dz\,d\theta$.



Setting up a triple integral in cylindrical coordinates over a conical region.

Solution:

a. The cone is of radius 1 where it meets the paraboloid. Since $z=2-x^2-y^2=2-r^2$ and $z=\sqrt{x^2+y^2}=r$ (assuming r is nonnegative), we have $2-r^2=r$. Solving, we have $r^2+r-2=(r+2)(r-1)=0$. Since $r\geq 0$, we have $r=1$. Therefore $z=1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z=1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.
Thus, we can describe the region as
Equation:

$$E=\left\{(r,\theta,z)|0\leq\theta\leq 2\pi,0\leq r\leq 1,r\leq z\leq 2-r^2\right\}.$$

Hence the integral for the volume is
Equation:

$$V=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r}^{z=2-r^2} r\,dz\,dr\,d\theta.$$

b. We can also write the cone surface as $r=z$ and the paraboloid as $r^2=2-z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z=1$ divides the region into two regions. Then the region can be described as
Equation:

$$\begin{aligned}E&=\{(r,\theta,z)|0\leq\theta\leq 2\pi,0\leq z\leq 1,0\leq r\leq z\}\\&\cup\left\{(r,\theta,z)|0\leq\theta\leq 2\pi,1\leq z\leq 2,0\leq r\leq\sqrt{2-z}\right\}.\end{aligned}$$

Now the integral for the volume becomes
Equation:

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta.$$

Note:
Exercise:

Problem: Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Solution:

$E = \left\{(r,\theta,z)|0 \le \theta \le 2\pi, 0 \le z \le 1, z \le r \le 2 - z^2\right\}$ and $V = \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$

Hint

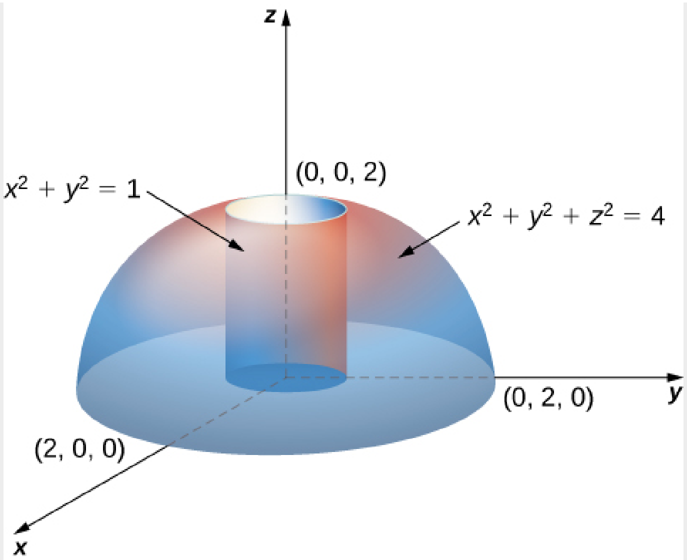
Note that θ is independent of r and z.

Example:
Exercise:

Problem:
Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ ([link]). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\,dr\,d\theta$
b. $dr\,dz\,d\theta$.



Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution:

a. Note that the equation for the sphere is
Equation:

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is
Equation:

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E
Equation:

$$E = \left\{(r, \theta, z) | 0 \leq z \leq \sqrt{4 - r^2}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\right\}$$

Hence the integral for the volume is
Equation:

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[rz|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\,d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2} \right) dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $\left(1, \sqrt{3}\right)$ in the rz-plane

Equation:

$$E_1 = \left\{ (r, \theta, z) | 0 \le r \le \sqrt{4 - r^2}, \sqrt{3} \le z \le 2, 0 \le \theta \le 2\pi \right\}$$

and

Equation:

$$E_2 = \left\{ (r, \theta, z) | 0 \le r \le 1, 0 \le z \le \sqrt{3}, 0 \le \theta \le 2\pi \right\}.$$

Hence the integral for the volume is

Equation:

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\,dr\,dz\,d\theta \\ &= \sqrt{3}\pi + \left(\tfrac{16}{3} - 3\sqrt{3} \right)\pi = 2\pi \left(\tfrac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

Note:
Exercise:

Problem: Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Solution:

$E_2 = \left\{ (r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq \sqrt{4 - r^2} \right\}$ and

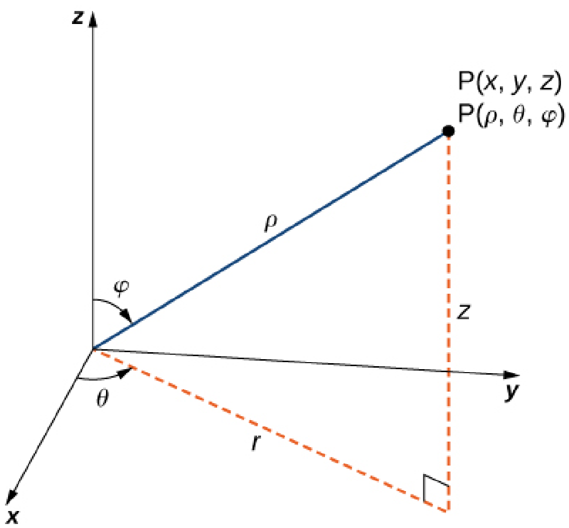
$$V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r \, d\theta \, dz \, dr.$$

Hint

A figure can be helpful. Note that θ is independent of r and z.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP ([link]). Note that $\rho \geq 0$ and $0 \leq \varphi \leq \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.



The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:
Equation:

$$x = \rho \sin \varphi \cos \theta, y = \rho \sin \varphi \sin \theta, \text{ and } z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:
Equation:

$$\rho^2 = x^2 + y^2 + z^2, \tan\theta = \frac{y}{x}, \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are
Equation:

- $r = \rho\sin\varphi$
- $\theta = \theta$
- $z = \rho\cos\varphi$

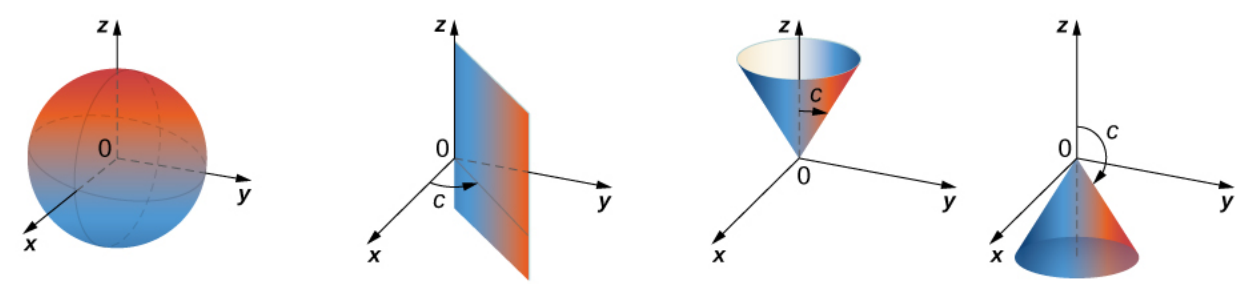
These equations are used to convert from spherical coordinates to cylindrical coordinates

and
Equation:

- $\rho = \sqrt{r^2+z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

These equations are used to convert from cylindrical coordinates to spherical coordinates.

The following figure shows a few solid regions that are convenient to express in spherical coordinates.



$0 < c < \frac{\pi}{2}$ $\frac{\pi}{2} < c < \pi$

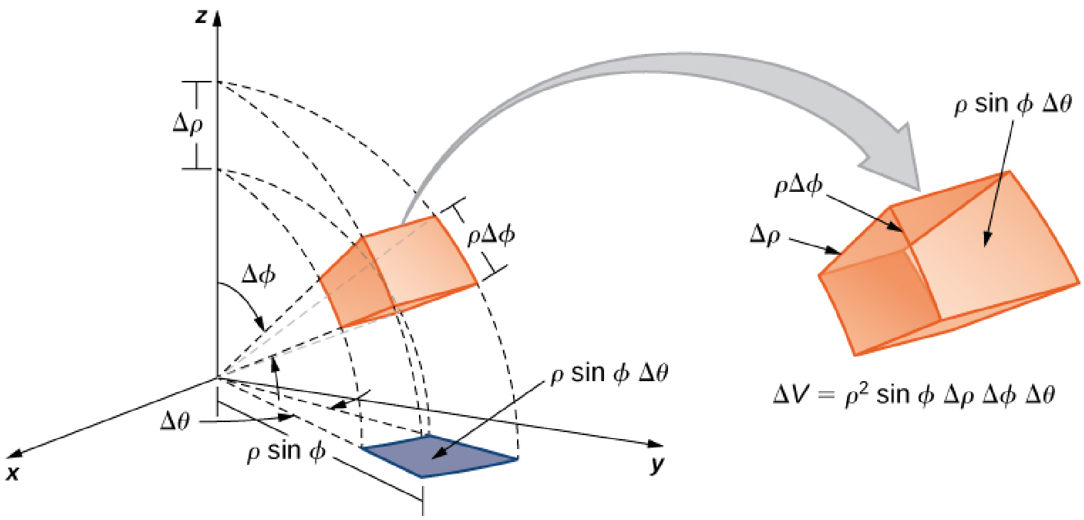
Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m and n subdivisions such that $\Delta\rho = \frac{b-a}{l}, \Delta\theta = \frac{\beta-\alpha}{m}, \Delta\varphi = \frac{\psi-\gamma}{n}$.

Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\overset{*}{\rho}_{ijk}, \overset{*}{\theta}_{ijk}, \overset{*}{\varphi}_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)(\rho\Delta\varphi)(\rho \sin\varphi\Delta\theta)$, , as shown in the following figure.



The volume element of a box in spherical coordinates.

Note:
Definition
The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,
Equation:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\overset{*}{\rho}_{ijk}, \overset{*}{\theta}_{ijk}, \overset{*}{\varphi}_{ijk})(\overset{*}{\rho}_{ijk})^2 \sin\varphi\Delta\rho\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini’s theorem takes the following form.

Note:
Fubini’s Theorem for Spherical Coordinates
If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then
Equation:

$$\iiint_B f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example:
Exercise:

Problem:
Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral $\displaystyle\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/2} \int_{p=0}^{\rho=1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$

Solution:

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:
Equation:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi \, d\varphi \int_0^1 \rho^2 d\rho = (2\pi)(1)\left(\frac{1}{3}\right) = \frac{2\pi}{3}.$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho, \theta, \varphi)$ over a general solid region
Equation:

$$E = \{(\rho, \theta, \varphi) | (\rho, \theta) \in D, u_1(\rho, \theta) \le \varphi \le u_2(\rho, \theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is
Equation:

$$\iiint_E f(\rho,\theta,\varphi)dV = \iint_D \left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)d\varphi\right] dA.$$

In particular, if $D = \{(\rho,\theta)|g_1(\theta) \le \rho \le g_2(\theta), \alpha \le \theta \le \beta\}$, then we have
Equation:

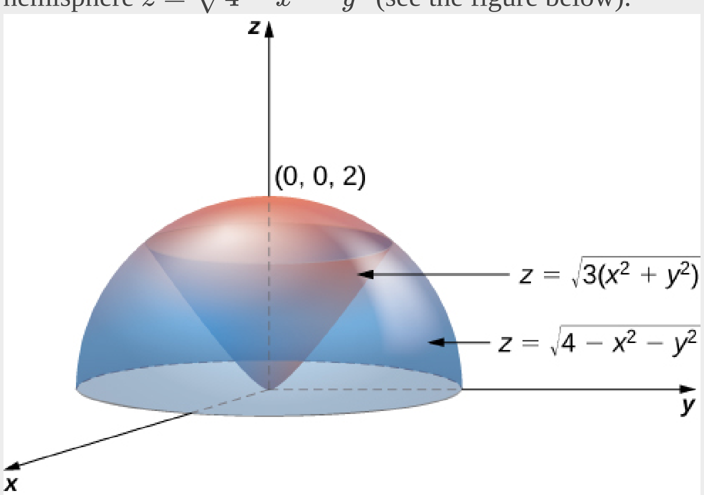
$$\iiint_E f(\rho,\theta,\varphi)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$ (see the figure below).



A region bounded below by a cone and above by a hemisphere.

Solution:

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3\left(x^2 + y^2\right)}$ or $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ or $\tan \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is $V\left(E\right) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\varphi=\pi/6} \int\limits_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$

Note:
Exercise:

Problem:

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Solution:

$$V\left(E\right) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\varphi=\pi/3} \int\limits_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

Hint

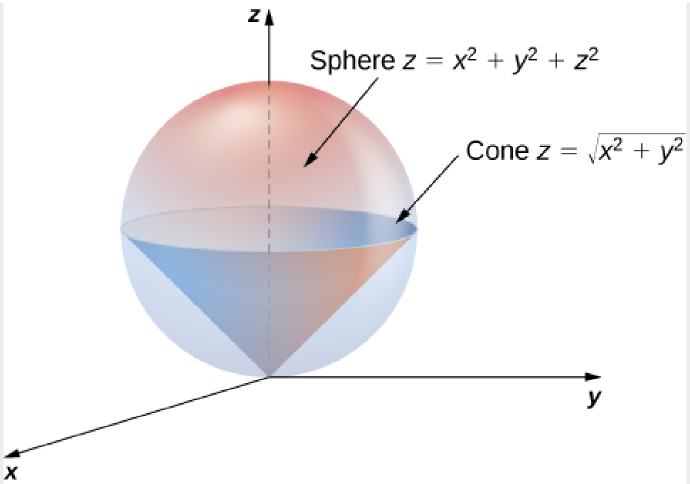
Follow the steps of the previous example.

Example:
Exercise:

Problem:
Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ ([link]). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$,
b. $d\varphi \, d\rho \, d\theta$.



A region bounded below by a cone and above by a sphere.

Solution:

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.
For the sphere:
Equation:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos \varphi \\ \rho &= \cos \varphi. \end{aligned}$$

For the cone:
Equation:

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho \cos \varphi &= \sqrt{\rho^2 \sin^2 \varphi \cos^2 \phi + \rho^2 \sin^2 \varphi \sin^2 \phi} \\ \rho \cos \varphi &= \sqrt{\rho^2 \sin^2 \varphi \left(\cos^2 \phi + \sin^2 \phi\right)} \\ \rho \cos \varphi &= \rho \sin \varphi \\ \cos \varphi &= \sin \varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes
Equation:

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are
Equation:

$$\begin{gathered} 0 \leq \varphi \leq \pi/4 \\ 0 \leq \rho \leq \cos\varphi. \end{gathered}$$

The curve $\rho = \cos\varphi$ meets the line $\varphi = \pi/4$ at the point $\left(\pi/4, \sqrt{2}/2\right)$. Thus, to change the order of integration, we need to use two pieces:
Equation:

$$\begin{array}{lcl} 0 \leq \rho \leq \sqrt{2}/2 & \text{and} & \sqrt{2}/2 \leq \rho \leq 1 \\ 0 \leq \varphi \leq \pi/4 & & 0 \leq \varphi \leq \cos^{-1}\rho. \end{array}$$

Hence the integral for the volume of the solid region E becomes
Equation:

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

In each case, the integration results in $V\left(E\right) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example:
Exercise:

Problem:
Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:
Equation:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy.$$

Solution:

The ranges of the variables are
Equation:

$$\begin{aligned} -1 &\leq y \leq 1 \\ 0 &\leq x \leq \sqrt{1-y^2} \\ x^2+y^2 &\leq z \leq \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are
Equation:

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1.$$

The limits of z are $r^2 \leq z \leq r$, hence
Equation:

$$\int\limits_{y=-1}^{y=1} \int\limits_{x=0}^{x=\sqrt{1-y^2}} \int\limits_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int\limits_{\theta=-\pi/2}^{\theta=\pi/2} \int\limits_{r=0}^{r=1} \int\limits_{z=r^2}^{z=r} r\,(r\cos\theta)\,(r\sin\theta) z\,dz\,dr\,d\theta.$$

Example:
Exercise:

Problem:
Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:
Equation:

$$\int\limits_{y=0}^{y=3} \int\limits_{x=0}^{x=\sqrt{9-y^2}} \int\limits_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy.$$

Solution:

The ranges of the variables are
Equation:

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ, we need to find where the cone and the sphere intersect, so solve the equation

Equation:

$$\begin{aligned} r^2 + z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2 + z^2 &= 18 \\ z^2 + z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives

Equation:

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

Equation:

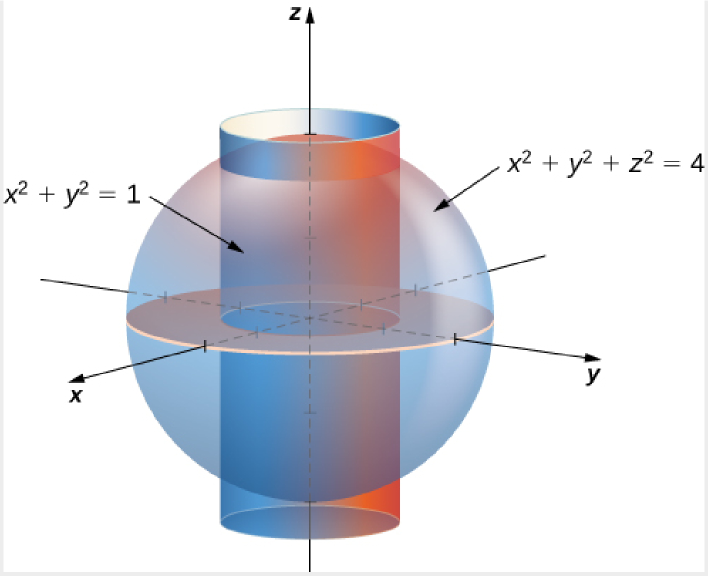
$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

Note:

Exercise:

Problem:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.



Solution:

Rectangular: $\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx.$

Cylindrical: $\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$

Spherical: $\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi.$

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

Example:
Exercise:

Problem:
Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.

(credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution:

We calculate the volume of the ball in the first octant, where $x \geq 0, y \geq 0$, and $z \geq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

Equation:

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq r, 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

Equation:

$$\begin{aligned} V &= \iiint\limits_D dx\,dy\,dz = 8 \int\limits_{\theta=0}^{\theta=\pi/2} \int\limits_{\rho=0}^{\rho=\pi} \int\limits_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta\,d\varphi\,d\rho\,d\theta \\ &= 8 \int\limits_{\varphi=0}^{\varphi=\pi/2} d\varphi \int\limits_{\rho=0}^{\rho=r} \rho^2 d\rho \int\limits_{\theta=0}^{\theta=\pi/2} \sin\theta\,d\theta \\ &= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523{,}600\,\text{ft}^3$.

For the next example we find the volume of an ellipsoid.

Example:
Exercise:

Problem:
Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Solution:

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \geq 0, y \geq 0$, and $z \geq 0$ and then multiply the result by 8.

In this case the ranges of the variables are
Equation:

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq \frac{\pi}{2}, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:
Equation:

$$x = a\rho \cos\varphi \sin\theta, y = b\rho \sin\varphi \sin\theta, \text{ and } z = c\rho \cos\theta.$$

Then the volume of the ellipsoid becomes
Equation:

$$\begin{aligned} V &= \iiint\limits_D dx\, dy\, dz \\ &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc\rho^2 \sin\theta\, d\varphi\, d\rho\, d\theta \\ &= 8abc \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta\, d\theta \\ &= 8abc \left(\frac{\pi}{2}\right) \left(\frac{1}{3}\right) (1) \\ &= \frac{4}{3}\pi abc. \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$.

Solution:

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75$ ft, $b = 80$ ft, and $c = 90$ ft in the result from [link]. Hence the volume of the ellipsoid is
Equation:

$$V_{\text{ellipsoid}} = \frac{4}{3}\pi(75)(80)(90) \approx 2{,}262{,}000 \text{ ft}^3.$$

From [link], the volume of the sphere is
Equation:

$$V_{\text{sphere}} \approx 523{,}600 \text{ ft}^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$ is approximately
Equation:

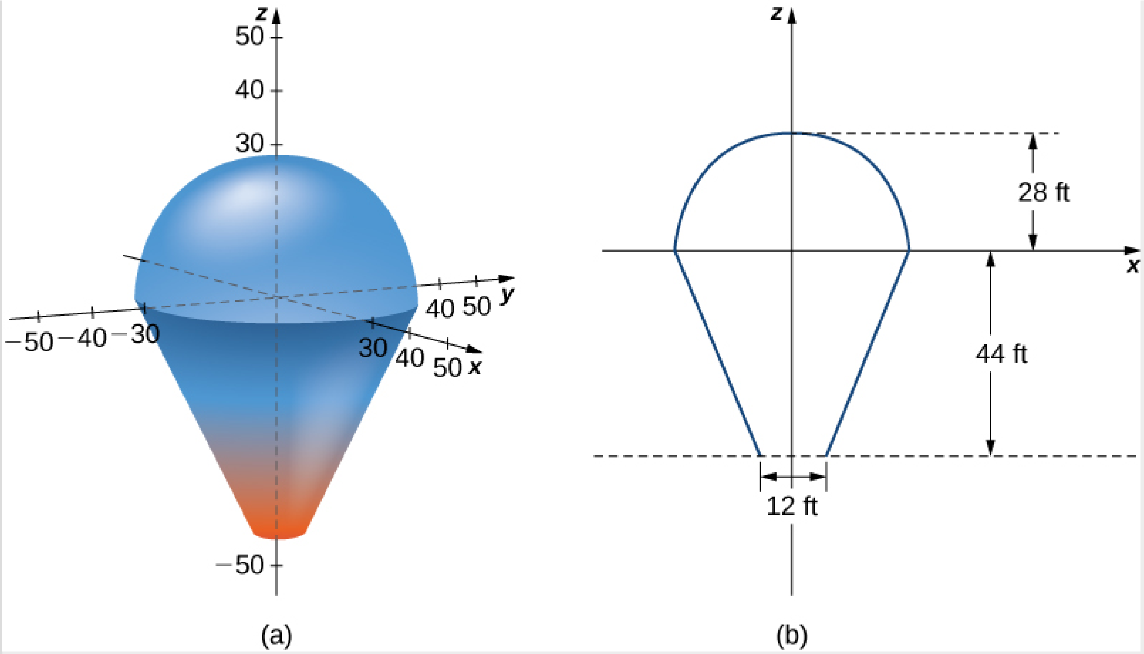
$$V_{\text{Hemisferic}} = V_{\text{ellipsoid}} - V_{\text{sphere}} = 1{,}738{,}400 \text{ ft}^3.$$

Note:
Hot air balloons
Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28 feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 6 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.



(a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

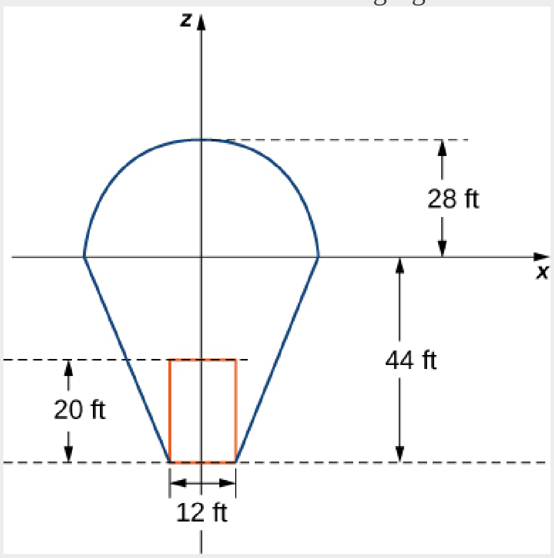
1. Find the volume of the balloon in two ways.
   a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)
   b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

   In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function
   Equation:

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure.



Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

Equation:

$$H(r, \theta, z) = -2z - 48.$$

Then the temperature of the air in the column is given by

Equation:

$$T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

Equation:

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral
  Equation:

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f\,(r,\theta,z)r\,dz\,dr\,d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral
  Equation:

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)}\int_{\varphi=u_1(r,\theta)}^{\varphi=u_2(r,\theta)} f\,(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

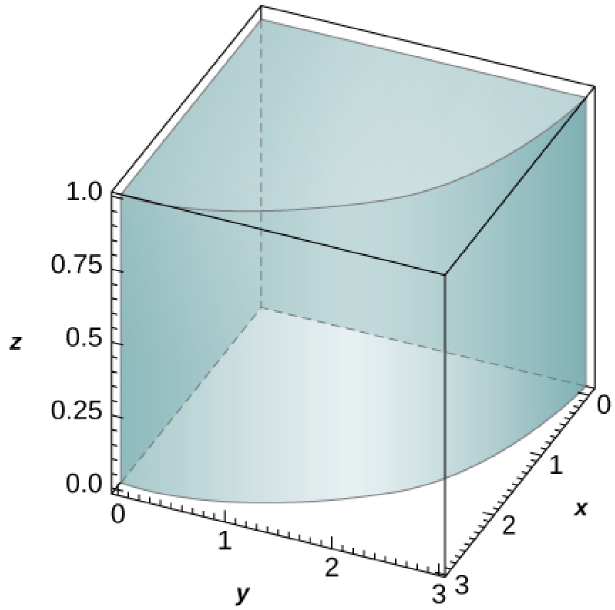
Key Equations

- **Triple integral in cylindrical coordinates**
  $\iiint\limits_B g\,(x,y,z)dV = \iiint\limits_B g\,(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz = \iiint\limits_B f\,(r,\theta,z)r\,dr\,d\theta\,dz$
- **Triple integral in spherical coordinates**
  $\iiint\limits_B f\,(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f\,(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

In the following exercises, evaluate the triple integrals $\iiint\limits_E f(x,y,z)dV$ over the solid E.

Exercise:

Problem: $f(x,y,z)=z$, $B=\left\{(x,y,z)\middle|x^2+y^2\le 9, x\ge 0, y\ge 0, 0\le z\le 1\right\}$



Solution:

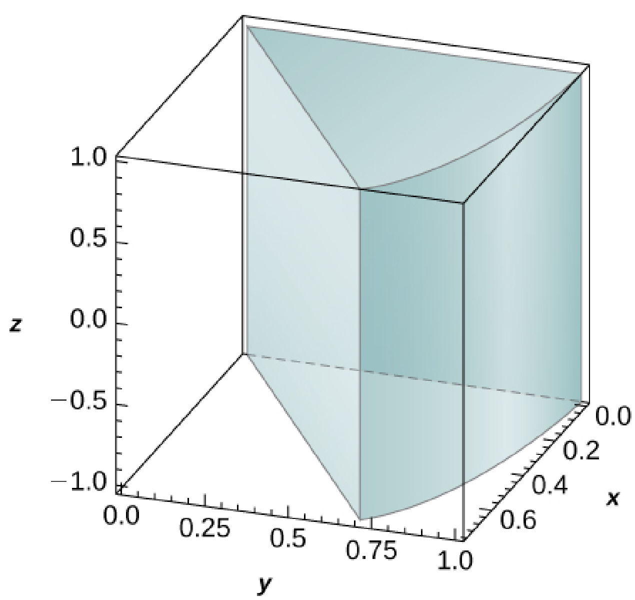
$\frac{9\pi}{8}$

Exercise:

Problem: $f(x,y,z) = xz^2$, $B = \left\{(x,y,z) \middle| x^2 + y^2 \leq 16, x \geq 0, y \leq 0, -1 \leq z \leq 1\right\}$

Exercise:

Problem: $f(x,y,z) = xy$, $B = \left\{(x,y,z) \middle| x^2 + y^2 \leq 1, x \geq 0, x \geq y, -1 \leq z \leq 1\right\}$



Solution:

$\frac{1}{8}$

Exercise:

Problem: $f(x, y, z) = x^2 + y^2$, $B = \{(x, y, z) | x^2 + y^2 \leq 4, x \geq 0, x \leq y, 0 \leq z \leq 3\}$

Exercise:

Problem: $f(x, y, z) = e^{\sqrt{x^2+y^2}}$, $B = \left\{(x, y, z) \middle| 1 \leq x^2 + y^2 \leq 4, y \leq 0, x \leq y\sqrt{3}, 2 \leq z \leq 3\right\}$

Solution:

$\frac{\pi e^2}{6}$

Exercise:

Problem: $f(x, y, z) = \sqrt{x^2 + y^2}$, $B = \{(x, y, z) | 1 \leq x^2 + y^2 \leq 9, y \leq 0, 0 \leq z \leq 1\}$

Exercise:

Problem:

a. Let B be a cylindrical shell with inner radius a, outer radius b, and height c, where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r) + h(z)$, where f and h are differentiable functions. If $\int_a^b \tilde{f}(r)dr = 0$ and $\tilde{h}(0) = 0$, where $\tilde{f}$ and $\tilde{h}$ are antiderivatives of f and h, respectively, show that
Equation:

$$\iiint_B F(x, y, z)dV = 2\pi c\left(b\tilde{f}(b) - a\tilde{f}(a)\right) + \pi\left(b^2 - a^2\right)\tilde{h}(c).$$

b. Use the previous result to show that $\iiint_B \left(z + \sin\sqrt{x^2 + y^2}\right) dx\, dy\, dz = 6\pi^2(\pi - 2)$, where B is a cylindrical shell with inner radius π, outer radius 2π, and height 2.

Exercise:

Problem:

a. Let B be a cylindrical shell with inner radius a, outer radius b, and height c, where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r)g(\theta)h(z)$, where $f, g,$ and h are differentiable functions. If $\int_a^b \tilde{f}(r)dr = 0$, where $\tilde{f}$ is an antiderivative of f, show that
Equation:

$$\iiint\limits_B F(x,y,z)dV = \left[b\widetilde{f}(b) - a\widetilde{f}(a)\right]\ [\tilde{g}(2\pi) - \tilde{g}(0)]\ \left[\widetilde{h}(c) - \widetilde{h}(0)\right],$$

where $\tilde{g}$ and $\widetilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that $\iiint\limits_B z\sin\sqrt{x^2+y^2}dx\,dy\,dz = -12\pi^2$, where B is a cylindrical shell with inner radius π, outer radius 2π, and height 2.

In the following exercises, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.
b. Convert the integral $\iiint\limits_E f(x,y,z)dV$ to cylindrical coordinates.

Exercise:

Problem:

E is bounded by the right circular cylinder $r = 4\sin\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=16$.

Solution:

a. $E = \left\{(r,\theta,z)\middle|0\le\theta\le\pi, 0\le r\le 4\sin\theta, 0\le z\le\sqrt{16-r^2}\right\}$; b. $\int\limits_0^{\pi}\int\limits_0^{4\sin\theta}\int\limits_0^{\sqrt{16-r^2}} f(r,\theta,z)r\,dz\,dr\,d\theta$

Exercise:

Problem:

E is bounded by the right circular cylinder $r=\cos\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=9$.

Exercise:

Problem:

E is located in the first octant and is bounded by the circular paraboloid $z=9-3r^2$, the cylinder $r=\sqrt{3}$, and the plane $r(\cos\theta+\sin\theta)=20-z$.

Solution:

a. $E = \left\{(r,\theta,z)\middle|0\le\theta\le\frac{\pi}{2}, 0\le r\le\sqrt{3}, 9-r^2\le z\le 10-r(\cos\theta+\sin\theta)\right\}$; b. $\int\limits_0^{\pi/2}\int\limits_0^{\sqrt{3}}\int\limits_{9-r^2}^{10-r(\cos\theta+\sin\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta$

Exercise:

Problem:

E is located in the first octant outside the circular paraboloid $z = 10 - 2r^2$ and inside the cylinder $r = \sqrt{5}$ and is bounded also by the planes $z = 20$ and $\theta = \frac{\pi}{4}$.

In the following exercises, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.
b. Convert the integral $\iiint\limits_{B} f(x, y, z) dV$ into cylindrical coordinates and evaluate it.

Exercise:

Problem: $f(x, y, z) = \frac{1}{x+3}$, $E = \{(x, y, z) | 0 \le x^2 + y^2 \le 9, x \ge 0, y \ge 0, 0 \le z \le x + 3\}$

Solution:

a. $E = \{(r, \theta, z) | 0 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}, 0 \le z \le r\cos\theta + 3\}$, $f(r, \theta, z) = \frac{1}{r\cos\theta + 3}$; b.
$$\int_0^3 \int_0^{\pi/2} \int_0^{r\cos\theta + 3} \frac{r}{r\cos\theta + 3} dz\, d\theta\, dr = \frac{9\pi}{4}$$

Exercise:

Problem: $f(x, y, z) = x^2 + y^2$, $E = \{(x, y, z) | 0 \le x^2 + y^2 \le 4, y \ge 0, 0 \le z \le 3 - x\}$

Exercise:

Problem: $f(x, y, z) = x$, $E = \{(x, y, z) | 1 \le y^2 + z^2 \le 9, 0 \le x \le 1 - y^2 - z^2\}$

Solution:

a. $y = r\cos\theta, z = r\sin\theta, x = z$,
$E = \{(r, \theta, z) | 1 \le r \le 3, 0 \le \theta \le 2\pi, 0 \le z \le 1 - r^2\}$, $f(r, \theta, z) = z$; b.
$$\int_1^3 \int_0^{2\pi} \int_0^{1-r^2} zr\, dz\, d\theta\, dr = \frac{356\pi}{3}$$

Exercise:

Problem: $f(x, y, z) = y$, $E = \{(x, y, z) | 1 \le x^2 + z^2 \le 9, 0 \le y \le 1 - x^2 - z^2\}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

Exercise:

Problem: E is above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and below the plane $z = 1$.

Solution:

π

Exercise:

Problem: E is below the plane $z = 1$ and inside the paraboloid $z = x^2 + y^2$.

Exercise:

Problem: E is bounded by the circular cone $z = \sqrt{x^2 + y^2}$ and $z = 1$.

Solution:

$\frac{\pi}{3}$

Exercise:

Problem:

E is located above the xy-plane, below $z = 1$, outside the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$, and inside the cylinder $x^2 + y^2 = 2$.

Exercise:

Problem:

E is located inside the cylinder $x^2 + y^2 = 1$ and between the circular paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2$.

Solution:

π

Exercise:

Problem:

E is located inside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and inside the circular cone $z = \sqrt{x^2 + y^2}$.

Exercise:

Problem:

E is located outside the circular cone $x^2 + y^2 = (z - 1)^2$ and between the planes $z = 0$ and $z = 2$.

Solution:

$\frac{4\pi}{3}$

Exercise:

Problem:

E is located outside the circular cone $z = 1 - \sqrt{x^2 + y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z = 0$ and $z = 2$.

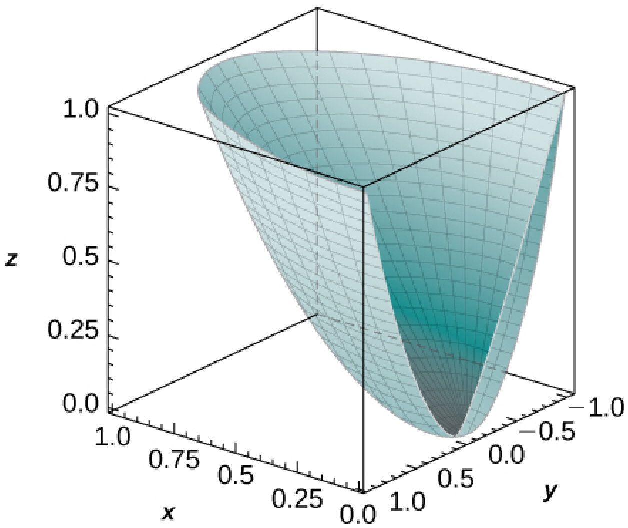
Exercise:

Problem:

[T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r\,dz\,dr\,d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Solution:

$V = \frac{\pi}{12} \approx 0.2618$



Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{0}^{\pi/2}\int_{0}^{1}\int_{r^4}^{r} r\,dz\,dr\,d\theta$. Find the volume V of the solid Round your answer to four decimal places.

Exercise:

Problem:

Convert the integral $\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

Solution:

$$\int_0^1 \int_0^\pi \int_{r^2}^r zr^2 \cos\theta \, dz \, d\theta \, dr$$

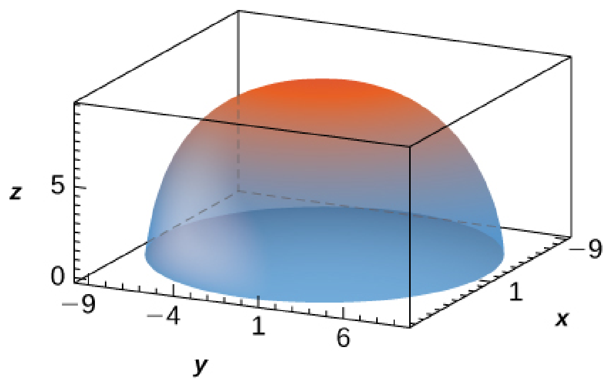
Exercise:

Problem: Convert the integral $\int_0^2 \int_0^x \int_0^1 (xy + z) dz \, dx \, dy$ into an integral in cylindrical coordinates.

In the following exercises, evaluate the triple integral $\iiint_B f(x, y, z) dV$ over the solid B.

Exercise:

Problem: $f(x, y, z) = 1$, $B = \{(x, y, z) | x^2 + y^2 + z^2 \le 90, z \ge 0\}$

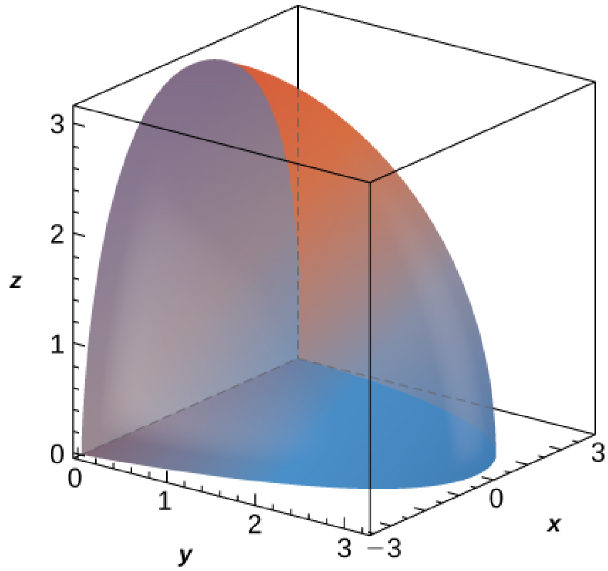


Solution:

$180\pi\sqrt{10}$

Exercise:

Problem: $f(x, y, z) = 1 - \sqrt{x^2 + y^2 + z^2}$, $B = \{(x, y, z) | x^2 + y^2 + z^2 \le 9, y \ge 0, z \ge 0\}$



Exercise:

Problem:

$f(x, y, z) = \sqrt{x^2 + y^2}$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ and below by the cone $2z^2 = x^2 + y^2$.

Solution:

$\frac{81\pi(\pi-2)}{16}$

Exercise:

Problem:

$f(x, y, z) = z$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 16$ with $z \geq 0$ and below by the cone $2z^2 = x^2 + y^2$.

Exercise:

Problem:

Show that if $F(\rho, \theta, \varphi) = f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B = \{(\rho, \theta, \varphi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \varphi \leq \psi\}$, then

Equation:

$$\iiint\limits_B F\, dV = \left(\int_a^b \rho^2 f(\rho) dr\right) \left(\int_\alpha^\beta g(\theta) d\theta\right) \left(\int_\gamma^\psi h(\varphi) \sin\varphi\, d\varphi\right).$$

Exercise:

Problem:

a. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x, y, z) = f(\rho)$, where

$\rho = \sqrt{x^2 + y^2 + z^2}$. Show that

Equation:

$$\iiint_B F(x,y,z)dV = 2\pi \int_a^b \rho^2 f(\rho)d\rho,$$

where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0 < a < b$ and F a spherical function defined on B.

b. Use the previous result to show that $\iiint_B \left(x^2 + y^2 + z^2\right)\sqrt{x^2 + y^2 + z^2}\, dV = 21\pi$, where

Equation:

$$B = \left\{(x,y,z) \middle| 1 \le x^2 + y^2 + z^2 \le 2, z \ge 0\right\}.$$

Exercise:

Problem:

a. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0 < a < b$. Consider F a function defined on B whose form in spherical coordinates (ρ, θ, φ) is $F\,(x,y,z) = f(\rho)\cos\varphi$. Show that if $g(a) = g(b) = 0$ and $\int_a^b h(\rho)d\rho = 0$, then

Equation:

$$\iiint_B F(x,y,z)dV = \frac{\pi^2}{4}[ah(a) - bh(b)],$$

where g is an antiderivative of f and h is an antiderivative of g.

b. Use the previous result to show that $\iiint_B \frac{z\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} dV = \frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In the following exercises, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.
b. Convert the integral $\iiint_B f\,(x,y,z)dV$ into cylindrical coordinates and evaluate it.

Exercise:

Problem: $f(x,y,z)=z$; $E=\left\{(x,y,z)\middle|0\le x^2+y^2+z^2\le 1, z\ge 0\right\}$

Exercise:

Problem: $f(x,y,z)=x+y$; $E=\left\{(x,y,z)\middle|1\le x^2+y^2+z^2\le 2, z\ge 0, y\ge 0\right\}$

Solution:

a. $f(\rho,\theta,\varphi)=\rho\sin\varphi(\cos\theta+\sin\theta)$, $E=\left\{(\rho,\theta,\varphi)\middle|1\le\rho\le 2, 0\le\theta\le\pi, 0\le\varphi\le\frac{\pi}{2}\right\}$; b. $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_1^{2}\rho^3\cos\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta=\frac{15\pi}{8}$

Exercise:

Problem: $f(x,y,z)=2xy$; $E=\left\{(x,y,z)\middle|\sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2}, x\ge 0, y\ge 0\right\}$

Exercise:

Problem: $f(x,y,z)=z$; $E=\left\{(x,y,z)\middle|x^2+y^2+z^2-2z\le 0, \sqrt{x^2+y^2}\le z\right\}$

Solution:

a. $f(\rho,\theta,\varphi)=\rho\cos\varphi$; $E=\left\{(\rho,\theta,\varphi)\middle|0\le\rho\le 2\cos\varphi, 0\le\theta\le\frac{\pi}{2}, 0\le\varphi\le\frac{\pi}{4}\right\}$; b. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\cos\varphi}\rho^3\sin\varphi\cos\varphi\,d\rho\,d\varphi\,d\theta=\frac{7\pi}{24}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

Exercise:

Problem: $E=\left\{(x,y,z)\middle|\sqrt{x^2+y^2}\le z\le\sqrt{16-x^2-y^2}, x\ge 0, y\ge 0\right\}$

Exercise:

Problem: $E=\left\{(x,y,z)\middle|x^2+y^2+z^2-2z\le 0, \sqrt{x^2+y^2}\le z\right\}$

Solution:

$\frac{\pi}{4}$

Exercise:

Problem:

Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho=1$ and inside the sphere $\rho=\cos\varphi$, with $\varphi\in\left[0,\frac{\pi}{2}\right]$.

Exercise:

Problem:

Use spherical coordinates to find the volume of the ball $\rho \leq 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

Solution:

$$9\pi\left(\sqrt{2} - 1\right)$$

Exercise:

Problem:

Convert the integral $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right) dz\, dx\, dy$ into an integral in spherical coordinates.

Exercise:

Problem:

Convert the integral $\int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right)^2 dz\, dy\, dx$ into an integral in spherical coordinates.

Solution:

$$\int_{0}^{\pi/2} \int_{0}^{\pi/2} \int_{0}^{4} \rho^6 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Exercise:

Problem:

Convert the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx$ into an integral in spherical coordinates and evaluate it.

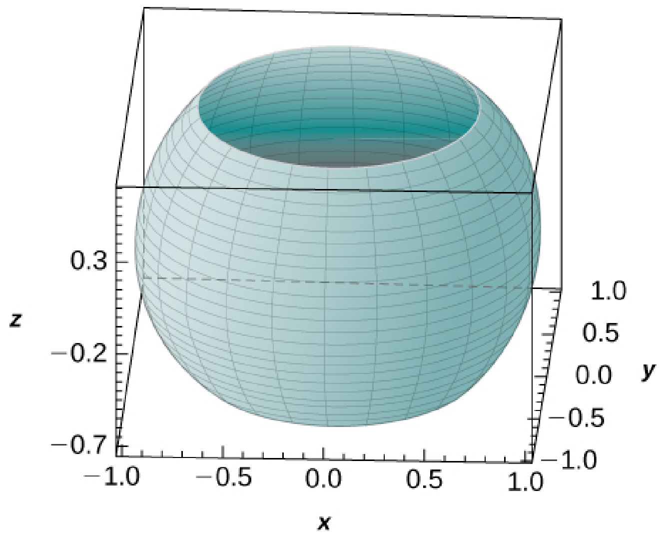
Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi} \int_{5\pi/6}^{\pi/6} \int_{0}^{2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Solution:

$V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$



Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_0^{2\pi} \int_{3\pi/4}^{\pi/4} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Exercise:

Problem:

[T] Use a CAS to evaluate the integral $\iiint_E (x^2 + y^2) dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y$.

Solution:

$\frac{343\pi}{32}$

Exercise:

Problem: **[T]**

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}} dV$, where E is bounded by the spheres $4x^2 + 4y^2 + 4z^2 = 1$ and $x^2 + y^2 + z^2 = 1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

Exercise:

Problem:

Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Solution:

$$\int_0^{2\pi} \int_2^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta; \int_{\pi/6}^{5\pi/6} \int_0^{2\pi} \int_{2\csc\varphi}^{4} \rho^2 \sin\rho\,d\rho\,d\theta\,d\varphi$$

Exercise:

Problem:

Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Exercise:

Problem:

The power emitted by an antenna has a power density per unit volume given in spherical coordinates by

$p(\rho, \theta, \varphi) = \frac{P_0}{\rho^2} \cos^2\theta \sin^4\varphi$,where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P = \iiint\limits_B p(\rho, \theta, \varphi) dV$. Find the total power P.

Solution:

$P = \frac{32P_0\pi}{3}$ watts

Exercise:

Problem:

Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho, \theta, \varphi) = \frac{30}{\rho^2} \cos^2\theta \sin^4\varphi$.

Exercise:

Problem:

A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x, y, z) = k\sqrt{x^2 + y^2 + z^2} \frac{\mu\text{C}}{\text{cm}^3}$, where $k > 0$. The total charge contained in B is given by $Q = \iiint\limits_B q(x, y, z) dV$. Find the total charge Q.

Solution:

$Q = kr^4 \pi \mu C$

Exercise:

Problem:

Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x, y, z) = 20\sqrt{x^2 + y^2 + z^2}\,\frac{\mu\,\mathrm{C}}{\mathrm{cm}^3}$.

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

Equation:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_{ijk}^{*}, \theta_{ijk}^{*}, z_{ijk}^{*}) r_{ijk}^{*} \Delta r \Delta\theta \Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

Equation:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_{ijk}^{*}, \theta_{ijk}^{*}, \varphi_{ijk}^{*}) (\rho_{ijk}^{*})^2 \sin\varphi \Delta\rho \Delta\theta \Delta\varphi$$

Change of Variables in Multiple Integrals

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as $\int_2^3 x(x^2-4)^5 dx$, we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\, dx$ or $x\, dx = \frac{1}{2} du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes $\int_0^5 \frac{1}{2} u^5 du$ and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,
Equation:

$$\int_a^b f(x)dx = \int_c^d f(g(u))g\prime(u)du,$$

Where $x = g(u), dx = g\prime(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

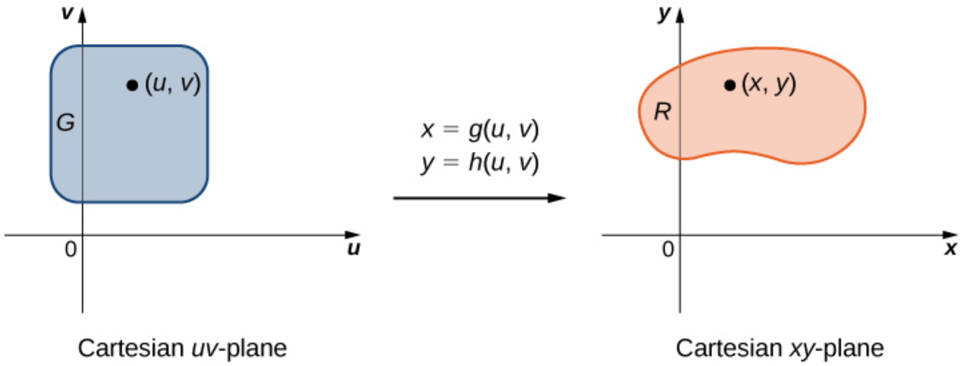
A similar result occurs in double integrals when we substitute $x = h(r, \theta) = r \cos\theta$, $y = g(r, \theta) = r \sin\theta$, and $dA = dx\, dy = r\, dr\, d\theta$. Then we get
Equation:

$$\iint_R f(x, y)dA = \iint_S f(r\cos\theta, r\sin\theta)r\, dr\, d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a **transformation** or mapping.

Planar Transformations

A **planar transformation** T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, [link] shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v)$, or sometimes we write $x = x(u, v)$ and $y = y(u, v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.



The transformation of a region G in the uv-plane into a region R in the xy-plane.

Note:
Definition
A transformation $T: G \to R$, defined as $T(u,v) = (x,y)$, is said to be a **one-to-one transformation** if no two points map to the same image point.

To show that T is a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain $(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

[link] shows the mapping $T(u,v) = (x,y)$ where x and y are related to u and v by the equations $x = g(u,v)$ and $y = h(u,v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

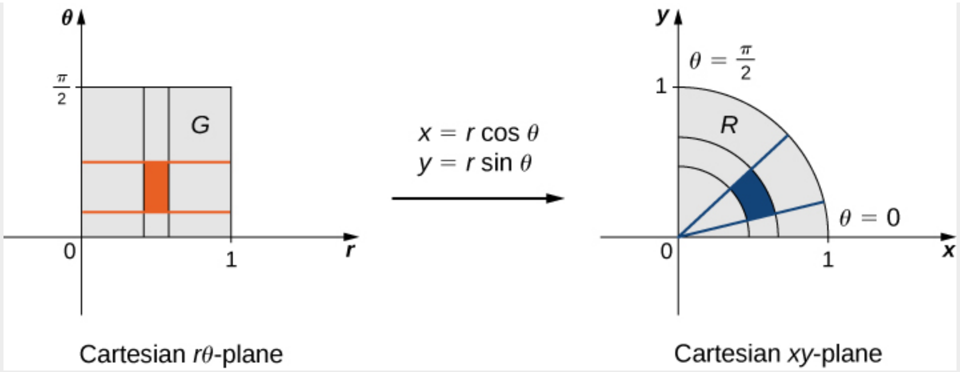
Example:
Exercise:

Problem:
Determining How the Transformation Works

Suppose a transformation T is defined as $T(r,\theta) = (x,y)$ where $x = r\cos\theta, y = r\sin\theta$. Find the image of the polar rectangle $G = \{(r,\theta) | 0 < r \le 1, 0 \le \theta \le \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x,y)$.

Solution:

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane ([link]). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.



A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T\left(r_1, \theta_1\right) = T\left(r_2, \theta_2\right)$ and show as a consequence that $\left(r_1, \theta_1\right) = \left(r_2, \theta_2\right)$. In this case, we have

Equation:

$$\begin{array}{rcl} T\left(r_1, \theta_1\right) & = & T\left(r_2, \theta_2\right), \\ \left(x_1, y_1\right) & = & \left(x_1, y_1\right), \\ \left(r_1 \cos \theta_1, r_1 \sin \theta_1\right) & = & \left(r_2 \cos \theta_2, r_2 \sin \theta_2\right), \\ r_1 \cos \theta_1 & = & r_2 \cos \theta_2, r_1 \sin \theta_1 = r_2 \sin \theta_2. \end{array}$$

Dividing, we obtain

Equation:

$$\begin{array}{rcl} \frac{r_1 \cos \theta_1}{r_1 \sin \theta_1} & = & \frac{r_2 \cos \theta_2}{r_2 \sin \theta_2} \\ \frac{\cos \theta_1}{\sin \theta_1} & = & \frac{\cos \theta_2}{\sin \theta_2} \\ \tan \theta_1 & = & \tan \theta_2 \\ \theta_1 & = & \theta_2 \end{array}$$

since the tangent function is one-one function in the interval $0 \leq \theta \leq \pi/2$. Also, since $0 < r \leq 1$, we have $r_1 = r_2, \theta_1 = \theta_2$. Therefore, $\left(r_1, \theta_1\right) = \left(r_2, \theta_2\right)$ and T is a one-to-one transformation from G into R.

To find $T^{-1}\left(x, y\right)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$. Thus $T^{-1}\left(x, y\right) = \left(r, \theta\right)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

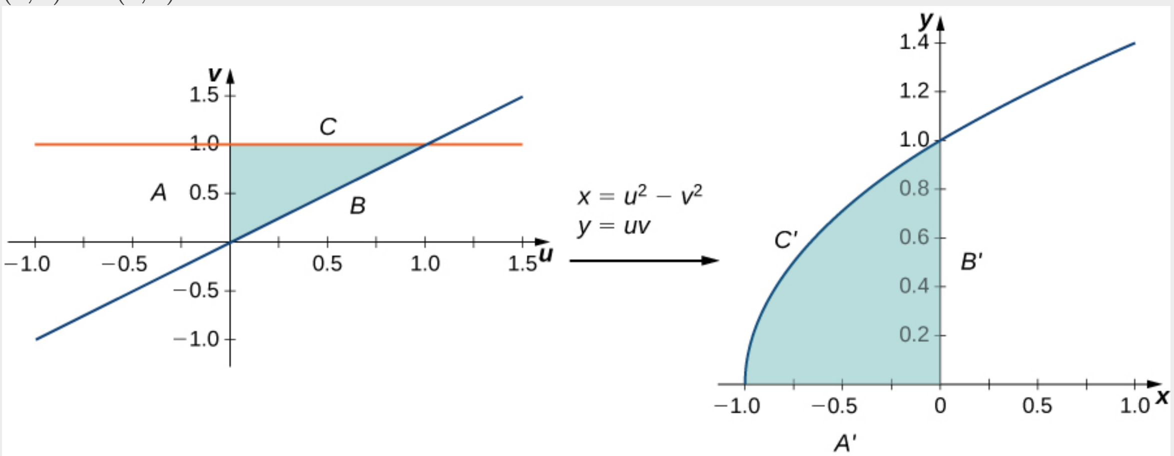
Example:
Exercise:

Problem:
Finding the Image under T

Let the transformation T be defined by $T\left(u, v\right) = \left(x, y\right)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $\left(0, 0\right)$, $\left(0, 1\right)$, and $\left(1, 1\right)$.

Solution:

The triangle and its image are shown in [link]. To understand how the sides of the triangle transform, call the side that joins $(0,0)$ and $(0,1)$ side A, the side that joins $(0,0)$ and $(1,1)$ side B, and the side that joins $(1,1)$ and $(0,1)$ side C.



A triangular region in the uv-plane is transformed into an image in the xy-plane.

For the side A: $u=0, 0 \leq v \leq 1$ transforms to $x=-v^2, y=0$ so this is the side $A\prime$ that joins $(-1,0)$ and $(0,0)$.

For the side B: $u=v, 0 \leq u \leq 1$ transforms to $x=0, y=u^2$ so this is the side B' that joins $(0,0)$ and $(0,1)$.

For the side C: $0 \leq u \leq 1, v=1$ transforms to $x=u^2-1, y=u$ (hence $x=y^2-1$) so this is the side C' that makes the upper half of the parabolic arc joining $(-1,0)$ and $(0,1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Note:
Exercise:

Problem:

Let a transformation T be defined as $T(u,v)=(x,y)$ where $x=u+v, y=3v$. Find the image of the rectangle $G=\{(u,v): 0 \leq u \leq 1, 0 \leq v \leq 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x,y)$.

Solution:

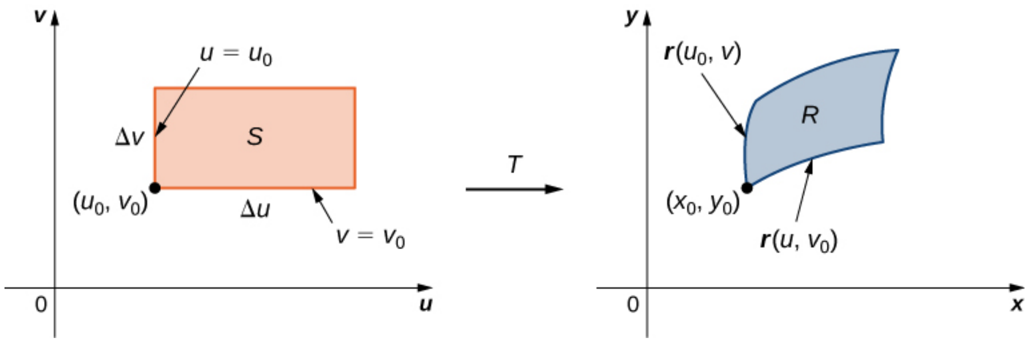
$T^{-1}(x,y)=(u,v)$ where $u=\frac{3x-y}{3}$ and $v=\frac{y}{3}$

Hint

Follow the steps of [link].

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u, and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u, v) = (g(u, v), h(u, v))$, where $x = g(u, v)$ and $y = h(u, v)$, be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (see the following figure).



A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$, we have the position vector $\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$ of the image of the point (u, v). Suppose that (u_0, v_0) is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$. The line $v = v_0$ maps to the image curve with vector function $\mathbf{r}(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

Equation:

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $\mathbf{r}(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

Equation:

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}.$$

Now, note that

Equation:

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u} \text{ so } \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u.$$

Similarly,

Equation:

$$\mathbf{r}_v = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v} \text{ so } \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v\mathbf{r}_v$ and $\Delta u\mathbf{r}_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u\mathbf{r}_u \times \Delta v\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v$. In determinant form, the cross product is

Equation:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}.$$

Since $|\mathbf{k}| = 1$, we have $\Delta A \approx |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta u\Delta v$.

Note:

Definition

The **Jacobian** of the C^1 transformation $T(u,v) = (g(u,v), h(u,v))$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

Equation:

$$J(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right).$$

Using the definition, we have

Equation:

$$\Delta A \approx J(u,v)\Delta u\Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v.$$

Note that the Jacobian is frequently denoted simply by

Equation:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}.$$

Note also that

Equation:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example:

Exercise:

Problem:
Finding the Jacobian

Find the Jacobian of the transformation given in [link].

Solution:

The transformation in the example is $T\left(r,\theta\right)=\left(r\cos\theta,r\sin\theta\right)$ where $x=r\cos\theta$ and $y=r\sin\theta$. Thus the Jacobian is

Equation:

$$\begin{array}{cc} J\left(r,\theta\right) & =\frac{\partial(x,y)}{\partial(r,\theta)}=\left|\begin{array}{cc}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{array}\right|=\left|\begin{array}{cc}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{array}\right|\\ & =r\cos^2\theta+r\sin^2\theta=r\left(\cos^2\theta+\sin^2\theta\right)=r.\end{array}$$

Example:
Exercise:

Problem:
Finding the Jacobian

Find the Jacobian of the transformation given in [link].

Solution:

The transformation in the example is $T\left(u,v\right)=\left(u^2-v^2,uv\right)$ where $x=u^2-v^2$ and $y=uv$. Thus the Jacobian is

Equation:

$$J\left(u,v\right)=\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}=\left|\begin{array}{cc}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{array}\right|=\left|\begin{array}{cc}2u & v\\ -2v & u\end{array}\right|=2u^2+2v^2.$$

Note:
Exercise:

Problem: Find the Jacobian of the transformation given in the previous checkpoint: $T\left(u,v\right)=\left(u+v,2v\right)$.

Solution:

$$J\left(u,v\right)=\frac{\partial(x,y)}{\partial(u,v)}=\left|\begin{array}{cc}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{array}\right|=\left|\begin{array}{cc}1 & 1\\ 0 & 2\end{array}\right|=2$$

Hint

Follow the steps in the previous two examples.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

Equation:

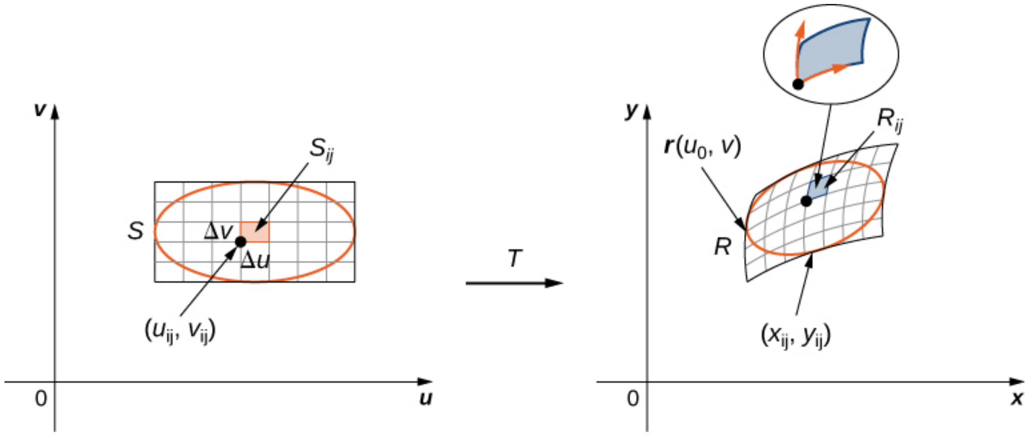
$$\Delta A \approx J(u, v)\Delta u, \Delta v.$$

Now let’s go back to the definition of double integral for a minute:

Equation:

$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A.$$

Referring to [link], observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u, v) = (x, y)$.



The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

Equation:

$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_{ij}, v_{ij}), h(u_{ij}, v_{ij}))\, |J(u_{ij}, v_{ij})| \Delta u \Delta v$$

Notice this is exactly the double Riemann sum for the integral

Equation:

$$\iint\limits_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

Note:
Change of Variables for Double Integrals
Let $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane; it maps S into the region R in the xy-plane. If f is continuous on R, then
Equation:

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing
Equation:

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

when we use the substitutions $x=g(u,v)$ and $y=h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example:
Exercise:

Problem:
Changing Variables from Rectangular to Polar Coordinates

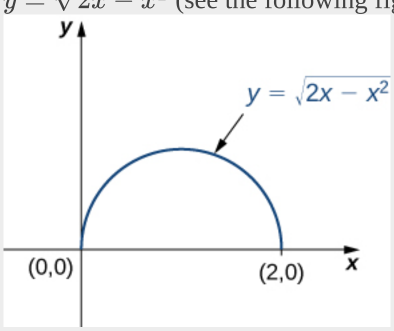
Consider the integral
Equation:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

Use the change of variables $x=r\cos\theta$ and $y=r\sin\theta$, and find the resulting integral.

Solution:

First we need to find the region of integration. This region is bounded below by $y=0$ and above by $y=\sqrt{2x-x^2}$ (see the following figure).



Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is, $y^2 + (x-1)^2 = 1$. In polar coordinates, the circle is $r = 2\cos\theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r,\theta) = r$, as shown in [link]. Since $r \ge 0$, we have $|J(r,\theta)| = r$.

The integrand $\sqrt{x^2 + y^2}$ changes to r in polar coordinates, so the double iterated integral is

Equation:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r\,|J(r,\theta)|\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta.$$

Note:
Exercise:

Problem:

Considering the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$, use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Solution:

$$\int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta$$

Hint

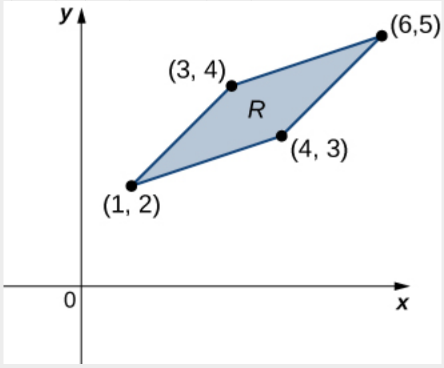
Follow the steps in the previous example.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example:
Exercise:

Problem:
Changing Variables

Consider the integral $\iint_R (x-y)dy\,dx$, where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ ([link]). Make appropriate changes of variables, and write the resulting integral.



The region of integration for the given integral.

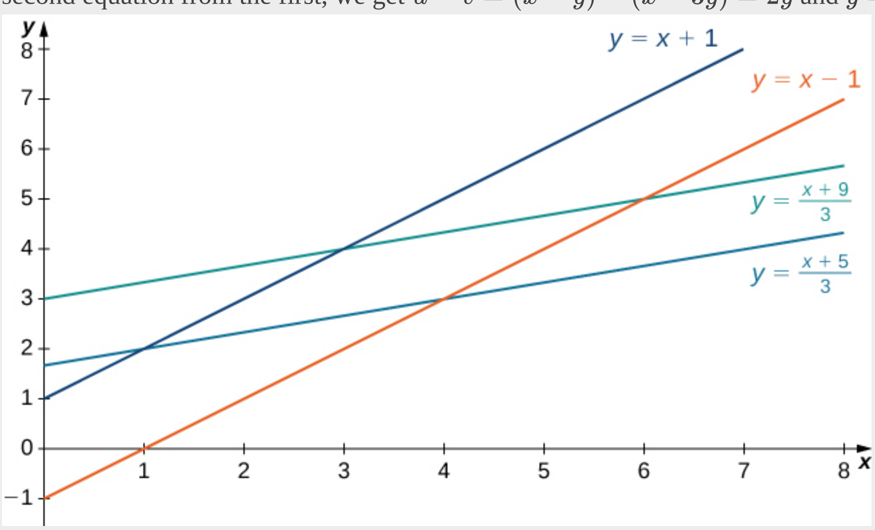
Solution:

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x-y+1=0$, $x-y-1=0$, $x-3y+5=0$, and $x-3y+9=0$ ([link]). Another way to look at them is $x-y=-1$, $x-y=1$, $x-3y=-5$, and $x-3y=9$.

Clearly the parallelogram is bounded by the lines $y=x+1$, $y=x-1$, $y=\frac{1}{3}(x+5)$, and $y=\frac{1}{3}(x+9)$.

Notice that if we were to make $u=x-y$ and $v=x-3y$, then the limits on the integral would be $-1\le u\le 1$ and $-9\le v\le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u-v=(3x-3y)-(x-3y)=2x$. Then we have $x=\frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u-v=(x-y)-(x-3y)=2y$ and $y=\frac{u-v}{2}$.



A parallelogram in the xy-plane that we want to transform by a change in

variables.

Thus, we can choose the transformation

Equation:

$$T\left(u,v\right)=\left(\frac{3u-v}{2},\frac{u-v}{2}\right)$$

and compute the Jacobian $J\left(u,v\right)$. We have

Equation:

$$J\left(u,v\right)=\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}3/2 & -1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{3}{4}+\frac{1}{4}=-\frac{1}{2}.$$

Therefore, $\left|J\left(u,v\right)\right|=\frac{1}{2}$. Also, the original integrand becomes

Equation:

$$x-y=\frac{1}{2}[3u-v-u+v]=\frac{1}{2}[3u-u]=\frac{1}{2}[2u]=u.$$

Therefore, by the use of the transformation T, the integral changes to

Equation:

$$\iint\limits_R (x-y)dy\,dx=\int\limits_{-9}^{-5}\int\limits_{-1}^{1}J\left(u,v\right)u\,du\,dv=\int\limits_{-9}^{-5}\int\limits_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute.

Note:
Exercise:

Problem:

Make appropriate changes of variables in the integral $\iint\limits_R \frac{4}{\left(x-y\right)^2}dy\,dx$, where R is the trapezoid bounded by the lines $x-y=2, x-y=4, x=0,$ and $y=0$. Write the resulting integral.

Solution:

$x=\frac{1}{2}(v+u)$ and $y=\frac{1}{2}(v-u)$ and $\int\limits_{-4}^{4}\int\limits_{-2}^{2}\frac{4}{u^2}\left(\frac{1}{2}\right)du\,dv.$

Hint

Follow the steps in the previous example.

We are ready to give a problem-solving strategy for change of variables.

Note:
Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g\left(u, v\right)$ and $y = h\left(u, v\right)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J\left(u, v\right)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J\left(u, v\right)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example:
Exercise:

Problem:
Evaluating an Integral

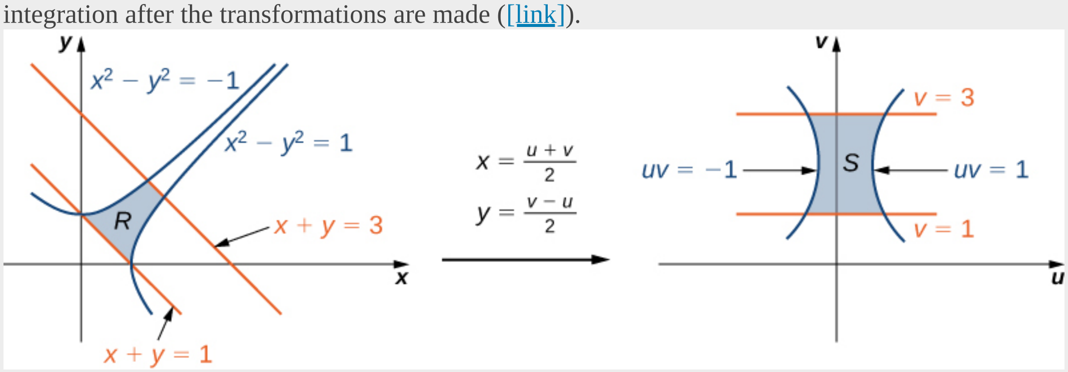
Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral
Equation:

$$\iint\limits_{R} (x - y)e^{x^2 - y^2} dA,$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in [link]).

Solution:

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made ([link]).



Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region [link]) as

Equation:

$$S = \left\{(u, v) | 1 \leq v \leq 3, \frac{-1}{v} \leq u \leq \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

Equation:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

Equation:

$$\iint_R (x - y)e^{x^2 - y^2} dA = \frac{1}{2} \int_1^3 \int_{-1/v}^{1/v} ue^{uv} du\, dv.$$

Doing the evaluation, we have

Equation:

$$\frac{1}{2} \int_1^3 \int_{-1/v}^{1/v} ue^{uv} du\, dv = \frac{4}{3e} \approx 0.490.$$

Note:
Exercise:

Problem:

Using the substitutions $x = v$ and $y = \sqrt{u + v}$, evaluate the integral $\iint_R y \sin\left(y^2 - x\right) dA$ where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$, and $y = 0$.

Solution:

$\frac{1}{2}(\sin 2 - 2)$

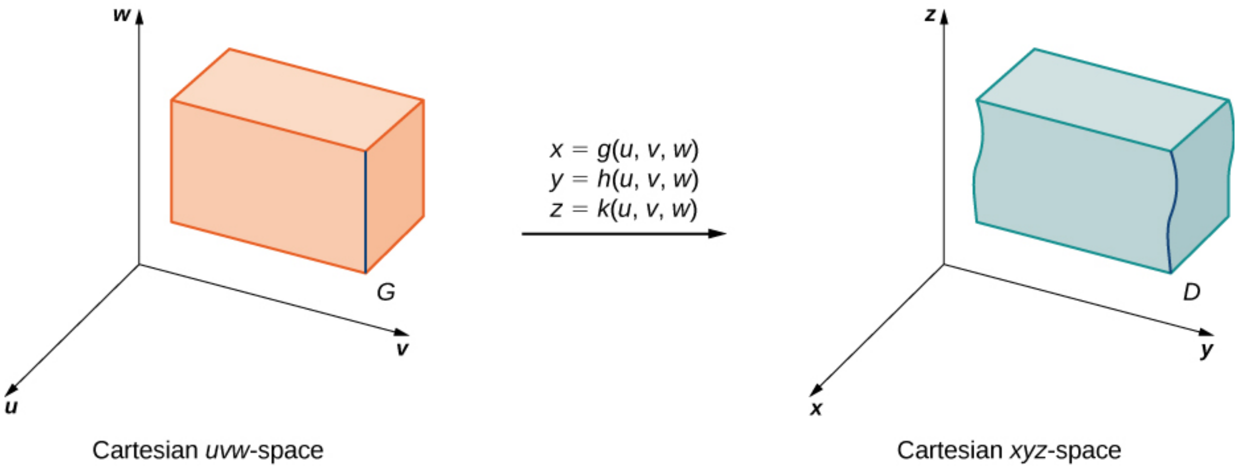
Hint

Sketch a picture and find the limits of integration.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space ([link]) by a one-to-one C^1 transformation $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.



A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:
Equation:

$$F(x, y, z) = F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w).$$

Now we need to define the Jacobian for three variables.

Note:
Definition
The Jacobian determinant $J(u, v, w)$ in three variables is defined as follows:
Equation:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as
Equation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Note:
Change of Variables for Triple Integrals
Let $T\left(u,v,w\right) = \left(x,y,z\right)$ where $x = g\left(u,v,w\right), y = h\left(u,v,w\right)$, and $z = k\left(u,v,w\right)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-plane into the region D in the xyz-plane. As in the two-dimensional case, if F is continuous on D, then
Equation:

$$\begin{aligned} \iiint\limits_R F\left(x,y,z\right)dV &= \iiint\limits_G F\left(g\left(u,v,w\right), h\left(u,v,w\right), k\left(u,v,w\right)\right)\left|\frac{\partial\left(x,y,z\right)}{\partial\left(u,v,w\right)}\right| du\, dv\, dw \\ &= \iiint\limits_G H\left(u,v,w\right)\left|J\left(u,v,w\right)\right| du\, dv\, dw. \end{aligned}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example:
Exercise:

Problem:
Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution:

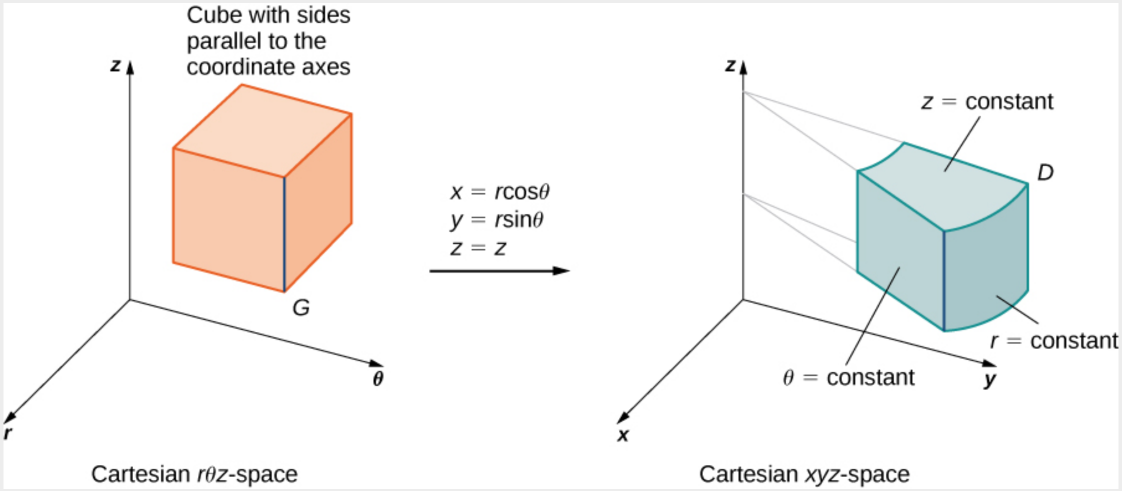
a. For cylindrical coordinates, the transformation is $T\left(r,\theta,z\right) = \left(x,y,z\right)$ from the Cartesian $r\theta z$-plane to the Cartesian xyz-plane ([link]). Here $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The Jacobian for the transformation is
Equation:

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

We know that $r \geq 0$, so $|J(r,\theta,z)| = r$. Then the triple integral is
Equation:

$$\iiint\limits_D f(x,y,z)dV = \iiint\limits_G f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz.$$



The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

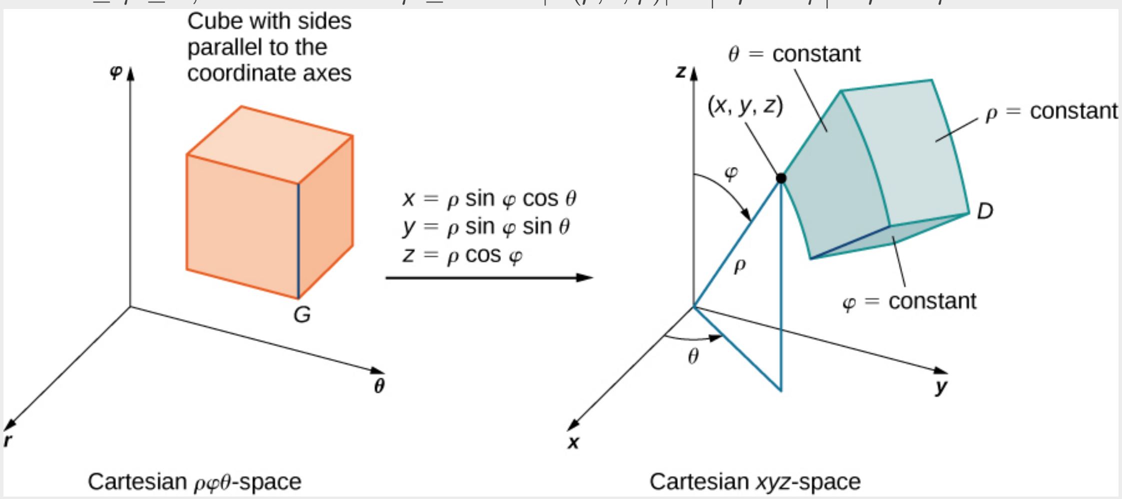
b. For spherical coordinates, the transformation is $T(\rho,\theta,\varphi) = (x,y,z)$ from the Cartesian $p\theta\varphi$-plane to the Cartesian xyz-plane ([link]). Here $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. The Jacobian for the transformation is
Equation:

$$J(\rho,\theta,\varphi) = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & -\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\theta & 0 & -\rho\sin\varphi \end{vmatrix}.$$

Expanding the determinant with respect to the third row:
Equation:

$$\begin{aligned}
&= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\
&= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) \\
&\quad -\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\
&= -\rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) - \rho^2\sin\varphi\sin^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\
&= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi \\
&= -\rho^2\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = -\rho^2\sin\varphi.
\end{aligned}$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho,\theta,\varphi)| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi$.



The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

Equation:

$$\iiint_D f(x,y,z)\,dV = \iiint_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Let’s try another example with a different substitution.

Example:
Exercise:

Problem:
Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

Equation:

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

in xyz-space by using the transformation

Equation:

$$u = (2x - y)/2, v = y/2, \text{and } w = z/3.$$

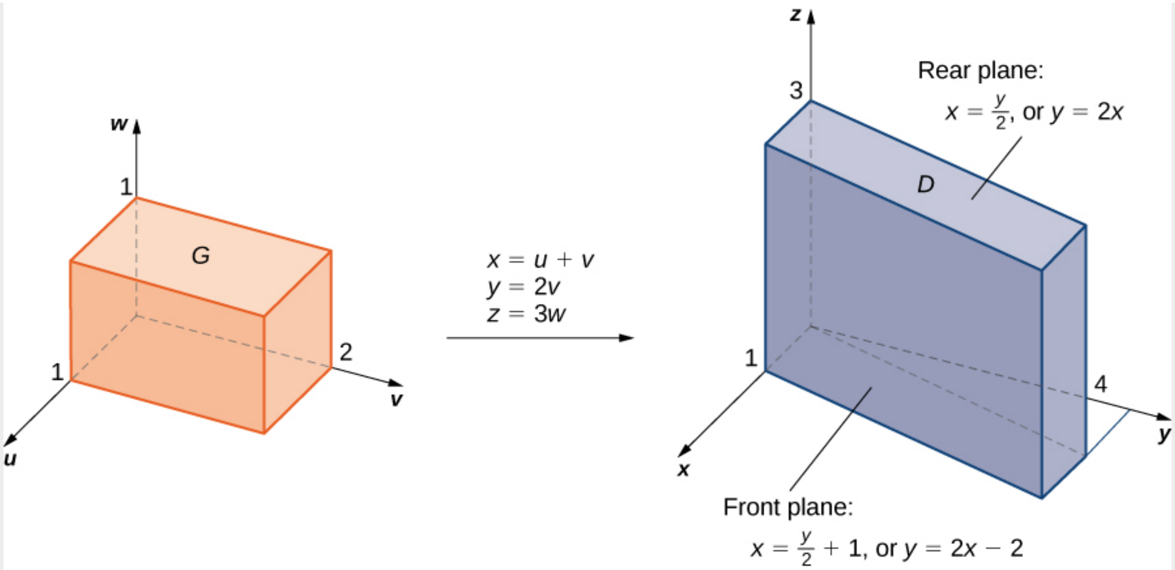
Then integrate over an appropriate region in uvw-space.

Solution:

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space ([link]). Clearly G in xyz-space is bounded by the planes $x = y/2, x = (y/2) + 1, y = 0, y = 4, z = 0,$ and $z = 4$. We also know that we have to use $u = (2x - y)/2, v = y/2,$ and $w = z/3$ for the transformations. We need to solve for $x, y,$ and z. Here we find that $x = u + v, y = 2v,$ and $z = 3w$.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	**Corresponding equations in uvw for the region G**	**Limits for the integration in uvw**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$



The region G in uvw-space is transformed to region D in xyz-space.

Now we can calculate the Jacobian for the transformation:
Equation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes
Equation:

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.
Equation:

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)\,|J(u,v,w)|du\,dv\,dw = \int_0^1 \int_0^2 \int_0^1 (u+v+w)\,|6|du\,dv\,dw$$

$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)du\,dv\,dw = 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\,dw$$

$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + w\right) dv\,dw = 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$

$$= 6\int_0^1 (3+2w)dw = 6\left[3w + w^2\right]_0^1 = 24.$$

Note:
Exercise:

Problem: Let D be the region in xyz-space defined by $1 \leq x \leq 2, 0 \leq xy \leq 2,$ and $0 \leq z \leq 1.$

Evaluate $\iiint\limits_D \left(x^2y + 3xyz\right) dx\, dy\, dz$ by using the transformation $u = x, v = xy,$ and $w = 3z.$

Solution:

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8$$

Hint

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$) is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then $\iint\limits_R f(x,y)dA = \iint\limits_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$
- If F is continuous on R, then
  Equation:

$$\begin{aligned}\iiint\limits_R F(x,y,z)dV &= \iiint\limits_G F(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw \\ &= \iiint\limits_G H(u,v,w)\,|J(u,v,w)|du\,dv\,dw.\end{aligned}$$

In the following exercises, the function $T : S \rightarrow R, T(u,v) = (x,y)$ on the region $S = \{(u,v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$ bounded by the unit square is given, where $R \subset \mathrm{R}^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.
b. Find the images of the vertices of the unit square S through the function T.
c. Determine the image R of the unit square S and graph it.

Exercise:

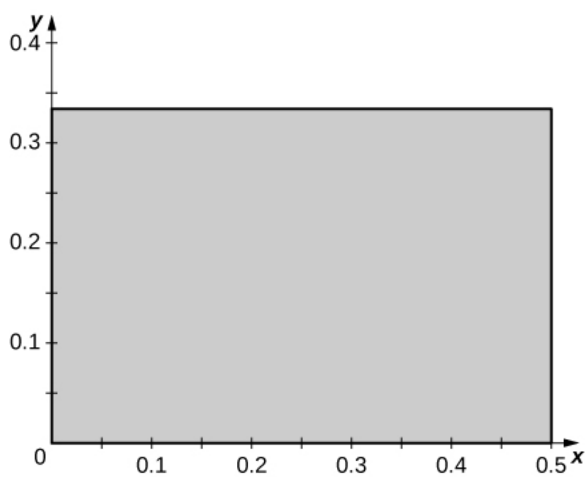
Problem: $x = 2u, y = 3v$

Exercise:

Problem: $x = \frac{u}{2}, y = \frac{v}{3}$

Solution:

a. $T(u,v) = (g(u,v), h(u,v)), x = g(u,v) = \frac{u}{2}$ and $y = h(u,v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = \frac{1}{2}$, $g_v(u,v) = 0, h_u(u,v) = 0$ and $h_v(u,v) = \frac{1}{3}$ are continuous on S; b. $T(0,0) = (0,0)$, $T(1,0) = \left(\frac{1}{2}, 0\right), T(0,1) = \left(0, \frac{1}{3}\right)$, and $T(1,1) = \left(\frac{1}{2}, \frac{1}{3}\right)$; c. R is the rectangle of vertices $(0,0)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.



Exercise:

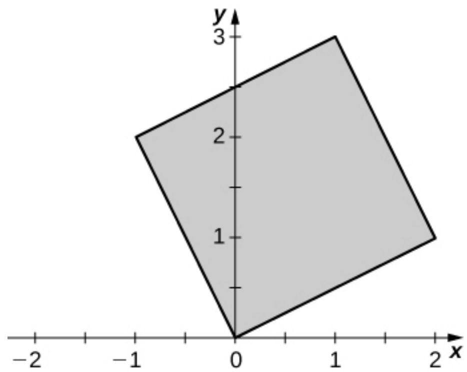
Problem: $x = u - v, y = u + v$

Exercise:

Problem: $x = 2u - v, y = u + 2v$

Solution:

a. $T(u, v) = (g(u, v), h(u, v)), x = g(u, v) = 2u - v$, and $y = h(u, v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 2$, $g_v(u, v) = -1$, $h_u(u, v) = 1$, and $h_v(u, v) = 2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (2, 1)$, $T(0, 1) = (-1, 2)$, and $T(1, 1) = (1, 3)$; c. R is the parallelogram of vertices $(0, 0), (2, 1), (1, 3), \text{and } (-1, 2)$ in the xy-plane; see the following figure.



Exercise:

Problem: $x = u^2, y = v^2$

Exercise:

Problem: $x = u^3, y = v^3$

Solution:

a. $T(u, v) = (g(u, v), h(u, v)), x = g(u, v) = u^3$, and $y = h(u, v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 3u^2$, $g_v(u, v) = 0$, $h_u(u, v) = 0$, and $h_v(u, v) = 3v^2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (1, 0)$, $T(0, 1) = (0, 1)$, and $T(1, 1) = (1, 1)$; c. R is the unit square in the xy-plane; see the figure in the answer to the previous exercise.

In the following exercises, determine whether the transformations $T : S \to R$ are one-to-one or not.

Exercise:

Problem: $x = u^2, y = v^2$, where S is the rectangle of vertices $(-1, 0), (1, 0), (1, 1), \text{and } (-1, 1)$.

Exercise:

Problem: $x = u^4, y = u^2 + v$, where S is the triangle of vertices $(-2, 0), (2, 0), \text{and } (0, 2)$.

Solution:

T is not one-to-one: two points of S have the same image. Indeed, $T(-2, 0) = T(2, 0) = (16, 4)$.

Exercise:

Problem: $x = 2u, y = 3v$, where S is the square of vertices $(-1, 1), (-1, -1), (1, -1), \text{and } (1, 1)$.

Exercise:

Problem: $T(u, v) = (2u - v, u)$, where S is the triangle of vertices $(-1, 1), (-1, -1)$, and $(1, -1)$.

Solution:

T is one-to-one: We argue by contradiction. $T(u_1, v_1) = T(u_2, v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u_2$ and $v_1 = v_2$.

Exercise:

Problem: $x = u + v + w, y = u + v, z = w$, where $S = R = \mathrm{R}^3$.

Exercise:

Problem: $x = u^2 + v + w, y = u^2 + v, z = w$, where $S = R = \mathrm{R}^3$.

Solution:

T is not one-to-one: $T(1, v, w) = (-1, v, w)$

In the following exercises, the transformations $T : S \to R$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

Exercise:

Problem: $x = 4u, y = 5v$, where $S = R = \mathrm{R}^2$.

Exercise:

Problem: $x = u + 2v, y = -u + v$, where $S = R = \mathrm{R}^2$.

Solution:

$u = \frac{x-2y}{3}, v = \frac{x+y}{3}$

Exercise:

Problem: $x = e^{2u+v}, y = e^{u-v}$, where $S = \mathrm{R}^2$ and $R = \{(x, y) | x > 0, y > 0\}$

Exercise:

Problem: $x = \ln u, y = \ln(uv)$, where $S = \{(u, v) | u > 0, v > 0\}$ and $R = \mathrm{R}^2$.

Solution:

$u = e^x, v = e^{-x+y}$

Exercise:

Problem: $x = u + v + w, y = 3v, z = 2w$, where $S = R = \mathrm{R}^3$.

Exercise:

Problem: $x = u + v, y = v + w, z = u + w$, where $S = R = \mathrm{R}^3$.

Solution:

$u = \frac{x-y+z}{2}, v = \frac{x+y-z}{2}, w = \frac{-x+y+z}{2}$

In the following exercises, the transformation $T : S \to R, T\left(u, v\right) = \left(x, y\right)$ and the region $R \subset \mathrm{R}^2$ are given. Find the region $S \subset \mathrm{R}^2$.

Exercise:

Problem: $x = au, y = bv, R = \left\{(x, y)|x^2 + y^2 \le a^2b^2\right\}$, where $a, b > 0$

Exercise:

Problem: $x = au, y = bv, R = \left\{(x, y)|\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$, where $a, b > 0$

Solution:

$S = \left\{(u, v)|u^2 + v^2 \le 1\right\}$

Exercise:

Problem: $x = \frac{u}{a}, y = \frac{v}{b}, z = \frac{w}{c}, R = \left\{(x, y)|x^2 + y^2 + z^2 \le 1\right\}$, where $a, b, c > 0$

Exercise:

Problem: $x = au, y = bv, z = cw, R = \left\{(x, y)|\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1, z > 0\right\}$, where $a, b, c > 0$

Solution:

$R = \left\{(u, v, w)|u^2 - v^2 - w^2 \le 1, w > 0\right\}$

In the following exercises, find the Jacobian J of the transformation.

Exercise:

Problem: $x = u + 2v, y = -u + v$

Exercise:

Problem: $x = \frac{u^3}{2}, y = \frac{v}{u^2}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $x = e^{2u-v}, y = e^{u+v}$

Exercise:

Problem: $x = ue^v, y = e^{-v}$

Solution:

-1

Exercise:

Problem: $x = u \cos(e^v), y = u \sin(e^v)$

Exercise:

Problem: $x = v \sin(u^2), y = v \cos(u^2)$

Solution:

$2uv$

Exercise:

Problem: $x = u \cosh v, y = u \sinh v, z = w$

Exercise:

Problem: $x = v \cosh\left(\frac{1}{u}\right), y = v \sinh\left(\frac{1}{u}\right), z = u + w^2$

Solution:

$\frac{v}{u^2}$

Exercise:

Problem: $x = u + v, y = v + w, z = u$

Exercise:

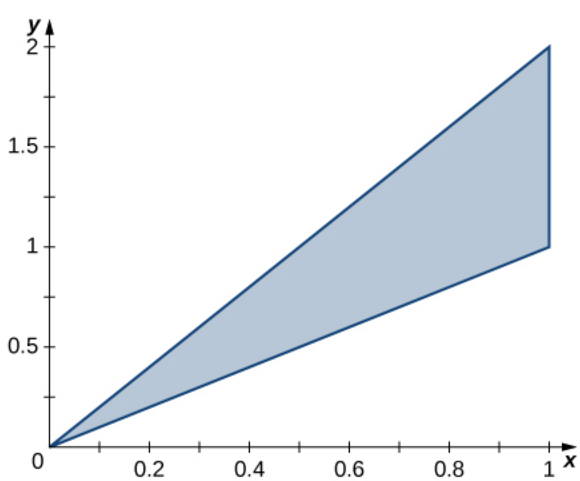
Problem: $x = u - v, y = u + v, z = u + v + w$

Solution:

2

Exercise:

Problem: The triangular region R with the vertices $(0,0)$, $(1,1)$, and $(1,2)$ is shown in the following figure.

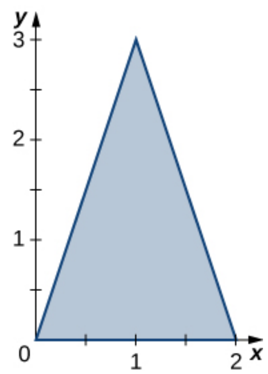


a. Find a transformation $T : S \to R$, $T(u,v) = (x,y) = (au + bv, cu + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0)$, $T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

Exercise:

Problem: The triangular region R with the vertices $(0,0)$, $(2,0)$, and $(1,3)$ is shown in the following figure.



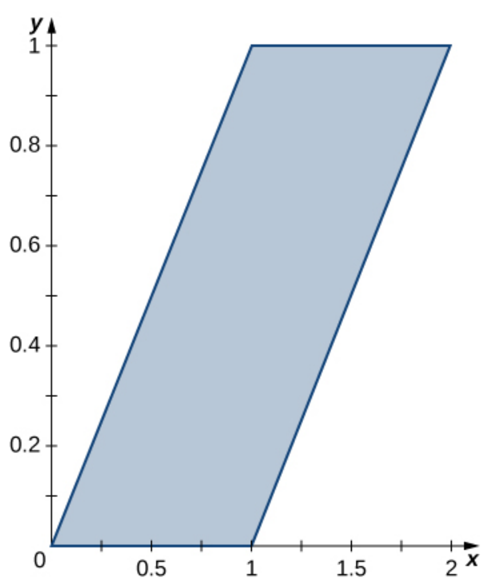
a. Find a transformation $T: S \to R$, $T(u,v) = (x,y) = (au + bv, cu + dv)$, where a, b, c and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0)$, $T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.
b. Use the transformation T to find the area $A(R)$ of the region R.

Solution:

a. $T(u,v) = (2u + v, 3v)$; b. The area of R is

$$A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\, dy = \int_0^1 \int_0^{1-u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du = \int_0^1 \int_0^{1-u} 6 dv\, du = 3.$$

In the following exercises, use the transformation $u = y - x$, $v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the following figure.



Exercise:

Problem: $\iint_R (y-x)dA$

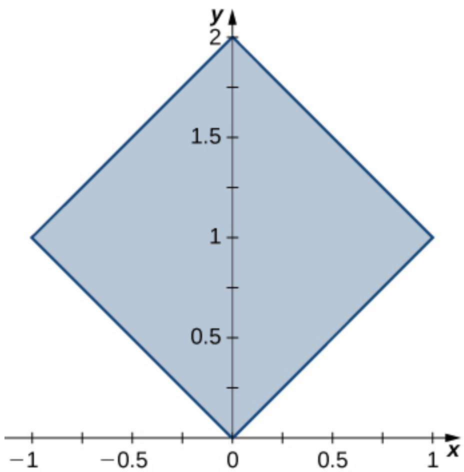
Exercise:

Problem: $\iint_R (y^2-xy)dA$

Solution:

$-\frac{1}{4}$

In the following exercises, use the transformation $y-x=u, x+y=v$ to evaluate the integrals on the square R determined by the lines $y=x, y=-x+2, y=x+2$, and $y=-x$ shown in the following figure.



Exercise:

Problem: $\iint_R e^{x+y}dA$

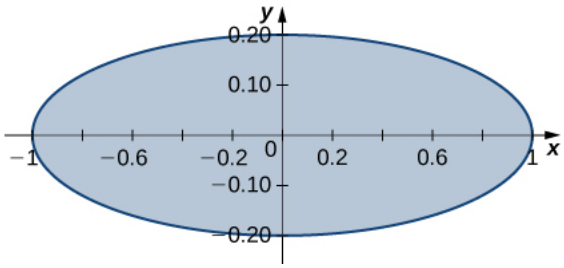
Exercise:

Problem: $\iint_R \sin(x-y)dA$

Solution:

$-1+\cos 2$

In the following exercises, use the transformation $x=u, 5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the following figure.



Exercise:

Problem: $\iint\limits_R \sqrt{x^2 + 25y^2}\, dA$

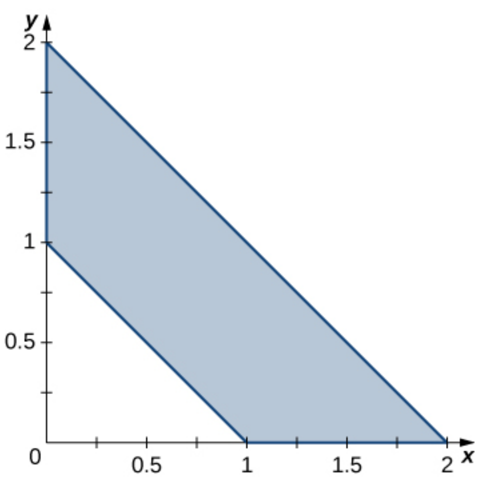
Exercise:

Problem: $\iint\limits_R \left(x^2 + 25y^2\right)^2 dA$

Solution:

$\frac{\pi}{15}$

In the following exercises, use the transformation $u = x + y, v = x - y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1, 0), (2, 0), (0, 2), \text{and } (0, 1)$ shown in the following figure.



Exercise:

Problem: $\iint\limits_R \left(x^2 - 2xy + y^2\right)e^{x+y} dA$

Exercise:

Problem: $\iint\limits_R \left(x^3 + 3x^2y + 3xy^2 + y^3\right) dA$

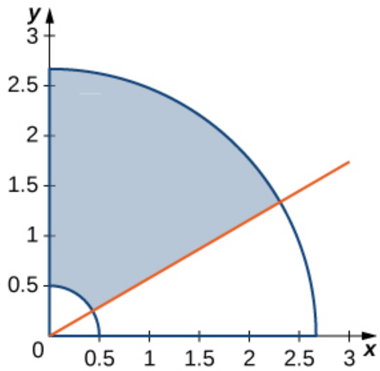
Solution:

$\frac{31}{5}$

Exercise:

Problem:

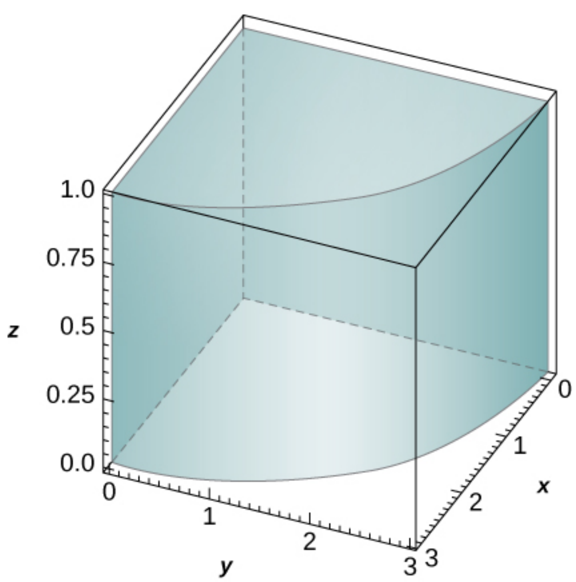
The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



Exercise:

Problem:

The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0, z = 1, x = 0,$ and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



Solution:

$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$; $S = [0, 3] \times \left[0, \frac{\pi}{2}\right] \times [0, 1]$ in the $r\theta z$-space

Exercise:

Problem:

Show that $\iint_R f\left(\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}\right)dA = 2\pi\sqrt{15}\int_0^1 f(\rho)\rho\, d\rho$, where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipse $5x^2+3y^2=15$.

Exercise:

Problem:

Show that $\iiint_R f\left(\sqrt{16x^2+4y^2+z^2}\right)dV = \frac{\pi}{2}\int_0^1 f(\rho)\rho^2 d\rho$, where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipsoid $16x^2+4y^2+z^2=1$.

Exercise:

Problem:

[T] Find the area of the region bounded by the curves $xy=1, xy=3, y=2x$, and $y=3x$ by using the transformation $u=xy$ and $v=\frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

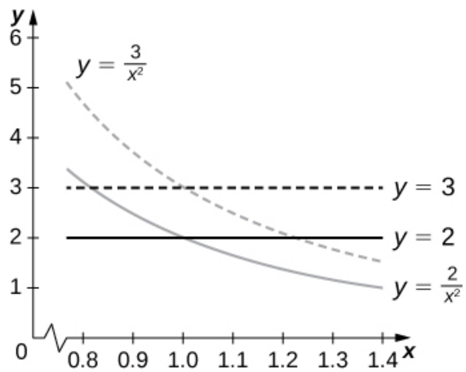
Exercise:

Problem:

[T] Find the area of the region bounded by the curves $x^2y=2, x^2y=3, y=x$, and $y=2x$ by using the transformation $u=x^2y$ and $v=\frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Solution:

The area of R is $10-4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



Exercise:

Problem:

Evaluate the triple integral $\int_0^1\int_1^2\int_z^{z+1}(y+1)dx\,dy\,dz$ by using the transformation $u=x-z$, $v=3y$, and $w=\frac{z}{2}$.

Exercise:

Problem:

Evaluate the triple integral $\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5-4y)dx\,dz\,dy$ by using the transformation $u = x - 3z, v = 4y,$ and $w = z$.

Solution:

8

Exercise:

Problem:

A transformation $T : \mathrm{R}^2 \to \mathrm{R}^2, T(u, v) = (x, y)$ of the form $x = au + bv, y = cu + dv$, where $a, b, c,$ and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

Exercise:

Problem:

The transformation $T_\theta : \mathrm{R}^2 \to \mathrm{R}^2, T_\theta(u, v) = (x, y)$, where $x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\cos\theta$, is called a rotation of angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta{}^{-1} = T_{-\theta}$, where $T_{-\theta}$ is the rotation of angle $-\theta$.

Exercise:

Problem:

[T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.
b. Evaluate the integral $\iint\limits_S e^{-2uv} du\,dv$. Round your answer to two decimal places.

Exercise:

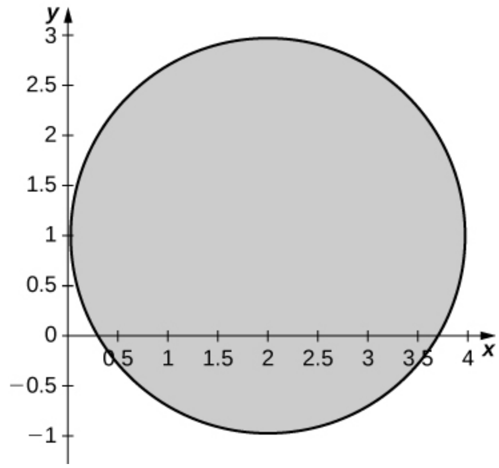
Problem:

[T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, \ldots, 4$, defined by $T_1(u, v) = (u, -v)$, $T_2(u, v) = (-u, v), T_3(u, v) = (-u, -v)$, and $T_4(u, v) = (v, u)$ are called reflections about the x-axis, y-axis, origin, and the line $y = x$, respectively.

a. Find the image of the region $S = \left\{(u, v) | u^2 + v^2 - 2u - 4v + 1 \leq 0\right\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.
b. Use a CAS to graph R.
c. Evaluate the integral $\iint\limits_S \sin(u^2) du\,dv$ by using a CAS. Round your answer to two decimal places.

Solution:

a. $R = \left\{(x, y) | y^2 + x^2 - 2y - 4x + 1 \leq 0\right\}$; b. R is graphed in the following figure;



c. 3.16

Exercise:

Problem:

[T] The transformation $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3, T_{k,1,1}(u, v, w) = (x, y, z)$ of the form $x = ku, y = v, z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral $\iiint\limits_S e^{-(4x^2+9y^2+25z^2)}\, dx\, dy\, dz$ on the solid $S = \{(x, y, z) | 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2}, y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

Exercise:

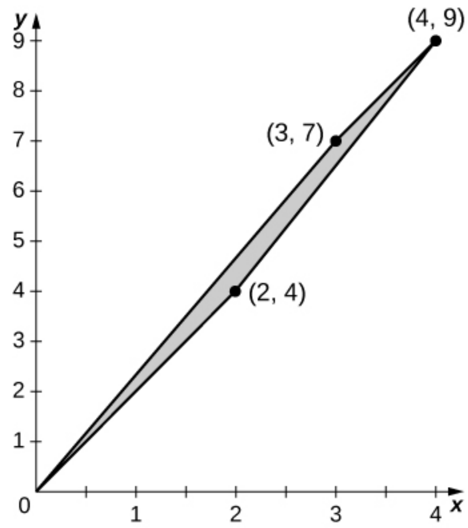
Problem:

[T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2, T_{a,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : \mathrm{R}^2 \to \mathrm{R}^2, T_{o,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.
b. Find the image R of the trapezoidal region S bounded by $u = 0, v = 0, v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.
c. Use a CAS to graph the image R in the xy-plane.
d. Find the area of the region R by using the area of region S.

Solution:

a. $T_{0,2} \circ T_{3,0}(u, v) = (u + 3v, 2u + 7v)$; b. The image S is the quadrilateral of vertices $(0, 0), (3, 7), (2, 4), \text{and } (4, 9)$; c. S is graphed in the following figure;



d. $\frac{3}{2}$

Exercise:

Problem:

Use the transformation, $x = au, y = av, z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

Exercise:

Problem:

Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Solution:

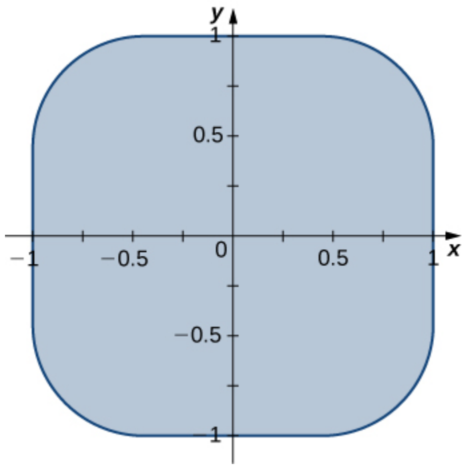
$\frac{2662}{3\pi} \simeq 282.45 \text{ in}^3$

Exercise:

Problem:

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where *a*, *b*, and *n* are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1, b = 2, n = 4$ and $n = 6$, respectively.
b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$, also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A\left(R\right)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

Exercise:

Problem:

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

Solution:

$A\left(R\right) \simeq$ 83,999.2

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.
Exercise:

Problem: $\int\limits_a^b \int\limits_c^d f\left(x, y\right) dy\, dx = \int\limits_c^d \int\limits_a^b f\left(x, y\right) dy\, dx$

Exercise:

Problem: Fubini’s theorem can be extended to three dimensions, as long as f is continuous in all variables.

Solution:

True.

Exercise:

Problem: The integral $\int\limits_0^{2\pi} \int\limits_0^1 \int\limits_r^1 dz\, dr\, d\theta$ represents the volume of a right cone.

Exercise:

Problem: The Jacobian of the transformation for $x = u^2 - 2v, y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Solution:

False.

Evaluate the following integrals.

Exercise:

Problem: $\iint\limits_R \left(5x^3y^2 - y^2\right) dA, R = \{(x, y)|0 \le x \le 2, 1 \le y \le 4\}$

Exercise:

Problem: $\iint\limits_D \frac{y}{3x^2 + 1} dA, D = \{(x, y)|0 \le x \le 1, -x \le y \le x\}$

Solution:

0

Exercise:

Problem: $\iint\limits_D \sin\left(x^2 + y^2\right) dA$ where D is a disk of radius 2 centered at the origin

Exercise:

Problem: $\int\limits_0^1 \int\limits_y^1 xye^{x^2} dx\, dy$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $\int\limits_{-1}^1 \int\limits_0^z \int\limits_0^{x-z} 6dy\, dx\, dz$

Exercise:

Problem: $\iiint\limits_R 3y\, dV$, where $R = \left\{(x, y, z)|0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{9 - y^2}\right\}$

Solution:

1.475

Exercise:

Problem: $\int_0^2 \int_0^{2\pi} \int_r^1 r\, dz\, d\theta\, dr$

Exercise:

Problem: $\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\, d\varphi\, d\theta$

Solution:

$\frac{52}{3}\pi$

Exercise:

Problem: $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$

For the following problems, find the specified area or volume.

Exercise:

Problem: The area of region enclosed by one petal of $r = \cos(4\theta)$.

Solution:

$\frac{\pi}{16}$

Exercise:

Problem: The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

Exercise:

Problem: The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

Solution:

93.291

Exercise:

Problem:

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

Exercise:

Problem: $\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

Solution:

$\left(\frac{8}{15}, \frac{8}{15}\right)$

Exercise:

Problem: $\rho(x, y) = (y+1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

Exercise:

Problem: $\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

Solution:

$\left(0, 0, \frac{8}{5}\right)$

Exercise:

Problem:

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

Exercise:

Problem:

If the compacted trash used to build Mount Holly on average has a density 400 lb/ft^3, find the amount of work required to build the mountain.

Solution:

$1.452\pi \times 10^{15}$ ft-lb

Exercise:

Problem:

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density 400 lb/ft^3 and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

Exercise:

Problem:

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature $^{\circ}C$
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Inside.shtml

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Solution:

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately $2800^{\circ}C$

Exercise:

Problem:

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu/hbase/geophys/earthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Exercise:

Problem: Find the volume when you revolve the region around the x-axis.

Solution:

$\frac{\pi}{3}$

Exercise:

Problem: Find the volume when you revolve the region around the y-axis.

Glossary

Jacobian

the Jacobian $J\left(u,v\right)$ in two variables is a $2\ \times\ 2$ determinant:

Equation:

$$J\left(u,v\right)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix};$$

the Jacobian $J\left(u,v,w\right)$ in three variables is a $3\ \times\ 3$ determinant:

Equation:

$$J(u,v,w)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w}\end{vmatrix}$$

one-to-one transformation

a transformation $T:G\to R$ defined as $T\left(u,v\right)=\left(x,y\right)$ is said to be one-to-one if no two points map to the same image point

planar transformation

a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation

a function that transforms a region G in one plane into a region R in another plane by a change of variables